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A note on the Rankin-Selberg method for Siegel cusp forms of genus 2

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1 Introduction and Notations

In [K-S] Kohnen and Skoruppa introduced and studied a new type of Dirichlet series, which is associated with the Fourier-Jacobi expansion of a pair F, G of Siegel cusp forms of the same weight and genus 2. The proof is based on the Rankin-Selberg method. In particular, it was shown that this Dirichlet series is equal to the Spinor zeta function attached to F up to constant on condition that F is a Hecke eigenform and G is in the "Maass space".

In the present note we extend a part of results in [K-S] to the case of *any level*. As an application, we give a new proof of meromorphic coutinuation of the Spinor zeta function attached to a Siegel cusp form F of any level (on a condition for Fourier coefficients of F), and find certain functoinal equation satisfied by the Spinor zeta function of any level >1. We also prove the Spinor zeta function of F times a simple meromorphic function is entire if F is not in a certain Maass space, which was proved in the level 1 case in [Ev 2], [K-S], [O].

We remark that it is relatively easy to study Kohnen-Skoruppa's Dirichlet series, even in the case of higher level (or even in the case of half-integral weight), because of its simple integral representation.

Notations. We use standard notations, found in [Ei-Z]. We let $\Gamma^g := \mathrm{Sp}_g(\mathbf{Z})$ be integral symplectic $2g \times 2g$-matrices and set

$$\Gamma_0^g(N) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \middle| \, C \equiv O \pmod N \right\},$$

where A, B, C, D are $g \times g$-matrices. We let $\Gamma^{1,J}(N)$ be the semi-direct product of $\Gamma_0^1(N)$ and $\mathbf{Z}^2$ (see [Ei-Za, p.9]), which is called the Jacobi group of level N.

$\mathcal{H}_g$ denotes the Siegel upper half space of genus g consisting of complex $g\times g$-matrices with positive definite imaginary part. We often write

$$Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathcal{H}_2,\ X = \mathrm{Re}(Z) = \begin{pmatrix} u & x \\ x & u' \end{pmatrix},\ Y = \mathrm{Im}(Z) = \begin{pmatrix} v & y \\ y & v' \end{pmatrix}.$$

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We usually set $|Y| = \det Y$.

Let k be an even integer > 2. Γ^2 acts on $\mathcal{H}_2$ by

$$\gamma\langle Z\rangle := (AZ+B)(CZ+D)^{-1} \quad \left(\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^2,\ Z \in \mathcal{H}_2\right),$$

and acts on any function $F(Z)$ on $\mathcal{H}_2$ by

$$F|_k\gamma(Z) := \det(CZ+D)^{-k} F(\gamma\langle Z\rangle).$$

$\Gamma^{1,J}(N)$ acts on any function $\phi(\tau,z)$ on $\mathcal{H}_1 \times \mathbf{C}$ by

$$\phi|_{k,m}\gamma(\tau,z) = \frac{1}{(c\tau+d)^k}\mathbf{e}\left(m\left(\frac{-cz^2}{c\tau+d} + \lambda^2\frac{a\tau+b}{c\tau+d} + \frac{2\lambda z}{c\tau+z}\right)\right)\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z+\lambda(a\tau+b)}{c\tau+d} + \mu\right)$$

$$(\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \lambda, \mu\right) \in \Gamma^{1,J}(N),\ (\tau,z) \in \mathcal{H}_1 \times \mathbf{C}),$$

where m denotes an integer ≥ 0.

We write simply $\mathbf{e}(*)$ for $\exp(2\pi i *)$.

Definition. Let χ be a Dirichlet character modulo N. A *Siegel modular form* of integral weight k, level N and character χ is a holomorphic function on $\mathcal{H}_2$ satisfying

(i) $F|_k\gamma = \chi(\det D)\, F \quad (\forall\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^2(N))$

and the vector space of all such functions F is denoted by $M_k(N,\chi)$. If $F \in M_k(N,\chi)$ satisfies

(ii) $\Phi(F|_k\gamma) = 0 \quad (\forall\gamma \in \Gamma^2,\ \Phi$ is the Siegel operator, cf. [A, p.75]),

F is called a *Siegel cusp form* and the vector space of all such functions F is denoted by $S_k(N,\chi)$. *A Jacobi cusp form* ϕ of weight k, level N, character χ and index m is a holomorphic function on $\mathcal{H}_1 \times \mathbf{C}$ satisfying

(i)' $\phi|_{k,m}\gamma = \chi(d)\,\phi \quad (\forall\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda,\mu)\right) \in \Gamma^{1,J}(N))$

(ii)' $\phi|_{k,0}\gamma = \sum_{\substack{n,r \in \frac{1}{N_\gamma}\mathbf{Z} \\ D=r^2-4mn<0}} c(D,r)q^n\zeta^r \quad (\forall\gamma \in \Gamma^1,\ N_\gamma$ is a natural number depending on $\gamma)$

and the vector space of all such functions ϕ is denoted by $J_{k,m}^{\text{cusp}}(N,\chi)$.

