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Complex projective structures and the marked length rigidity

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Abstract

In this note we show that the marked length spectrum with respect to the Thurston metric on the complex projective structure determines the complex structure.

1 Introduction

A surface can be endowed with either a real or a complex projective structure modeled on $\mathbb{R}P^2$ or $\mathbb{C}P^1$ respectively [2], [3]. So a (complex) projective structure is a coordinate system into open sets in $\mathbb{C}P^1$ so that the transition maps are projective maps i.e., the restriction of elements in $PSL(2,\mathbb{C})$. The projective structure on the surface of genus g is a holomorphic bundle over the Teichmuller space with fibre quadratic differentials on each hyperbolic structure (equivalently a complex structure).

W. Thurston described a projective structure as a bent structure of a hyperbolic structure along some measured lamination. Embed a hyperbolic plane in hyperbolic 3-space and bend the plane along the lifts of a closed geodesic in the surface. What we see in the ideal boundary ($\mathbb{C}P^1$) is inserting a flat annulus along the geodesic. In this point of view, Thurston described the geometry of the projective structures and gave a concrete construction of the projective structure from a hyperbolic structure. See [5] for related topics. He also defined a Thurston (pseudo)-metric (projective metric) parallel to the Kobayashi metric (hyperbolic metric) on a Riemann surface.

We will use this Thurston metric to measure the length of geodesics and the area of the projective structure. Since this metric measures only the length of the tangent vectors, it is a Finsler metric if non-degenerate. Even though there are examples that a Finsler and a hyperbolic metric can have the same marked length spectrum and are not isometric, in our situation, by the dint of the simple description of the Thurston metric, a projective structure having the same marked length spectrum with a hyperbolic structure must be isometric to that hyperbolic structure. We will deduce the problem to showing rather a simple fact in hyperbolic geometry. After the description of the Thurston metric, it will be clear that a projective structure with the same volume (in an appropriate sense) to a hyperbolic structure is itself hyperbolic.

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[2]*Key words and phrases.* projective structure, Thurston metric, marked length spectrum.

2 Thurston metric and a grafting

2.1 Thurston metric

Definition 1 *Let M be a $\mathbb{C}P^1$ manifold. For each tangent vector $v \in T_xM$, define the length of the vector v by*

$$t(v) = \inf_{f:\triangle\to M} \rho(f^*v),$$

where the infimum is taken over all projective immersion $f : \triangle \to M$ and ρ is the hyperbolic metric on the unit disc $\triangle$.

Note that since the Kobayashi metric is defined as the infimum over all holomorphic immersion, generally Thurstion metric is larger than the Kobayashi metric. But the Thurston metric has the following properties:

1. If two metrics are non-degenerate and coincide at a nonzero tangent vector, then they coincide on the entire tangent space TM.
2. If $t(v) \neq 0$ at $v \in T_zM$, then there is a projective map $f : \triangle \to M$ realizing $t(v)$, and this map is unique up to precomposition of automorphism of $\triangle$.

Note that the hyperbolic metric (Kobayashi metric) does not distinguish some projective structures but projective metric does.

2.2 Grafting

Let $X = H^2/\Gamma$ be a hyperbolic surface and γ be a simple closed geodesic. Take a lower half plane model of H^2 and arrange that a lift of γ is the lower half of y-axis. Let γ represent the element $\gamma(z) = e^{l(\gamma)}z$ where $l(\gamma)$ is the length of γ. Next take the sector $\Theta = \{re^{i\rho} | 0 < r < \infty, 0 \leq \rho \leq \theta\}$ with the projective structure inherited from $\mathbb{C}P^1$. Then $A_\theta = \Theta/\langle\gamma\rangle$ is a flat annulus with height θ and circumference $l(\gamma)$.

Cut X along γ and paste back A_θ. Then the hyperbolic structure on X and the projective structure on A_θ match along the copies of γ to give a new projective structure. This process is called the grafting along γ according to Thurston. Since the simple closed curves with weights are dense in the space of measured lamination, above construction extends to an arbitrary measured lamination.

In terms of Thurston metric t, it is not difficult to see that t agrees with a hyperbolic metric on the original piece and with a flat metric on A_θ. Thurston showed the following theorem, which is the starting point of our argument in this note.

