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GENERALIZED ISOMETRIC SPHERES OF ELEMENTS OF $PU(1,n;\mathbf{C})$

SHIGEYASU KAMIYA

神谷 茂保 (岡山理大 工)

Let G be a discrete subgroup of $PU(1,n;\mathbf{C})$. For a boundary point y of the Siegel domain, we define the generalized isometric sphere $I_y(f)$ of an element f of $PU(1,n;\mathbf{C})$. By using the generalized isometric spheres of elements of G, we construct a fundamental domain $P_y(G)$ for G, which is regarded as a generalization of the Ford domain. And we show that the Dirichlet polyhedron $D(w)$ for G with center w convereges to $P_y(G)$ as $w \to y$.

1. First let us recall some definitions and notation. Let $\mathbf{C}$ be the field of complex numbers. Let $V = V^{1,n}(\mathbf{C})$ denote the vector space $\mathbf{C}^{n+1}$, together with the unitary structure defined by the Hermitian form

$$\widetilde{\Phi}(z^*, w^*) = -(\overline{z_0^*}w_1^* + \overline{z_1^*}w_0^*) + \sum_{j=2}^{n} \overline{z_j^*}w_j^*$$

for $z^* = (z_0^*, z_1^*, z_2^*, ..., z_n^*), w^* = (w_0^*, w_1^*, w_2^*, ..., w_n^*)$ in V. An automorphism g of V, that is a linear bijection such that $\widetilde{\Phi}(g(z^*), g(w^*)) = \widetilde{\Phi}(z^*, w^*)$ for z^*, w^* in V, will be called a unitary transformation. We denote the group of all unitary transformations by $U(1,n;\mathbf{C})$. Set $PU(1,n;\mathbf{C}) = U(1,n;\mathbf{C})/(center)$. Let $V_0 = \{w^* \in V | \ \widetilde{\Phi}(w^*, w^*) = 0\}$ and $V_- = \{w^* \in V | \ \widetilde{\Phi}(w^*, w^*) < 0\}$. It is clear that V_0 and V_- are invariant under $U(1,n;\mathbf{C})$. Set $V^* = V_- \cup V_0 - \{0\}$. Let $\pi : V^* \longrightarrow \pi(V^*)$ be the projection map defined by $\pi(w_0^*, w_1^*, w_2^*, \cdots, w_n^*) = (w_1, w_2, \cdots, w_n)$, where $w_j = w_j^*/w_0^*$ for $j = 1, 2, \ldots, n$. We write ∞ for $\pi(0, 1, 0, \ldots, 0)$. We may identify $\pi(V_-)$ with the Siegel domain

$$H^n = \{w = (w_1, w_2, ..., w_n) \in \mathbf{C}^n \ | \ Re(w_1) > \frac{1}{2}\sum_{j=2}^{n} |w_j|^2\}.$$

An element g in $PU(1,n;\mathbf{C})$ acts on the Siegel domain H^n and its boundary ∂H^n. In H^n, we can introduce the hyperbolic metric d (see [3] and [6]). An element of $PU(1,n;\mathbf{C})$ is an isometry of H^n with respect to d. Denote $H^n \cup \partial H^n$ by $\overline{H^n}$. The *H-coordinates* of a point $(w_1, w_2, ..., w_n) \in \overline{H^n} - \{\infty\}$ are defined by $(k, t, w')_H \in (\mathbf{R}^+ \cup \{0\}) \times \mathbf{R} \times \mathbf{C}^{n-1}$ such that $k = Re(w_1) - \frac{1}{2}\sum_{j=2}^{n} |w_j|^2$, $t = Im(w_1)$ and $w' = (w_2, ..., w_n)$. The *Cygan metric* $\rho(p, q)$ for $p = (k_1, t_1, w')_H$ and $q = (k_2, t_2, W')_H$ is given by

$$\rho(p, q) = |\{\frac{1}{2}\|W' - w'\|^2 + |k_2 - k_1|\} + i\{t_1 - t_2 + Im(\overline{w'}W')\}|^{\frac{1}{2}},$$

where $\overline{w'}W' = \sum_{j=2}^{n} \overline{w_j}W_j$.

Let $f = (a_{ij})_{1\le i,j\le n+1} \in PU(1,n;\mathbf{C})$ with $f(\infty) \ne \infty$. We define the *isometric sphere* $I(f)$ of f by

$$I(f) = \{w = (w_1, w_2, ..., w_n) \in \overline{H^n} \mid \quad |\tilde{\Phi}(W,Q)| = |\tilde{\Phi}(W, f^{-1}(Q))|\},$$

where $Q = (0,1,0,...,0)$, $W = (1, w_1, w_2, ..., w_n)$ in V^* (see [5]). It follows that the isometric sphere $I(f)$ is the sphere in the Cygan metric with center $f^{-1}(\infty)$ and radius $R_f = \sqrt{1/|a_{12}|}$, that is,

$$I(f) = \left\{ w = (k,t,w')_H \in (\mathbf{R}^+ \cup \{0\}) \times \mathbf{R} \times \mathbf{C}^{n-1} \mid \rho(w, f^{-1}(\infty)) = \sqrt{\frac{1}{|a_{12}|}} \right\}.$$

Fix $y \in \partial H^n$ such that $f(y) \ne y$. Let γ be an element of $PU(1,n;\mathbf{C})$ with $\gamma(y) = \infty$. We define the generalized isometric sphere $I_y(f)$ at y of f as

$$I_y(f) = \gamma^{-1}(I_{\gamma f \gamma^{-1}}) = \{z \in \overline{H^n} | \ \rho(\gamma(z), \gamma f^{-1} \gamma^{-1}(\infty)) = R_{\gamma f \gamma^{-1}}\}$$

(see [1]). Note that if $y = \infty$, then $I_\infty(f)$ is the usual isometric sphere $I(f)$. The definition above does not depend on the choice of the element γ such that $\gamma(y) = \infty$.