The Petersson inner product on these spaces are normalized by

$$\langle F,G\rangle_N := \int_{\Gamma_0^2(N)\backslash\mathcal{H}_2} F(Z)\,\bar{G}(Z)\,|Y|^{k-3}\,dX\,dY$$

$$(F,G \in M_k(N,\chi),\ Z = X+iY \in \mathcal{H}_2, \text{ One of } F,G \text{ is in } S_k(N,\chi)),$$

$$\langle\phi,\psi\rangle_N := \int_{\Gamma^{1,J}(N)\backslash\mathcal{H}_1\times\mathbf{C}} \phi(\tau,z)\,\bar{\psi}(\tau,z)\,v^{k-3}\exp\left(-\frac{4\pi m y^2}{v}\right) du\,dv\,dx\,dy$$

$$(\phi,\psi \in J_{k,m}^{\text{cusp}}(N,\chi),\ \tau = u+iv \in \mathcal{H}_1,\ z = x+iy \in \mathbf{C}).$$

2 Statement of Result

Definition. Take $F \in S_k(N,\chi)$, $G \in M_k(N,\chi)$ and a natural number M which divides N. For $\gamma \in \Gamma^2 = \mathrm{Sp}_2(\mathbf{Z})$, we write the Fourier-Jacobi expansions of $F|_k\gamma$ and $G|_k\gamma$ by

$$F|_k\gamma = \sum_{n\geq 1} \phi_{n,\gamma}(\tau,z)\,\mathbf{e}\left(\frac{n\tau'}{N}\right) \quad\text{and}\quad G|_k\gamma = \sum_{n\geq 1} \psi_{n,\gamma}(\tau,z)\,\mathbf{e}\left(\frac{n\tau'}{N}\right).$$

Then we define a *Dirichlet series* $D_{F,G,M}(s)$ as $\zeta(2s-2k+4)$ times

$$\sum_{n\geq 1}\left\{\int_{\mathcal{F}} \sum_{\gamma\in\Gamma_0^2(N)\backslash\Gamma_0^2(M)} \phi_{n,\gamma}(\tau,z)\,\bar{\psi}_{n,\gamma}(\tau,z)\exp\left(-\frac{4\pi n y^2}{vN}\right) v^{k-3}\,du\,dv\,dx\,dy\right\} n^{-s}, \tag{1}$$

on the assumption that $D_{F,G,M}(s)$ converges for sufficiently large $\mathrm{Re}(s)$, where $\mathcal{F}$ is a fundamental domain $\Gamma^{1,J}(M)\backslash\mathcal{H}_1\times\mathbf{C}$. We define its gamma factor by

$$D^*_{F,G,M}(s) := (2\pi)^{-2s}\,\Gamma(s)\,\Gamma(s-k+2)\,D_{F,G,M}(s).$$

In a special case of $M = N$, the Dirichlet series above is an obvious generalization of Rankin's Dirichlet series in the case of genus 1 (cf. [R]). In fact, if we write the Fourier-Jacobi expansions of F and G by

$$F(Z) = \sum_{n\geq 1}\phi_n(\tau,z)\,\mathbf{e}(n\tau') \quad\text{and}\quad G(Z) = \sum_{n\geq 1}\psi_n(\tau,z)\,\mathbf{e}(n\tau'),$$

then

$$D_{F,G,N}(s) = \frac{1}{N^s}\zeta(2s-2k+4)\sum_{n\geq 1}\frac{\langle\phi_n,\psi_n\rangle_N}{n^s}.$$

On the other hand, if $F(Z) \in S_k(N,\chi)$ is a Hecke eigenform with

$$T(n)F = \lambda_F(n)F$$

for all the Hecke operators $T(n)$ with $(n,N) = 1$, we can associate with F the *Spinor zeta function* $Z_F(s)$ which has an Euler product of the form

$$Z_F(s) := \prod_{\substack{p:\text{prime}\\(p,N)=1}} Q_{F,p}(\bar{\chi}(p)p^{-s}) \quad (\mathrm{Re}(s)\gg 0),$$

$$\begin{aligned} Q_{F,p}(t) := \{1 &- \lambda_F(p)t + (\lambda_F(p)^2 - \lambda_F(p^2) - \chi(p^2)p^{2k-4})t^2 \\ &- \chi(p^2)\lambda_F(p)p^{2k-3}t^3 + \chi(p^4)p^{4k-6}t^4\}^{-1}, \end{aligned} \tag{2}$$

see [A, (4.3.35), Proposition 3.3.35, Exercise 3.3.38 and (4.4.21)]. We define its gamma factor by

$$Z^*_F(s) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)Z_F(s).$$

Note that the gamma factor of $D_{F,G,M}(s)$ concides with that of $Z_F(s)$.

The modular forms which play an important role in relating (1) to (2) are Poincaré series. First, for a negative discriminant $D = r^2 - 4n$, we define the D-th *Jacobi Poincaré series* $P_{D,N}(\tau, z)$ of level N and index 1 by

$$\lambda_{k,D} P_{D,N}(\tau, z) := \sum_{\gamma \in \Gamma^{1,J}(\infty)\backslash\Gamma^{1,J}(N)} \bar{\chi}(\gamma)\mathbf{e}(n\tau + rz)|_{k,1}\gamma \in J_{k,1}^{\text{cusp}}(N, \chi), \tag{3}$$

where we write $\lambda_{k,D} := \frac{1}{2}\Gamma(k-\frac{3}{2})(\pi|D|)^{-k+3/2}$, $\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \lambda, \mu\right) \in \Gamma^{1,J}(N)$ and $\Gamma^{1,J}(\infty) := \left(\begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix}, 0, \mu\right) \subset \Gamma^{1,J}(N)$. Next, we define a *Siegel modular form* $\mathcal{P}_{D,N}(Z) \in M_k(N, \chi)$ as the image of $P_{D,N}(\tau, z)$ under the Maass lifting (for the definition, see (6) in the section 3).

Now let us state our main result.