Theorem 1 *This grafting map is a homeomorphism from $T_g \times \mathcal{ML}$ onto P_g where T_g is a Teichmuller space, $\mathcal{ML}$ the space of measured lamination, P_g the set of projective structures.*

3 Hyperbolic structure is minimal volume

In general, if N is just a norm and g is an inner product on $\mathbb{R}^n$, then the volume element

$$dvol_N = \frac{Vol_g(B_g^1)}{Vol_g(B_N^1)} dvol_g$$

where B_g^1, B_N^1 denote the unit ball with respect to g and N respectively, is independent of the choice of the inner product g. So if N is a finsler metric, it has a volume element in this way. But in our case, it is clear from the description of the Thurston metric that we can take the metric as natural one on each pieces, namely the hyperbolic metric on hyperbolic piece and the flat metric on the grafted flat pieces. The area is then obviously the sum of hyperbolic pieces and the flat pieces. Since the sum of hyperbolic pieces is always $-2\pi\chi(S)$, the area is strictly larger than the one of hyperbolic structure whenever it is grafted. This shows that every nontrivial (meaning non-hyperbolic) projective structure has strictly larger volume than the hyperbolic structure.

Note that the topological entropy of a flow on a compact metric space measures the exponential complexity of the dynamics of the flow. So the geodesic flow on the flat metric has topological entropy zero.

Let M be a projective structure obtained from the hyperbolic structure (S, g) grafted along the measured lamination L. Denote t_M the Thurston metric.

Proposition 1 *We have the following Besson-Courtois-Gallot [1] inequality.*

$$Vol(M)h(t_M)^2 \geq Vol(S)h(g)^2$$

where $h(g)$ denote the topological entropy with respect to the metric g. The equality holds iff M is a hyperbolic structure g.

Proof: Since $h(t_M) = h(g) = 1$, by the above comments, the claim follows. ∎

4 Hyperbolic structure has the marked length rigidity

Let M be a projective structure parametrized by (g, L) by the Thurston theorem, where g is a hyperbolic structure and L is a measured lamination. In this section we show that if M has the same marked length spectrum with some hyperbolic structure, then M is actually g. First we prove a simple lemma.

Lemma 1 *For any closed curve c, $l_t(c) \geq l_g(c)$. Equality holds only if c does not intersect L.*

Proof: As in Figure 1, if c intersects the annulus, replace it by the curve above the annulus followed by the segment a along the boundary of the annulus and the curve below the annulus. Then since the length of a is less than the diagonal d in Euclidean geometry, the new curve constructed has strictly small length than c in M. If c meets the annulus orthogonally, the new curve is just the one with the segment in the annulus taken out, and in this case also, it has the smaller length than c in M. Pull this new curve tight in the hyperbolic structure g to get a geodesic representative. This shows that $l_t(c) \geq l_g(c)$ and the equality holds only if there is no annulus intersecting c. ∎

Suppose M has the same marked length spectrum with a hyperbolic structure h. Then this lemma implies that for any closed curve c, $l_g(c) \leq l_h(c)$ and the strict inequality whenever c intersects L transversely. We will show that this is impossible.

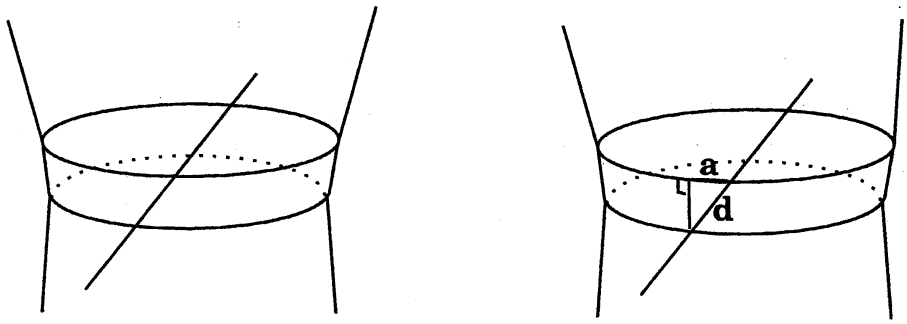

Figure 1: Replacing a curve by a shorter one

Theorem 2 *Let $M = (g, L)$ be a projective structure having the same marked length spectrum with a hyperbolic structure h. Then $M = g = h$.*

Proof: We can assume that L consists of simple closed curves since these kinds of measured laminations are dense in $\mathcal{ML}$. Let $L = (c_1, \cdots, c_k)$. Then one can extend this to $(c_1, \cdots, c_k, d_{k+1}, \cdots, d_{3g(M)-3})$, where $g(M)$ is the genus of M, to cut g along these curves into pairs of pants. Since M and h have the same marked length spectrum, all the pairs of pants in h have the same lengths of cuffs with the corresponding ones in g. Consider two pairs of pants glued along, say c_1. This has four cuffs and c_1 as a waist. Denote it S_h and S_g respectively. Since all the cuffs and the waist c_1 have the same length, S_h and S_g differ only by the twisting parameter along c_1. By Lemma 1, any closed curve in S_h crossing c_1 transversely has the strictly larger length than in S_g. We claim this is not possible, which is shown in the next section. ■

