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Homogeneous Manifolds of Negative Curvature and Harmonic Maps

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1 Complex hyperbolic spaces

We begin with a brief review of the geometry of complex hyperbolic spaces, which provides us the prototype of the manifolds we are going to study. For details a good reference is Chen and Greenberg [2].

Let $\mathbf{C}H^n$ denote the complex hyperbolic n-space, which is defined to be the unit ball $B^n = \{z \in \mathbf{C}^n \mid |z| < 1\}$ with the Bergman metric g_B. It is well-known that the group $SU(1,n)$ acts transitively on $\mathbf{C}H^n$ as isometries, which are linear fractional transformations on B^n preserving g_B. The isotropy subgroup K at a point $p \in \mathbf{C}H^n$ is isomorphic to $U(n)$ and is a maximal compact subgroup in an Iwasawa decomposition of $SU(1,n)$. In fact, this decomposition is given as $SU(1,n) = N \cdot A \cdot K$, where N is a 2-step nilpotent subgroup, called the Heisenberg group, and A is a 1-dimensional abelian subgroup. If we set $S = N \cdot A$, which is a solvable subgroup of $SU(1,n)$, then S acts simply transitively on $\mathbf{C}H^n$. Consequently, we can identify $\mathbf{C}H^n$ with a solvable Lie group S with a left invariant metric $\langle\ ,\ \rangle$. Moreover, it is known that the Heisenberg group N acts on the boundary sphere S^{2n-1} of B^n, which is the set of points at infinity of $\mathbf{C}H^n$, transitively except a point.

Summing up, we have the following

Fact 1 (1) *The complex hyperbolic n-space $\mathbf{C}H^n$ has a structure of a solvable Lie group S and its Bergman metric g_B is identified with a left invariant metric $\langle\ ,\ \rangle$ on S.*

(2) *The set of points at infinity S^{2n-1} of $\mathbf{C}H^n$ is identified with the one-point compactification of the Heisenberg group N, which is the nilpotent part of S.*

2 Homogeneous manifolds of negative curvature

Now, let $M = (M^n, g)$ be a Hadamard n-manifold, that is, a complete, simply connected Riemannian n-manifold of nonpositive curvature. It is well-known that M is diffeomorphic to the Euclidean n-space $\mathbf{R}^n$ and we can define a point at infinity of M to be an asymptote class of geodesic rays in M. Let $M(\infty)$ denote the set of points at infinity of M. Then we know that with a suitable topology, called the cone topology, $M(\infty)$ is homeomorphic to the $(n-1)$-sphere S^{n-1} and by attaching $M(\infty)$ to M we get a natural compactification $\overline{M} = M \cup M(\infty)$, which is homeomorphic to the closed n-ball $\overline{B^n}$. See [5] for details.

Suppose now that M is homogeneous, that is, the isometry group $\mathrm{Isom}(M)$ acts transitively on M. Then, by a result of Wolf [9] and Heintze [6], it is known that there is a solvable subgroup S of $\mathrm{Isom}_0(M)$, the identity component of $\mathrm{Isom}(M)$, which acts simply transitively on M. Therefore we can identify M with a solvable Lie group S with a left invariant metric $\langle\ ,\ \rangle$. Moreover, if we assume M is of strictly negative curvature $K < 0$, then it follows that S is a one-dimensional solvable extension of a nilpotent Lie group.

In fact, let $\mathbf{s}$ denote the Lie algebra of S and $\mathbf{n} = [\mathbf{s}, \mathbf{s}]$ be its derived algebra. Then, since $\mathbf{s}$ is solvable, $\mathbf{n}$ is a nilpotent subalgebra of $\mathbf{s}$, and the curvature condition $K < 0$ implies that the orthogonal complement $\mathbf{n}^\perp$ of $\mathbf{n}$ is one-dimensional, that is, $\mathbf{n}^\perp = \mathbf{R}\{H\}$ with a choice of a generator H. Corresponding to the direct sum decomposition $\mathbf{s} = \mathbf{n} + \mathbf{R}\{H\}$, S is decomposed as a semidirect product $S = N \cdot \mathbf{R}$ of the nilpotent subgroup N with Lie algebra $\mathbf{n}$ and the real line $\mathbf{R}$. Thus S is diffeomorphic to the product manifold $N \times \mathbf{R}$, and by identifying $(n, s) \in N \times \mathbf{R}$ with $(n, y = e^s) \in N \times \mathbf{R}_+$ we get a *generalized* Cayley transform

$$\Psi : N \times \mathbf{R}_+ \to S,$$

identifying S with the half space $N \times \mathbf{R}_+$, the product manifold of N with the positive half line $\mathbf{R}_+$. Moreover, under this identification, it is known that the set of points at infinity $M(\infty)$ of M naturally corresponds to $N \times \{0\}$ except a point defined by asymptotic geodesic rays in the $\mathbf{R}_+$ direction.