Theorem. *Let F be a Siegel cusp form in $S_k(N, \chi)$ (k:even integer> 2). For a natural number M dividing N such that χ is defined modulo M, we define a trace of F by*

$$\mathrm{Tr}_M^N(F) := \sum_{\gamma \in \Gamma_0^2(N)\backslash\Gamma_0^2(M)} F|_k\gamma(Z) \in S_k(M, \chi).$$

Suppose that $\mathrm{Tr}_M^N(F)$ is a non-zero Hecke eigenform. Then for any negative fundamental discriminant D and a Siegel modular form $\mathcal{P}_{D,M}(Z) \in M_k(M, \chi)$ defined above, we have a relation

$$D_{F, \mathcal{P}_{D,M}, M}(s) = d_{\mathrm{Tr}_M^N(F), D}(s) Z_{\mathrm{Tr}_M^N(F)}(s). \tag{4}$$

Here for $\mathrm{Tr}_M^N(F)(Z) = \sum_{Q>0} \tilde{A}(Q)\mathbf{e}(\mathrm{tr}QZ)$, by writing the indices of Fourier coefficients by integral ideals of some order in quadratic fields, we define a Dirichlet series

$$d_{\mathrm{Tr}_M^N(F), D}(s) := \frac{1}{N^s} \sum_{\Im | M^\infty} \tilde{A}(\Im) \mathrm{N}\Im^{-(s-k+2)} \quad (\mathrm{Re}(s) \gg 0), \tag{5}$$

where $\Im$ runs through all integral ideals of the maximal order in $\mathbf{Q}(\sqrt{D})$ such that each of the prime ideals which divides $\Im$ also divides M and $\mathrm{N}\Im$ denotes the norm of $\Im$. This Dirichlet series is also defined by a following meromorphic function on the whole s-plane:

$$d_{\mathrm{Tr}_M^N(F), D}(s) := \frac{1}{N^s h(D)} \sum_{\xi} \prod_{\wp | M} \left(1 - \frac{\bar{\xi}(\wp)}{\mathrm{N}\wp^{s-k+2}}\right)^{-1} \sum_{i=1}^{h(D)} \xi(\Im_i)\tilde{A}(\Im_i),$$

where $h(D)$ denotes the class number of $\mathbf{Q}(\sqrt{D})$, $\wp$ runs through all prime ideals dividing M of the maximal order in $\mathbf{Q}(\sqrt{D})$, $\{\Im_i\}_{i=1,\dots h(D)}$ denotes a set of representatives of ideal class group and ξ runs through all ideal class characters.

We shall write down our relation (4) in the special case of $M = N$. Let

$$F(Z) = \sum_{T>0} A(T)\mathbf{e}(\mathrm{tr}TZ) = \sum_{m>0} \phi_m(\tau, z)\mathbf{e}(m\tau') \in S_k(N, \chi)$$

be a non-zero Hecke eigenform for all the Hecke operators $T(n)$ with $(n, N) = 1$, then for any negative fundamental discriminant D we have an explicit relation

$$\zeta(2s - 2k + 4) \sum_{n \geq 1} \frac{\langle \phi_n, P_{D,N}|V_n \rangle_N}{n^s} = \sum_{\Im | N^\infty} \frac{A(\Im)}{\mathrm{N}\Im^{s-k+2}} \times Z_F(s),$$

where V_n denotes the n-th Hecke operator which maps $J_{k,1}^{\text{cusp}}(N, \chi)$ to $J_{k,n}^{\text{cusp}}(N, \chi)$ (see below).

3 Proof

The proof proceeds along the lines of the second proof of [K-S], which uses the "Maass lifting" of Jacobi Poincaré series and "Andrianov's formula".

We generalize Maass lifting as follows:

Theorem-Definition ((Saito-Kurokawa-)Maass lifting). (cf. [Ei-Za] and [M-Ra-V]) Let $\phi(\tau,z)$ be a Jacobi cusp form of index 1 in $J_{k,1}^{\text{cusp}}(N,\chi)$. Then we have a lifting map from $J_{k,1}^{\text{cusp}}(N,\chi)$ to $M_k(N,\chi)$ via

$$\phi(\tau,z)\mapsto \operatorname{Lift}(\phi):=\sum_{m\geq 1}\phi|V_m(\tau,z)\mathbf{e}(m\tau'),$$

where V_m is the m-th Hecke operator which maps $J_{k,1}^{\text{cusp}}(N)$ to $J_{k,m}^{\text{cusp}}(N)$ and defined by

$$(\phi|V_m)(\tau,z):=m^{k-1}\sum_{\substack{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma_0^1(N)\backslash M_2(\mathbf{Z})\\ ad-bc=m,\ c|N,\ (a,N)=1}}\chi(a)(c\tau+d)^{-k}\mathbf{e}\left(\frac{-mcz^2}{c\tau+d}\right)\phi\left(\frac{a\tau+b}{c\tau+d},\frac{mz}{c\tau+d}\right).$$

We call this map the *Maass lifting*. We call the image $\operatorname{Lift}(J_{k,1}^{\text{cusp}}(N,\chi))$ the *Maass space* of level N and character χ.

Before the proof, we give a definition.

Definition. We define the *Jacobi subgroup* of level N of $\Gamma_0^2(N)$ by

$$C_{2,1}(N):=\left\{\begin{pmatrix} a & 0 & b & \mu\\ \lambda' & 1 & \mu' & \kappa\\ c & 0 & d & -\lambda\\ 0 & 0 & 0 & 1\end{pmatrix}\in\Gamma_0^2(N)\right\},\ (\lambda',\ \mu')=(\lambda,\ \mu)\begin{pmatrix} a & b\\ c & d\end{pmatrix})$$

which is a central extension of $\Gamma^{1,J}(N)$ by $\mathbf{Z}$.