5 Geometry of a hyperbolic structure on the surface

Let S be a pair of pants. The hyperbolic structure of S is completely determined by the lengths of three cuffs since S can be decomposed into two copies of right-angled hexagons and a right-angled hexagon is determined by the lengths of three non-consecutive edges. Let S' be another pair of pants with the same cuff c with S. Glue S and S' along a cuff c. Denote it M. Then the hyperbolic structure of M is determined by the gluing parameter along the cuff. There is a natural base point of the hyperbolic structures on M. Let l be the arc meeting c orthogonally and separating the other two cuffs. This arc is unique. Let l' be such an arc in S'. If one glue S and S' so that one of the end point of l and the one the end point of l' match together, this hyperbolic structure is a base point in the set of hyperbolic structures. Any other hyperbolic structure can be described as the one glued along c rotated by the length t from this base structure. See Figure 2. What happens in the universal cover is the shifting along the lifts of c, which is called the *earthquake* by Thurston [4]. In Figure 3, the left hand picture shows the base hyperbolic structure and the right hand one the hyperbolic structure rotated by t. The geodesic a in the rotated structure is obtained as follows. The line consists of horizontal pieces, and vertical pieces on the lifts of c with length t, is a quasi-geodesic. It converges to two points on the ideal boundary of $H^2_{\mathbb{R}}$, and a is the unique geodesic connecting these two points. As the picture suggests, the angles from a to c counter-clockwise, is strictly decreasing. For a complete

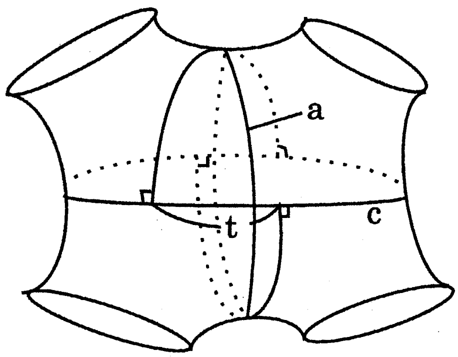

Figure 2: Hyperbolic structure on M

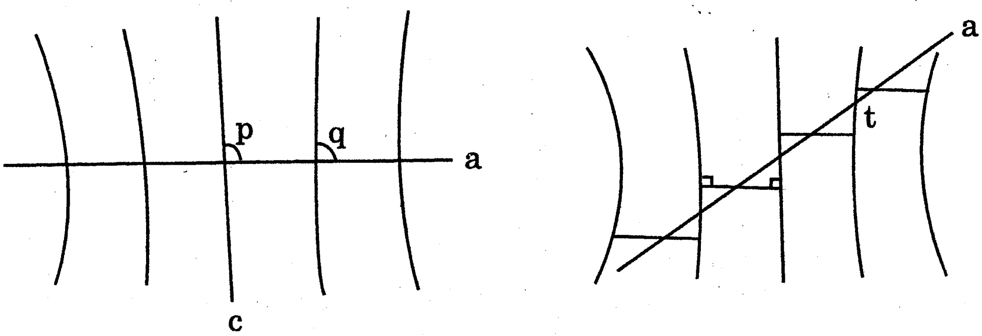

Figure 3: Earthquake along the lifts of c

proof see [4]. Furthermore the rate of change of the length of a at the base hyperbolic structure is $\cos p + \cos q$ in Figure 3. In this section we use the Kerckhoff theorem [4] to prove Theorem 2.

Theorem 3 *(Kerckhoff) Let γ be a closed curve and μ be a measured lamination. Let $E(t)$ be the earthquake map defined by $t\mu$ $(t \geq 0)$. Let g be a fixed hyperbolic structure and $g_t = E(t)(g)$. Let l_t be the geodesic length of γ in g_t and $\theta_i(t)$ be the angle measured from γ to μ counter-clockwise. Then*

$$\frac{dl_t}{dt}(t) = \sum_i \cos \theta_i(t)$$

and $\theta_i(t)$ is strictly decreasing.

What we want to show in this section is: For two hyperbolic structures g and h on M, it is not possible that, for any closed curve γ intersecting the waist c, $l_g(\gamma) < l_h(\gamma)$.

