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共形ケーラー等質空間について
(Conformal Kähler homogeneous spaces)

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Contents

Introduction

The purpose of this note is to show that the complex 2-dimensional locally conformal Kähler solvmanifold obtained by L. de Andres, Fernandez, Mencia and Cordero [ACFM] coincides bihomolophically with the Inoue surface equipped with the locally conformal Kähler structure constructed by Tricerri [TR]. In order to prove it, we supplement several facts related to the existence of locally conformal Kähler structure on compact complex surfaces.

1 Seifert manifolds with solvable fundamental groups

We collect several facts to complex 2-dimensional infrasolvmanifolds. Let $\mathcal{G}$ be a connected simply connected Lie group and $\mathrm{Aut}(\mathcal{G})$ its automorphism group. The affine group $\mathrm{A}(\mathcal{G})$ is defined to be the semidirect product $\mathcal{G} \rtimes \mathrm{Aut}(\mathcal{G})$ with group law $(g, \alpha) \cdot (h, \beta) = (g\alpha(h), \alpha\beta)$. Viewed $\mathcal{G}$ as a space, $\mathrm{A}(\mathcal{G})$ acts on $\mathcal{G}$ by $(g, \alpha)(x) = g\alpha(x)$ for $x \in \mathcal{G}$. Let $\mathcal{K}$ be a maximal compact subgroup of $\mathrm{Aut}(\mathcal{G})$. Form $\mathrm{E}(\mathcal{G}) = \mathcal{G} \rtimes \mathcal{K}$. It is a closed subgroup of $\mathrm{A}(\mathcal{G})$. Suppose that $\mathcal{G}$ is a connected simply connected solvable Lie group S. If π is a discrete uniform subgroup of $\mathrm{E}(S)$, then π acts properly discontinuously on S with compact quotient. In addition, when π is torsionfree, the orbit space $\pi\backslash S$ is a compact smooth manifold. $\pi\backslash S$ is called a generalized solvmanifold. Let Γ be the intersection of π with $S(\subset \mathrm{E}(S))$. If Γ is uniform in S (*i.e.*, $\Gamma\backslash S$ is a compact solvmanifold), then the generalized solvmanifold $\pi\backslash S$ is said to be an infrasolvmanifold. An infrasolvmanifold is finitely covered by a solvmanifold under preserving the structure of the affine group $\mathrm{A}(S)$. In the case that $\mathcal{G}$ is a nilpotent Lie group N, the Bieberbach - Auslander theorem says that a generalized nilmanifold $\pi\backslash N$ is always an infranilmanifold. It is noted that a generalized solvmanifold need not be an infrasolvmanifold. However we see that a generalized solvmanifold is topologically an infrasolvmanifold. In fact, given a generalized solvmanifold $\pi\backslash S$, π is a discrete subgroup of $\mathrm{E}(S)$. As $\mathrm{E}(S)$ is an extension of the solvable Lie group S by a compact group $\mathcal{K}$, $\mathrm{E}(S)$ is an amenable Lie group. Therefore, π is a virtually polycyclic group of rank $\pi = \dim S$.

Conversely, given a virtually polycyclic group π of rank equal to $\dim S$, π can be realized as the fundamental group of an infrasolvmanifold $\pi\backslash S'$ by [AJ]. It is proved that there is a simply connected solvable Lie group S' such that an extension of S' by a finite group $S' \rtimes F$ contains π as a discrete uniform subgroup. On the other hand, such π is realized as the fundamental group of an injective Seifert fiber space $M(\pi)$ by the result of [KLR]. In this case, $M(\pi)$ is a (singular) fiber space over some k ($\geqq 2$)-torus with typical fiber a nilmanifold (exceptional fiber an infranilmanifold). Moreover, there is a smooth rigidity between such injective Seifert fiber spaces. A map which represents a rigidity between them can be chosen to be a fiber preserving diffeomorphism. It is known that a generalized solvmanifold admits a structure of injective Seifert fiber space. (Compare also [LR].) Applying the soomth rigidity, we have

Corollary 1.1 *A generalized solvmanifold* $\pi\backslash S$ *is diffeomorphic to an infrasolvmanifold* $\pi\backslash S'$.

2 Supplement to complex 2-infrasolvmanifolds

Suppose that M is a closed aspherical complex surface whose fundamental group is virtually polycyclic. The classification of Enriques and Kodaira (cf.

[BPV]) implies that M is biholomorphic to a complex surface of type VII_0, a hyperelliptic surface C^2/π ($\pi \subset \mathrm{E}_C(2)$), or a primary (resp. secondary) Kodaira surface $S^1 \underset{F}{\times} Nil^3/\Delta$ (where F is a finite cyclic subgroup and Δ is a nilpotent subgroup of rank 3.) Bogomolov's assertion has been shown in [LYZ] that a complex surface of type VII_0 is one of the Inoue surfaces S_M, S_N^+, S_N^-. By the classification of 4-dimensional Riemannian homogeneous geometries by Wall [WA], there exist solvable Lie groups Sol_0^4, Sol_1^4, or $Sol_1^{4'}$ (cf. §4) whose quotients are identified biholomorphically with the Inoue surfaces. It is noticed that the Inoue surface S_M is modeled on Sol_0^4, the other Inoue surfaces S_N^+ ($t \in \mathbb{R}$), S_N^- are modeled on Sol_1^4, and S_N^+ ($t \notin \mathbb{R}$) is modeled on $Sol_1^{4'}$. Note that $\mathrm{E}(Sol_0^4) = Sol_0^4 \rtimes \mathrm{U}(1)$, $\mathrm{E}(Sol_1^4) = Sol_1^4$, $\mathrm{E}(Sol_0^{4'}) = Sol_0^{4'}$. In summary we obtain that (Compare §4.)