Proof. The proof is a direct generalization of [Ei-Z, Theorem 6.2 and Theorem 4.2]. By straightforward calculations, we see $\phi|V_m$ transforms like a Jacobi form of index m. Therefore

$$\phi|V_m(\tau,z)\mathbf{e}(m\tau')$$

transforms like a Siegel modular form under the action of $C_{2,1}(N)$, hence a sum $\operatorname{Lift}(\phi)$ also does.

On the other hand, if we write the Fourier expansion of ϕ by

$$\phi(\tau,z)=\sum_{\substack{n,r\in\mathbf{Z}\\ r^2-4n<0}}c(r^2-4n,r)q^n\zeta^r\quad (q:=\mathbf{e}(\tau),\ \zeta=\mathbf{e}(z)),$$

then a standard calculation shows

$$\phi|V_m(\tau,z)=\sum_{\substack{n,r\in\mathbf{Z}\\ r^2-4mn<0}}\left(\sum_{a|(n,r,m)}\chi(a)a^{k-1}c\left(\frac{r^2-4mn}{a^2},\frac{r}{a}\right)\right)q^n\zeta^r,$$

hence we have

$$\mathrm{Lift}(\phi)\begin{pmatrix}\tau & z\\ z & \tau'\end{pmatrix} = \sum_{\begin{pmatrix}n & r/2\\ r/2 & m\end{pmatrix}>0}\left(\sum_{a|(n,r,m)}\chi(a)a^{k-1}c\left(\frac{r^2-4mn}{a^2},\frac{r}{a}\right)\right)q^n\zeta^r p^m \quad (p:=\mathbf{e}(\tau')).$$

Also we can easily see $\mathrm{Lift}(\phi)$ is symmetric in n and m, so we deduce that $\mathrm{Lift}(\phi)$ transforms like a Siegel modular form with respect to the matrix

$$V=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}.$$

Therefore $\mathrm{Lift}(\phi)$ satisfies the transformation law of Siegel modular forms by using Lemma 1 below on generators for $\Gamma_0^2(N)$.

□

Remark. We have not succeeded in proving $\mathrm{Lift}(\phi)$ is a cusp form $\in S_k(N,\chi)$ in general.

Lemma 1. *$\Gamma_0^2(N)$ is generated by $C_{2,1}(N)$ (the Jacobi subgroup of level N) and the element*

$$V=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}.$$

Proof. Any integral primitive vector $X={}^t(x_1,x_2,x_3,x_4)$ could be reduced by the left multiplication by the element of type

$$M(x,y,z)=\begin{pmatrix}1&0&0&x\\-y&1&x&z\\0&0&1&y\\0&0&0&1\end{pmatrix}$$

to a vector with g.c.d.$(x_2,x_4)=1$. Next using the element of type

$$\begin{pmatrix}1&0&0&0\\0&a&0&b\\0&0&1&0\\0&c&0&d\end{pmatrix},\ c\equiv 0\pmod N,$$

we may reduce the primitive vector X with $N|x_3,x_4$ to $X={}^t(x_1,x_2,x_3,0)$. Moreover X reduces to $(x_1,1,x_3,0)$ by using a matrix of type $M(x,y,z)$, and then by the left multiplication by the element of type

$$\begin{pmatrix}a&0&b&0\\0&1&0&0\\c&0&d&0\\0&0&0&1\end{pmatrix},\ c\equiv 0\pmod N,$$

X could be reduced to $X = (x_1, 1, 0, 0)$ (note that g.c.d.$(x_1, x_3) = 1$ and $N|x_3$).

For any element $\gamma = (X_1, X_2, X_3, X_4) \in \Gamma_0^2(N)$, we reduce the 2-th column vector X_2 to the form ${}^t(x_1, 1, 0, 0)$ and multiplying an element $VM(x,y,z)V$ finnaly to $(0,1,0,0)$. It is easily shown that this type matrix belongs to the parabolic subgroup $C_{2,1}(N)$, so Lemma 1 is proved.

□

We define a Siegel modular form as the Maass lifting of Jacobi Poicaré series defined in (3), i.e.

$$\mathcal{P}_{D,M}(Z) := \mathrm{Lift}(P_{D,M}) = \sum_{m\geq 1} (P_{D,M}|V_m)(\tau, z)\mathbf{e}(m\tau') \in M_k(M, \chi). \tag{6}$$

Now, we recall an important property of Jacobi Poincaré series:

Lemma 2. *$P_{D,N}(\tau, z)$ (the D-th Jacobi Poincaré series in $J_{k,1}^{\mathrm{cusp}}(N, \chi)$ defined in (3)) is characterized by*

$$\langle \phi, P_{D,N}\rangle_N = c(D, r) \quad (\forall \phi \in J_{k,1}^{\mathrm{cusp}}(N)),$$

where $c(D, r)$ denotes the (D, r)-th Fourier coefficient of ϕ, i.e.

$$\phi(\tau, z) = \sum_{\substack{n,r\in\mathbf{Z} \\ D=r^2-4n<0}} c(D, r)q^n\zeta^r \quad (q := \mathbf{e}(\tau),\ \zeta := \mathbf{e}(z)).$$

(Note that $c(D, r)$ depend only on $D = r^2 - 4n$ and r (mod 2)).