The most typical examples of homogeneous manifolds of strict negative curvature are the rank one Riemannian symmetric spaces of noncompact type, that is, real, complex or quaternion hyperbolic spaces and the Cayley hyperbolic plane. For these manifolds the nilpotent group N in the above description is in fact a 2-step nilpotent Lie group. Namely, the Lie algebra $\mathbf{n}$ of N satisfies $[\mathbf{n}, [\mathbf{n}, \mathbf{n}]] = \{0\}$, and hence is decomposed as $\mathbf{n} = \mathbf{n}_1 + \mathbf{n}_2$, where $\mathbf{n}_2 = [\mathbf{n}, \mathbf{n}]$ and $[\mathbf{n}_1, \mathbf{n}_2] = \{0\}$. Moreover, with a suitable choice of the generator H for $\mathbf{n}^\perp$, we see that $\mathbf{n}_1$ and $\mathbf{n}_2$ are the eigenspaces of the adjoint representation $\mathrm{ad}H$ on $\mathbf{n}$ with eigenvalues 1 and 2, respectively:

$$\mathbf{n}_i = \{X \in \mathbf{n} \mid \mathrm{ad}H(X) = iX\}, \qquad i = 1, 2.$$

For details, we refer the reader to [6].

Summing up these, we obtain

Fact 2 (1) *Each homogeneous Hadamard manifold $M = (M^n, g)$ has a structure of a solvable Lie group S and g is identified with a left invariant metric $\langle\ ,\ \rangle$.*

(2) *If M is of strictly negative curvature, then S is decomposed as a semidirect product $S = N \cdot \mathbf{R}$ of a nilpotent subgroup N and the real line $\mathbf{R}$.*

Moreover, M is realized as a half space $N \times \mathbf{R}_+$ under a generalized Cayley transform $\Psi : N \times \mathbf{R}_+ \to S$, and the set of points at infinity $M(\infty)$ of M is identified with the one-point compactification of N.

(3) *If M is, in particular, a rank one Riemannian symmetric space of noncompact type, then N is a 2-step nilpotent Lie group, and the Lie algebra $\mathbf{s}$ of S has an orthogonal decomposition*

$$\mathbf{s} = \mathbf{n}_1 + \mathbf{n}_2 + \mathbf{R}\{H\},$$

where $\mathbf{n} = \mathbf{n}_1 + \mathbf{n}_2$ is the Lie algebra of N and

$$\mathbf{n}_i = \{X \in \mathbf{n} \mid \mathrm{ad}H(X) = iX\}, \qquad i = 1, 2$$

with a suitable choice of H.

Remark 1 (1) In this context, real hyperbolic spaces $\mathbf{R}H^n$ are exceptional in the sense that $\mathbf{n}$ is abelian, that is, $\mathbf{n} = \mathbf{n}_1$ and $\mathbf{n}_2 = \{0\}$.

(2) In the case of complex hyperbolic spaces $\mathbf{C}H^n = (B^n, g_B)$, we see that the decomposition $\mathbf{n} = \mathbf{n}_1 + \mathbf{n}_2$ corresponds to the natural contact structure on the boundary sphere S^{2n-1}. In fact, if we identify S^{2n-1} with the one-point compactification of the Heisenberg group N as in Fact 1, and take the Hopf fibration $S^{2n-1} \to \mathbf{C}P^{n-1}$ of S^{2n-1} over the complex projective $(n-1)$-space $\mathbf{C}P^{n-1}$, then, under left translations by N, $\mathbf{n}_1$ defines the horizontal subspace at each tangent space of S^{2n-1} and $\mathbf{n}_2$ corresponds to the vertical subspace along the fibre.