Unless otherwise stated, we shall always take f, g, ... to be elements of $PU(1,n;\mathbf{C})$ fixing neither y nor ∞. Set

$$\alpha_y(f,z) = \frac{R_f \rho(y,z)}{\rho(z, f^{-1}(y))\rho(y, f(\infty))}.$$

We can write $I_y(f)$ as

$$I_y(f) = \{z \in \overline{H^n} | \ \alpha_y(f,z) = 1\}.$$

Put

$$Ext\ I_y(f) = \{z \in \overline{H^n} | \ \alpha_y(f,z) < 1\},$$
$$Int\ I_y(f) = \{z \in \overline{H^n} | \ \alpha_y(f,z) > 1\},$$

respectively.

Just as in the case of isometric spheres, we have

Proposition 1.1.
(1) $I_{f(y)}(f) = f(I_y(f)) = I_y(f^{-1})$;
(2) $f(Ext\ I_y(f)) \subset Int\ I_y(f^{-1})$;
(3) $f(Int\ I_y(f)) \subset Ext\ I_y(f^{-1})$.

Next we consider the location of fixed points of elements.

Proposition 1.2. *Let f be an element of $PU(1,n;\mathbf{C})$ with fixed point x. If f is elliptic or parabolic, then x lies on the isometric sphere $I(f^{-1})$ of f^{-1}. If f is loxodromic, then $I(f^{-1})$ does not contain x.*

Replacing isometric spheres by generalized isometric spheres leads to the same conclusion as in Proposition 1.2.

Proposition 1.3. *Let f be an element of $PU(1, n; \mathbf{C})$ with fixed point x. If f is elliptic or parabolic, then x lies on $I_y(f)$. If f is loxodromic, then $I_y(f)$ does not contain x.*

2. Let z_1, z_2 be two different points in H^n. Let $E(z_1, z_2)$ be the bisector of $\{z_1, z_2\}$, that is,

$$E(z_1, z_2) = \{w \in H^n |\ d(z_1, w) = d(z_2, w)\}$$

(see [5] for details). Let G be a discrete subgroup of $PU(1, n; \mathbf{C})$ and let w be any point of H^n that is not fixed by any element of G except the identity. The Dirichlet polyhedron $D(w)$ for G with center w is defined by

$$D(w) = \bigcap_{g \in G - \{id\}} H_g(w),$$

where $H_g(w) = \{z \in H^n |\ d(z, w) < d(z, g(w))\}$. We observe that

(1) $D(w)$ is not necessarily convex,
(2) $D(w)$ is star-shaped about w,
(3) $D(w)$ is locally finite

(see [2], [4], [11] and [12]).

Let $\Omega(G)$ be the ordinary set of G. Assume that $\infty \in \Omega(G)$ and its stability subgroup $G_\infty = \{identity\}$. Then there is a positive constant M such that $\rho(0, g(\infty)) \leq M$ for any element g of G. The same argument as in [4] leads to the following results.

(1) The radii of isometric spheres are bounded above.

(2) The number of isometric spheres with radii exceeding a given positive quantity is finite.

(3) Given any infinite sequence of distinct isometric spheres of elements of G, the radii being R_{g_1}, $R_{g_2}, \cdots$, then $\lim_{m \to \infty} R_{g_m} = 0$.

We show that the generalized isometric sphere $I_y(f)$ is closely related to the bisector $E(z, f^{-1}(z))$.

Proposition 2.1. *If $z \in H^n$ converges to $y \in \partial H^n$, then $E(z, f^{-1}(z))$ converges to $I_y(f)$.*

By using generalized isometric spheres, we can construct a fundamental domain.

Theorem 2.2. *Let G be a discrete subgroup of $PU(1, n; \mathbf{C})$. Let ∞ be a point of $\Omega(G)$ and let $G_\infty = \{identity\}$. Suppose that y is a point of $\Omega(G) \cap \partial H^n$ and that G_y consists only of the identity. Then*

$$P_y(G) = \bigcap_{f \in G - \{id\}} Ext\ I_y(f)$$

is a fundamental domain for G.

By Proposition 2.1 and Theorem 2.2, we obtain

Theorem 2.3. *Let G be a discrete subgroup of $PU(1,n;\mathbf{C})$. Let $z \in H^n$ and let $y \in \partial H^n \cap \Omega(G)$. Then $D(z) \to P_y(G)$ as $z \to y$.*

From the manner of constructing $P_y(G)$, we have

Corollary 2.4. *The fundamental domain $P_y(G)$ is locally finite.*

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Okayama University of Science
1-1 Ridai-cho, Okayama 700-0005 JAPAN
e-mail:kamiya@mech.ous.ac.jp