Proof. Tis is proved using the unfolding trick, in the same way of [G-K-Z, p.520].

□

For a half integral symmetric matrix $T = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ with $D := b^2 - 4ac$, we can associate with T a binary quadratic form

$$Q(x, y) = [a, b, c](x, y) = ax^2 + bxy + cy^2$$

of discriminant D, and a proper o-ideal of some order o of the quadratic field $\mathbf{Q}(\sqrt{D})$:

$$\Im = a\mathbf{Z} + \frac{-b + \sqrt{D}}{2}\mathbf{Z}.$$

We occasionally write $A(Q)$, $A(a, b, c)$ or $A(\Im)$ instead of $A(T)$ for Fourier coefficients of Siegel modular forms.

Proof of Theorem. We put the assumpsion that $D_{F,\mathcal{P}_{D,M},M}(s)$ converges sufficiently large $\mathrm{Re}(s)$ and put forward calculations, and later will remove the assumpsion by the convergence of Spinor zeta functions. Write the Fourier and the Fourier-Jacobi expansion of $\mathrm{Tr}_M^N(F)$ by

$$\mathrm{Tr}_M^N(F)(Z) = \sum_{T>0} \tilde{A}(T)\,\mathbf{e}(\mathrm{tr}TZ) = \sum_{m>0} \tilde{\phi}_m(\tau, z)\,\mathbf{e}(m\tau')$$

respectively, where T runs over all positive definite half integral matrices.

We recall the definition (6) of the Siegel modular form $\mathcal{P}_{D,N}(Z) \in M_k^*(N,\chi)$. We note that for any $\gamma \in \Gamma_0^2(M)$

$$\mathcal{P}_{D,M}|_k\gamma(Z) = \mathcal{P}_{D,M}(Z) = \sum_{m>0} P_{D,M}|V_m(\tau,z)\mathbf{e}(m\tau'),$$

so in the notations of (2) in Definition

$$\psi_{n,\gamma} = \begin{cases} 0 & n \text{ is not divisible by } N \\ P_{D,M}|V_m & \text{if } n = Nm \end{cases}.$$

Therefore the Nm-th coefficient of $\zeta(2s-2k+4)^{-1}D_{F,\mathcal{P}_{D,M},M}(s)$ is equal to

$$\int_{\mathcal{F}} \sum_{\gamma\in\Gamma_0^2(M)\backslash\Gamma_0^2(N)} \phi_{Nm,\gamma}(\tau,z)\,\bar{\psi}_{Nm,\gamma}(\tau,z)\exp\left(\frac{-4\pi m y^2}{v}\right)v^{k-3}\,du\,dv\,dx\,dy$$

$$= \langle \sum_{\gamma} \phi_{Nm,\gamma}, P_{D,M}|V_m\rangle_M.$$

We remark that $\sum_\gamma \phi_{Nm,\gamma}(\tau,z) = \tilde{\phi}_m(\tau,z)$ is nothing but the m-th Fourier-Jacobi coefficient of $\mathrm{Tr}_M^N(F)$ and it is a Jacobi form of index m and level M. Hence we can rewrite the above as

$$\langle \tilde{\phi}_m, P_{D,M}|V_m\rangle_M = \langle \tilde{\phi}_m|V_m^*, P_{D,M}\rangle_M,$$

where $V_m^* : J_{k,m}^{\mathrm{cusp}}(M,\chi) \to J_{k,1}^{\mathrm{cusp}}(M,\chi)$ denotes the adjoint operator of $V_m : J_{k,1}^{\mathrm{cusp}}(M,\chi) \to J_{k,m}^{\mathrm{cusp}}(M,\chi)$. Now we must calculate the action of V_m^* on Fourier coefficients explicitly.

Proposition 1. *Let $V_m^* : J_{k,m}^{\mathrm{cusp}}(N,\chi) \to J_{k,1}^{\mathrm{cusp}}(N,\chi)$ be the adjoint operator of $V_m : J_{k,1}^{\mathrm{cusp}}(N,\chi) \to J_{k,m}^{\mathrm{cusp}}(N,\chi)$ with respect to the Petersson inner products. Then we have*

$$\sum_{\substack{D<0,r\in\mathbf{Z}\\ D\equiv r^2 \pmod{4m}}} c(D,r)\mathbf{e}\left(\frac{r^2-D}{4m}\tau + rz\right)|V_m^*$$

$$= \sum_{\substack{D<0,r\in\mathbf{Z}\\ D\equiv r^2 \pmod 4}} \left(\sum_{d|m}\bar{\chi}(m/d)d^{k-2}\sum_{\substack{s \pmod{2d}\\ s^2\equiv D \pmod{4d}}} c\left(\frac{m^2}{d^2}D, \frac{m}{d}s\right)\right)\mathbf{e}\left(\frac{r^2-D}{4}\tau + rz\right).$$

(Here, $c(D,r)$ denotes the Fourier coefficient of a Jacobi form of index m and note that $c(D,r)$ depends only on D and r (mod $2m$).)

Proof. In our general case (i.e. level $N \geq 1$ and with character χ), we can proceed along the same calculation on [K-S, p.554-557].