(3) By a theorem of Kobayashi [7], every connected homogeneous Riemannian manifold of strictly negative curvature is simply connected.

3 Carnot spaces

Motivated by the observations in the previous sections, we now consider a more general class of homogeneous Riemannian manifolds of negative curvature which arises as a one-dimensional solvable extension of certain k-step nilpotent Lie groups, called *Carnot groups* ([8]).

More precisely, let S be a simply connected solvable Lie group satisfying the following conditions:

1. S is a semidirect product of a nilpotent Lie group N and the real line $\mathbf{R}$.

2. If $\mathbf{n}$ and $\mathbf{s} = \mathbf{n} + \mathbf{R}\{H\}$ denote the Lie algebras of N and S respectively, then $\mathbf{n}$ has a decomposition $\mathbf{n} = \sum_{i=1}^{k} \mathbf{n}_i$ into k-subspaces given by

$$\mathbf{n}_i = \{X \in \mathbf{n} \mid \mathrm{ad}H(X) = iX\}, \qquad i = 1, \ldots, k.$$

It is easy to see that, since $\mathrm{ad}H$ is a Lie algebra homomorphism, the above decomposition of $\mathbf{n}$ defines a graded Lie algebra structure of $\mathbf{n}$, that is, $[\mathbf{n}_i, \mathbf{n}_j] \subset \mathbf{n}_{i+j}$ with the convention $\mathbf{n}_i = \{0\}$ for $i > k$. Also, it follows from a result of Heintze [6] that S admits a left invariant metric g of strictly negative curvature.

Definition 1 We call a homogenous Riemannian manifold $M = (S, g)$ of negative curvature obtained as above a *k-term Carnot space.*

For example, real hyperbolic spaces are 1-term Carnot spaces, and complex or quaternion hyperbolic spaces and the Cayley hyperbolic plane are 2-term Carnot spaces.

Now, let $M = (S, g)$ be a k-term Carnot space. Then, as seen in Fact 2, via a generalized Cayley transform $\Psi : N \times \mathbf{R}_+ \to S$ mapping $(n, y) \in N \times \mathbf{R}_+$ to $n \cdot \exp sH \in S = N \cdot \mathbf{R}$, where $s = \log y$, M is realized as a half space $N \times \mathbf{R}_+$. This half space model of M clearly describes how fast the metric g blows up at infinity. In fact, we have the following proposition.

Proposition 1 (1) *On $N \times \mathbf{R}_+$ the metric g is written as a k-ply warped product metric*

$$\Psi^* g = \frac{1}{y^2} g_{\mathbf{n}_1} + \frac{1}{y^4} g_{\mathbf{n}_2} + \ldots + \frac{1}{y^{2k}} g_{\mathbf{n}_k} + \frac{dy^2}{y^2},$$

where $g_{\mathbf{n}_1} + g_{\mathbf{n}_2} + \ldots + g_{\mathbf{n}_k}$ is a left invariant metric on N and y is the coordinate on $\mathbf{R}_+$.

(2) $\mathbf{R}_+$ *directions* $(n = \text{const}, y)$ *define asymptotic geodesics of M and hence give rise to a point at infinity $\infty \in M(\infty)$. Moreover, $M(\infty) \setminus \{\infty\}$ is naturally identified with $N \times \{0\}$.*

Since asymptote classes of geodesic rays are preserved under isometries, the isometry group $\mathrm{Isom}(M)$ of M acts also on $M(\infty)$. Concerning this extended action of $\mathrm{Isom}(M)$, we have the following fact obtained by a combination of results due to Chen [1] and Druetta [4].

Fact 3 (1) *If M is a rank one symmetric space of noncompact type, then* $\mathrm{Isom}(M)$ *has no common fixed point in $M(\infty)$.*

(2) *If M is non-symmetric, then* $\mathrm{Isom}(M)$ *has a unique common fixed point $\gamma(\infty) \in M(\infty)$, and for any $p \neq \gamma(\infty)$ in $M(\infty)$, under the left translations by N as isometries, the orbit $N(p)$ of p coincides with $M(\infty) \setminus \{\gamma(\infty)\}$.*

As a consequence, if M is symmetric, then, via generalized Cayley transforms, half space models $N \times \mathbf{R}_+$ of M provide local coordinate charts at the boundary $M(\infty)$ of the compactification $\overline{M} = M \cup M(\infty)$ so that $\overline{M}$ admits a structure of a smooth manifold with boundary. On the other hand, if M is non-symmetric, then we have a unique half space model $\Psi : N \times \mathbf{R}_+ \to S$ of M.