Proof of Theorem 2: Since the Teichmuller space of M is $\mathbb{R}^+$, and for any closed curve γ intersecting the waist c, $l_g(\gamma) < l_h(\gamma)$, $h = E(t)(g)$ for some $t > 0$. But it is not hard to see that there is a closed curve γ such that $\sum_i \cos \theta_i(t) < 0$ with $\theta_i(t)$ almost π in the hyperbolic structure h (for example, a closed curve wraps around c many times and cross c from the bottom to the top). Since the angle $\theta_i(t)$ is strictly decreasing, $\sum_i \cos\theta_i(0) < 0$ in g. This means that the length of the curve γ was larger in g than in h, which is a contraction.

This proof shows that along the earthquake path, some curves get longer and some curves get shorter.

6 Marked length spectrum on Projective structures

This section attempts to prove the marked length rigidity between two non-trivial projective structures. This seems to be much harder than the previous section problem. A similar argument to lemma 1 shows that if one fixes a hyperbolic structure g and a measured lamination L, then (g, L) and $(g, \rho L)$ do not have the same marked length spectrum for $\rho \neq 1 > 0$. Also (g, L_1) and (g, L_2) have the different marked length spectrum if there is a closed curve which intersects only one of L_i. Specially (g, L_1) and (g, L_2) have different marked length spectrum if $i(L_1, L_2) \neq 0$. The following theorem will show that (g, L_1) and (g, L_2) have different marked length spectrum in any case.

In this section we prove the following simple theorem.

Theorem 4 *Let $(g, L_1), (h, L_2)$ be two projective structures having the same marked length spectrum and $i(L_1, L_2) = 0$. Then $g = h, L_1 = L_2$, i.e., the projective structures are the same.*

Proof: Assume L_i are weighted simple closed curves. Then we can extend $L_1 \cup L_2$ to the set of $3g-3$ simple closed curves to cut the surface into pairs of pants. If a subsurface M, which is a glued pants along a waist c, has an annulus along c in one projective structure and no annulus in the other structure, by the proof of the previous section, there should be no annulus and two hyperbolic structures on M are the same. The only non-trivial case is that both structures have annulus inserted. Denote the annuli A_g and A_h respectively. Note that the lengths of all the cuffs and the waist c are the same by the assumption. We still denote the hyperbolic structures on M by g and h respectively. Then one is obtained from the other by an earthquake. Let h be $E(t)(g)$ for some $t > 0$ without loss of generality. Then it is easy to see using the universal covering picture that the projective structure (M, L_2) is obtained from g by first inserting A_h and then earthquaking by the distance t. See Figure 4.

CASE I. A_h is thicker than A_g.

Choose a closed curve γ in (M, L_1) such that $\sum_i \cos\theta_i > 0$ with $\theta_i < \frac{\pi}{2}$ (again such a curve exists by wrapping around c many times and cross c from the top to the bottom). As noticed already, the length of γ in the projective structure obtained from g by inserting A_h, is longer than in (M, L_1) since A_h is thicker than A_g. Then by the Kerckhoff's formula, earthquaking by the distance t makes γ longer, which means that the length of γ in (M, L_2) is larger than in (M, L_1). This is a contradiction. So A_h cannot be thicker than A_g. By the same argument we conclude that $A_h = A_g$.

CASE II. $A_h = A_g$.

In this case also, if $E(t)(g) = h$, then the closed curve in (M, L_1) with $\sum_i \cos\theta_i > 0$ with $\theta_i < \frac{\pi}{2}$ is longer in (M, L_2) than in (M, L_1) since (M, L_2) is obtained from (M, L_1) by the earthquake along A_g by the distance $t > 0$. This shows that $g = h$. ■

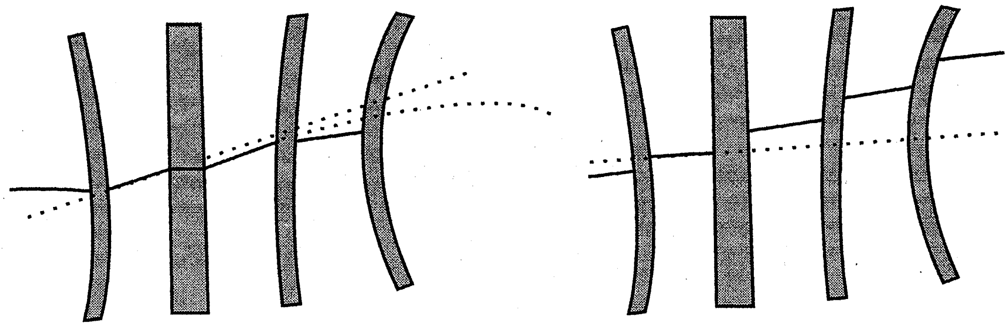

Figure 4: Earthquake of a projective structure

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