□

Using Proposition 1 and the characterization of $P_{D,M}$ in Lemma 2, we have

$$\langle \tilde{\phi}_m|V_m^*, P_{D,M}\rangle_M = \sum_{d|m,(m/d,M)=1} \bar{\chi}(m/d)d^{k-2}\sum_{s \pmod{2d},\ s^2\equiv D \pmod{4d}} \tilde{A}\left(\frac{m}{d}\left(\frac{s^2-D}{4d}, s, d\right)\right),$$

where $\tilde{A}(*)$ denotes the Fourier coefficients of $\mathrm{Tr}_M^N(F)$. Let $\{Q_i\}_{i=1,\dots,h}$ be a set of representatives of binary quadratic forms of discriminant r^2-4n and let

$$n(Q_i;d) := \sharp\left\{s \pmod{2d} \mid s^2 \equiv D \pmod{4d},\ [\frac{s^2-D}{4d}, s, d] \sim Q_i\right\}$$

be the number of $s \pmod{2d}$ such that $s^2 \equiv D \pmod{4d}$ and the quadratic form $Q(x,y) = \frac{s^2-D}{4d}x^2 + sxy + dy^2$ is equivalent to Q_i. Then we have

$$\langle \tilde{\phi}|V_m^*, P_{D,M}\rangle_M = \sum_{i=1}^{h}\sum_{d|m} \bar{\chi}(m/d)d^{k-2}n(Q_i;d)\tilde{A}\left(\frac{m}{d}Q_i\right).$$

By [Z, Proposition 3 (i)] we can see

$$\sum_{n\geq 1} n(Q_i;n)n^{-s} = \zeta_{Q_i}(s)\zeta(2s)^{-1},$$

where $\zeta_{Q_i}(s)$ is the (partial) zeta function of the class of Q_i (= the zeta function of the ideal class of $\mathbf{Q}(\sqrt{D})$ corresponding in the usual way to the class of Q_i), so we obtain

$$D_{F,\mathcal{P}_{D,M},M}(s) = N^{-s}\sum_{i=1}^{h}\zeta_{Q_i}(s-k+2)R_{Q_i,\mathrm{Tr}_M^N(F),M}(s), \tag{7}$$

with

$$R_{Q_i,\mathrm{Tr}_M^N(F),M}(s) := \sum_{n\geq 1,(n,M)=1} \bar{\chi}(n)\tilde{A}(nQ_i)n^{-s}.$$

We now recall *Andrinov's formula*, which is mentiond in [A, Theorem 4.3.16] in a most general form. Take any negative fundamental discriminant D and any Hecke eigenform $F(Z) = \sum_Q A(Q)\mathbf{e}(\mathrm{tr}QZ) \in S_k(M,\chi)$. Then for any class character ξ of the class group $H(D)$ and any completely multiplicative function ω on $\mathrm{N}_{(M)} := \{n \in \mathrm{N} | (n,M)=1\}$, it holds that

$$A_\xi(s)\prod_{\substack{\wp:\text{prime ideal}\\(\wp,M)=1}}\left(1-\frac{\chi(\mathrm{N}\wp)\omega(\mathrm{N}\wp)\xi(\wp)}{(\mathrm{N}\wp)^{s-k+2}}\right)\prod_{\substack{p:\text{prime}\\(p,M)=1}} Q_{F,p}(\omega(p)p^{-s})$$

$$= \sum_{i=1}^{h(D)}\xi(Q_i)\sum_{n\in\mathrm{N}_{(M)}}\frac{\omega(n)A(nQ_i)}{n^s},$$

with

$$A_\xi(s) := \sum_{i=1}^{h(D)}\xi(Q_i)A(Q_i),$$

where $h = h(D) = \sharp H(D)$ is the class number of discriminant D. Inverting this,

$$\sum_{n\in\mathrm{N}_{(M)}}\frac{\omega(n)A(nQ_i)}{n^s}$$

$$= \frac{1}{h}\prod_{(p,M)=1} Q_{F,p}(\omega(p)p^{-s})\sum_{\xi}\bar{\xi}(Q_i)A_\xi(s)\prod_{\substack{\wp:\text{prime ideal}\\(\wp,M)=1}}\left(1-\frac{\chi(\mathrm{N}\wp)\omega(\mathrm{N}\wp)\xi(\wp)}{(\mathrm{N}\wp)^{s-k+2}}\right).$$

Instituting this formula for $F = \mathrm{Tr}_M^N(F)$, $\omega = \bar{\chi}$ in (7), we have

$$D_{F,\mathcal{P}_{D,M},M}(s) = \frac{Z_{\mathrm{Tr}_M^N(F)}(s)}{N^s h} \sum_{i=1}^{h} \zeta_{Q_i}(s-k+2) \sum_{\xi} \bar{\xi}(Q_i)\tilde{A}_{\xi}(s) \prod_{\substack{\wp:\text{prime ideal} \\ (\wp,M)=1}} \left(1 - \frac{\xi(\wp)}{(\mathrm{N}\wp)^{s-k+2}}\right).$$

$$= \frac{Z_{\mathrm{Tr}_M^N(F)}(s)}{N^s h} \sum_{\xi} \prod_{\wp|M} \left(1 - \frac{\bar{\xi}(\wp)}{\mathrm{N}\wp^{s-k+2}}\right)^{-1} \tilde{A}_{\xi}(s),$$

since, by writing the above Euler product by $L(s,\xi)$, it holds $L(s,\bar{\xi}) = L(s,\xi)$. We note that

$$d_{F,D}(s) := \frac{1}{N^s h} \sum_{\xi} \prod_{\wp|M} \left(1 - \frac{\bar{\xi}(\wp)}{\mathrm{N}\wp^{s-k+2}}\right)^{-1} \tilde{A}_{\xi}(s)$$

is a meromorphic function on the whole s-plane. Expanding the right hand side we get

$$D_{F,\mathcal{P}_{D,M},M}(s) = \frac{Z_{\mathrm{Tr}_M^N(F)}(s)}{N^s h} \sum_{i=1}^{h} \tilde{A}(Q_i) \sum_{\xi} \sum_{\mathfrak{I}} \frac{\xi(\mathfrak{I}Q_i^{-1})}{\mathrm{N}\mathfrak{I}^{s-k+2}},$$

and summing up for ξ's, we have the relation (4) and the expression (5).