4 Harmonic maps

Let $M = (M, g)$ and $M' = (M', g')$ be Riemannian manifolds and $u : M \to M'$ a C^∞ map from M to M'. Then the differential du is a section of $T^*M \otimes u^{-1}TM'$, the tensor product of the cotangent bundle T^*M of M and the induced bundle $u^{-1}TM'$ obtained from the tangent bundle TM' of M' by u, and the Levi-Civita connections of M and M' define a natural connection ∇ on $T^*M \otimes u^{-1}TM'$. So we have ∇du as a section of $T^*M \otimes T^*M \otimes u^{-1}TM'$ and, taking the trace in the first two factors, we get the *tension field* $\tau(u) = \mathrm{Tr}(\nabla du)$ of u, which is a section of $u^{-1}TM'$. We call u a *harmonic map* if its tension field $\tau(u)$ vanishes identically.

From now on, let $M = (N \cdot \mathbf{R}, g)$ and $M' = (N' \cdot \mathbf{R}, g')$ be k-term Carnot spaces with $k \geq 2$. Recall that the Lie algebras $\mathbf{n}$ and $\mathbf{n}'$ of N and N' have decompositions $\mathbf{n} = \sum_{i=1}^{k} \mathbf{n}_i$ and $\mathbf{n}' = \sum_{i=1}^{k} \mathbf{n}'_i$ as graded Lie algebras, respectively. Moreover, as in Proposition 1, when identifying $M(\infty) \setminus \{\infty\}$ with $N \times \{0\}$ and $M'(\infty) \setminus \{\infty'\}$ with $N' \times \{0\}$, each subspace $\mathbf{n}_i$ and $\mathbf{n}'_i$ define, under left translations by N and N' respectively, distributions on the boundaries $M(\infty)$ and $M'(\infty)$, which we denote also by $\mathbf{n}$ and $\mathbf{n}'$.

Now, let $u : M \to M'$ be a proper C^∞ map from M to M', and V be a neighborhood of some boundary point $p \in N \times \{0\}$. Suppose that u extends to a C^k map from $V \cap \overline{M}$ into $\overline{M'}$, and denote the boundary value of u by $f : V \cap (N \times \{0\}) \to N' \times \{0\}$. We say that f is *nondegenerate* if it satisfies

$$df_p\left((\mathbf{n}_k)_p\right) \not\subset \sum_{j=1}^{k-1} (\mathbf{n}'_j)_{f(p)}$$

at any $p \in V \cap (N \times \{0\})$. Then we have the following

Theorem 1 *Suppose that* $u \in C^\infty(V \cap M, M') \cap C^k(V \cap \overline{M}, \overline{M'})$ *be a harmonic map with nondegenerate boundary value* $f \in C^k(V \cap (N \times \{0\}), N' \times \{0\})$*. Then* f *must satisfy for each* $1 \leq i \leq k$

$$df_p \left(\sum_{j=1}^{i} (\mathbf{n}_j)_p\right) \subset \sum_{j=1}^{i} (\mathbf{n}'_j)_{f(p)}$$

for any $p \in V \cap (N \times \{0\})$*.*

Theorem 1 claims that nondegenerate boundary values of proper harmonic maps between k-term Carnot spaces, having sufficient regularity up to the boundary, preserve the filtrations on the boundaries defined by distributions $\mathbf{n}_i$ and $\mathbf{n}'_i$. In fact, under the assumption of Theorem 1, we can inductively deduce the asymptotic behavior of derivatives, in the $\mathbf{R}_+$ direction, of u near the boundary. The details will appear elsewhere.

Remark 2 *When* $k = 2$*, the conclusion of Theorem* 1 *simply means that* $df_p((\mathbf{n}_1)_p) \subset (\mathbf{n}'_2)_{f(p)}$ *for any* $p \in V \cap (N \times \{0\})$*. This result has been proved by Donnelly in* [3] *under a weaker condition that* $u \in C^\infty(V \cap M, M') \cap C^1(V \cap \overline{M}, \overline{M'})$ *and without the nondegeneracy of* f*.*

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