Now we can remove the assumption on convergence of $D_{F,\mathcal{P}_{D,M},M}(s)$ for sufficiently large $\mathrm{Re}(s)$ by using convergence of $Z_F(s)$. This completes the proof of Theorem.

□

4 Applications

We summarize the known facts about the analytic properties for $D_{F,G,M}(s)$'s.

We define *Eisenstain series of Klingen-Siegel type* of weight 0 and level N by

$$E_{s,N}(Z) := \sum_{\gamma \in C_{2,1}(N)\backslash \Gamma_0^2(N)} \left(\frac{\det \mathrm{Im}\,\gamma\langle Z\rangle}{\mathrm{Im}\,\gamma\langle Z\rangle_1}\right)^s,$$

where $C_{2,1}(N)$ stands for the Jacobi subgroup of level N (see Definition in the section 3) and Z_1 denotes the left upper entry of $Z \in \mathcal{H}_2$. We define its gamma factor by

$$E^*_{s;N}(Z) := \pi^{-s}\Gamma(s)\zeta(2s) \prod_{p|N} \left(1 - \frac{1}{p^{2s}}\right) E_{s,N}(Z).$$

In this last section, for Siegel modular forms $F \in S_k(N,\chi), G \in M_k(N,\chi)$ and a natural number M dividing N, we put

$$D_{F,G;M}(s) := \prod_{p|M} \left(1 - \frac{1}{p^{2(s-k+2)}}\right) D_{F,G,M}(s), \quad D^*_{F,G;M}(s) := \prod_{p|M} \left(1 - \frac{1}{p^{2(s-k+2)}}\right) D^*_{F,G,M}(s),$$

$$Z_{F;N}(s) := \prod_{p|N} \left(1 - \frac{1}{p^{2(s-k+2)}}\right) Z_F(s), \quad Z^*_{F;N}(s) := \prod_{p|N} \left(1 - \frac{1}{p^{2(s-k+2)}}\right) Z^*_F(s).$$

Then $D_{F,G;M}(s)$ has a following integral representation:

Lemma 3 ([H 1, Lemma 2]). *We have*

$$N^s D^*_{F,G;M}(s) = \pi^{-k+2}\langle F E^*_{s-k+2;M}, G\rangle_N.$$

Also we can prove functional equations of Eisenstein series $E_{s,N}(Z)$ for arbitrary level:

Lemma 4. *Let N be a natural number. Then, the function $E^*_{s,N}(Z)$ has a meromorphic continuation to $\mathbf{C}$ with possible simple poles at $s = 0, 2$ and satisfies a functional equation*

$$\frac{1}{N^{2-s}} \sum_{d|N} d^{2(2-s)} E^*_{2-s,d}(Z) = \frac{1}{N^s} \sum_{d|N} d^{2s} E^*_{s,d}(Z),$$

or equivalently

$$E^*_{2-s,N}(Z) = \frac{1}{N^2} \sum_{e|N} e^{2s} \prod_{p|N/e} (1 - p^{2s-2}) E^*_{s,e}(Z).$$

Proof. (For details, see [H 3].) We will prove for any natural numbers m and N the formula

$$N^s E_{s,m}(NZ) = \sum_{\substack{(m,N)|d|N \\ (m,N/d)=1}} \prod_{\substack{p|d \\ (p,m)=1}} (p^{2s} - 1) \prod_{\substack{p|d \\ p|m}} p^{2s} \prod_{\substack{p^{f+1}||d \\ f\geq 1}} p^{2fs} E_{s,md}(Z) \tag{8}$$

by induction on N, and later specilize the formula (8) to $m = 1$.

By the reduction method found in [H 1, section 4], we can easily prove

$$N^s E_{s,m}(NZ) = -\sum_{1\neq M|N} \mu(M) \sum_{d|M} \mu(d) (N/M)^s E_{s,\mathrm{l.c.m.}(m,d)}((N/M)Z) + N^{2s} E_{s,mN}(Z),$$

where $\mu(*)$ denotes the Möbius function. We note that for a square-free number M with $(m, M) > 1$

$$\sum_{d|M} \mu(d) E_{s,\mathrm{l.c.m.}(m,d)}((N/M)Z) = \sum_{d_1|M/(m,M)} \mu(d_1) \sum_{d_2|(m,M)} \mu(d_2) E_{s,md_1}((N/M)Z) = 0,$$

then we have

$$N^s E_{s,m}(NZ) = -\sum_{\substack{1\neq M|N \\ (m,M)=1}} \mu(M) \sum_{d|M} \mu(d) (N/M)^s E_{s,md}((N/M)Z) + N^{2s} E_{s,mN}(Z).$$

Now by using the assumption of induction on N, we have

$$\begin{aligned}
& N^s E_{s,m}(NZ) \\
= & -\sum_{\substack{1\neq M|N \\ (m,M)=1}} \mu(M) \sum_{d|M} \mu(d) \sum_{\substack{(md,N/M)|e|N/M \\ (md,N/(Me))=1}} \prod_{\substack{p|e \\ (p,md)=1}} (p^{2s} - 1) \prod_{\substack{p|e \\ p|md}} p^{2s} \prod_{\substack{p^{f+1}||e \\ f\geq 1}} p^{2fs} E_{s,mde}(Z) \\
& + N^{2s} E_{s,mN}(Z) \\
= & -\sum_{\substack{M|N \\ (m,M)=1}} \mu(M) \sum_{d|M} \mu(d) \sum_{\substack{(md,N/M)|e|N/M \\ (md,N/(Me))=1}} \prod_{\substack{p|e \\ p|md}} p^{2s} \prod_{\substack{p^{f+1}||e \\ f\geq 1}} p^{2fs} E_{s,mde}(Z) \\
& + \sum_{\substack{(m,N)|e|N \\ (m,N/e)=1}} \prod_{\substack{p|e \\ (p,m)=1}} (p^{2s} - 1) \prod_{\substack{p|e \\ p|m}} p^{2s} \prod_{\substack{p^{f+1}||e \\ f\geq 1}} p^{2fs} E_{s,me}(Z) \\
& + N^{2s} E_{s,mN}(Z).
\end{aligned}$$

Now we can see the sum of the first and third lines on the RHS is equal to 0 by using the following Claim and get the formula (8).

Claim. We fix natural numbers d, e, m and N such that $de|N$, $(d,m)=1$ and d is square-free, then we have

$$\sum_{\substack{M\in\mathbf{N},\, d|M|N,\, (m,M)=1 \\ (md,N/M)|e|N/M,\, (md,N/(Me))=1}} \mu(M) = \begin{cases} \mu(d) & \text{if } de=N \\ 0 & \text{if } de<N \end{cases}.$$

Then the assertions for meromorphic continuation and poles are obvious by (8) and induction on N, and the symmetric functional equation follows by specilizing (8) to the case $m=1$ and using the functional equation $E^*_{2-s,1}(Z) = E^*_{s,1}(Z)$ (cf. [K-S, Main Lemma]). We can easily prove the other functional equation from the symmetic one.

□

By Lemma 3 and 4, we can deduce

Proposition 2 ([H 1, Proposition 1 and the section 4] and [H 3]). *All $D_{F,G;M}(s)$'s with $M|N$ have a meromorphic continuation to $\mathbf{C}$, are entire if $\langle F,G\rangle_N=0$ and otherwise has a simple pole at $s=k$ as its only singularity with the residue*

$$\mathrm{Res}_{s=k}D_{F,G;M}(s) = \frac{4^k\pi^{k+2}}{(k-1)!N^kM^2}\prod_{p|M}\left(1-\frac{1}{p^2}\right)\langle F,G\rangle_N.$$

Furthermore there exists a functional equation

$$N^{2(k-s)}D^*_{F,G;N}(2k-2-s) = \sum_{M|N} M^{2(s-k+2)}\prod_{p|N/M}\left(1-p^{2(s-k+1)}\right)D^*_{F,G;M}(s).$$

□

Using Proposition 2 and Theorem in the case of $M=N$ we have

Cororally 1. *Let $F\in S_k(N,\chi)$ be a non-zero Hecke eigenformof level N. Suppose that $d_{F,D}(s)$ defined by (5) is not identically zero for some fundamental discriminant D. Then $Z_{F;N}(s)$ has a meromorphic continuation to the whole s-plane, the possible poles of $d_{F,D}(s)Z_{F;N}(s)$ are $s=k$. If $d_{F,D}(k)\langle F,\mathcal{P}_{N,D}\rangle_N\neq 0$, then we have*

$$\frac{1}{\pi^{k+2}\langle F,\mathcal{P}_{N,D}\rangle_N}\mathrm{Res}_{s=k}Z_{F;N}(s) = \frac{4^k}{(k-1)!N^{k+2}d_{F,D}(k)}\prod_{p|N}\left(1-\frac{1}{p^2}\right)\in\mathbf{Q}(F,\mathrm{e}(1/h(D))),$$

where $\mathbf{Q}(F,\mathrm{e}(1/h(D)))$ is the field generated by the Fourier coefficients of F and a primitive $h(D)$-th root of unity over $\mathbf{Q}$.

Furtnhermore there exists a functional equation satisfied by the Spinor zeta function $Z_{F;N}(s)$ and the Dirichlet series $D_{F,\mathcal{P}_{M,D},M}(s)$'s with $M|N$. Explicitly, it holds

$$\begin{aligned} & N^{2(k-s)}d_{F,D}(2k-2-s)Z^*_{F;N}(2k-2-s) \\ = \; & N^{2(s-k+2)}d_{F,D}(s)Z^*_{F;N}(s) + \sum_{\substack{M|N \\ M\neq N}} M^{2(s-k+2)}\prod_{p|N/M}\left(1-p^{2(s-k+1)}\right)D^*_{F,G;M}(s). \end{aligned}$$

□

Remark. Similar results of Corollay 1 are given in [Ma] by the different method. For principal congruence subgroups. Similar results pf Corollary 1 are reported in [Ev 1, English transl. p.457] (without proof).

Cororally 2. (cf. [Ev 2], [K-S], [O].) *Let* $F \in S_k(N, \chi)$ *be a non-zero Hecke eigenform. Suppose F is in the orthogonal compliment of* $\mathrm{Lift}(J_{k,1}^{\mathrm{cusp}}(N, \chi))$ *(the Maass space, see the section 3), then* $d_{F,D}(s) Z_{F;N}(s)$ *is holomorphic for all s.*

□

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