Theorem 2.1 *Let M be a closed aspherical complex surface with virtually polycylic fundamental group. Then, M is biholomorphic to the complex euclidean space form C^2/π, an infranilmanifold $S^1 \underset{F}{\times} Nil^3/\Delta$, or the Inoue surfaces Sol_0^4/π, Sol_1^4/π, $Sol_1^{4'}/\pi$.*

Corollary 2.2 *If a generalized solvmanifold $\pi\backslash S$ admits a complex structure compatible with the the group $\mathrm{E}(S)$, then $\pi\backslash S$ is biholomorphic to one of C^2/π, $S^1 \underset{F}{\times} Nil^3/\Delta$, or Sol_0^4/π, Sol_1^4/π, $Sol_1^{4'}/\pi$. In particular, Sol_1^4/π, $Sol_1^{4'}/\pi$ are solvmanifolds with an infinite central subgroup. Sol_0^4/π is an infrasolvmanifold if and only if the projection of π into $\mathrm{U}(1)$ has a finite cyclic summand.*

3 Conformal Kähler homogeneous spaces

A conformal Kähler homogeneous space is a simply connected Kähler manifold X on which a finite dimensional Lie group G acts transitively as a group of conformal holomorphic transformations with compact stbilizer. Let $X = G/K$ where K is the stabilizer G_x at somepoint $x \in X$. If a subgroup Γ of G acts properly discontinuously and freely on X, then the orbit space is said to be a locally conformal Kähler homogeneous manifold. Especially when G happens to be a group of Kähler isometries of X, then a locally homogeneous spaceX/Γ is nothing but a Kähler maniofld. We are interested in the non-Kähler case, *that is,* G has a nontrivial conformal transformations.

Suppose that dim $X > 2$. Then a conformal transformation preserving its complex structure J on X must be a homothetic transformation. Let g be a Kähler metric on (X, J) and Ω be its fundamental 2- form. If $\alpha \in$

G, then $\alpha^*\Omega = \rho(\alpha) \cdot \Omega$ for some constant $\rho(\alpha) \in R^+$. Hence we have a continuous homomorphism $\rho : G \longrightarrow R^+$. By our hypothesis that G has a nontrivial homothetic summand, ρ is surjective. As the stabilizer G_x is compact, $\rho(G_x) = 1$. The map ρ naturally extends to a map $\hat{\rho} : X = G \cdot x \longrightarrow R^+$. Then we can define a Hermitian metric

$$h_p(Y, Z) = \frac{g_p(Y, Z)}{\hat{\rho}(p)}$$

for each $p \in X$ and arbitrary $Y, Z \in T_pX$. Since $\hat{\rho}(\alpha p) = \rho(\alpha)\hat{\rho}(p)$ for $\alpha \in G$, it follows that $h_{\alpha \cdot p}(\alpha_* Y, \alpha_* Z) = h_p(Y, Z)$. G acts on X as a group of holomorphic isometries with respect to h. Given a conformal Kähler homogeneous geometry (G, X), we obtain a G-invariant Hermitian metric h which is locally conformal to a Kähler metric. As a consequence, the compact locally conformal Kähler manifold $\Gamma \backslash X$ is also a locally homogeneous Riemannian manifold compatible with a preferable complex structure.

Now in the sense of Thurston, recall that a geometric complex manifold is a $2n$-dimensional manifold locally modeled on a Riemannian homogeneous geometry (G, X) compatible with the preferable complex structure on X. Here G is a finite dimensional Lie group which acts holomorphically and transitively on a simply connected complex manifold X whose stabilizer is compact.

4 Classification of compact geometric complex surfaces

It is known that 4-dimensional Riemannian homogeneous geometries consist of 19 isomorphism classes (cf. [FL]). Among them, Wall [WA] has determined that the 14 geometries (G, X) carry a complex structure invariant under the automorphism group G; the complex structure is unique up to isomorphism, except for the solvable geometry. He has further observed that out of the 14 geometries, only the 9 geometries (G, X) can admit a Kähler structure compatible with a geometric structure (*i.e.*, each element of G is a holomorphic transformation preserving its Kähler structure.) In the remaining cases, there is no Kähler structure compatible with G. (Compare Theorem 1.2 [WA].) Thus the problem is left to the remaining 5 geometries which Hermitian geometry is compatible with its homogeneous structure. Tricerri [TR] and Vaisman [VA3] took up this problem to find a locally conformal Kähler structure compatible with G (abbreviated to l.c. Kähler from now.)

The remaining 5 geometries are locally modeled on the products of the positive real numbers $\mathbb{R}^+$ with the sphere S^3, the Heisenberg nilpotent Lie group $\mathcal{N}$, or the complete simply connected Lorentz space of constant negative curvature $\tilde{\mathbb{H}}^{1,2}$, or locally modeled on one 4-dimensional solvable Lie group Sol_0^4, and the other solvable Lie group Sol_1^4 with two isomorphism classes of complex structures. Vaisman ([VA1],[VA2]) has observed that the compact complex surfaces $S^3 \times S^1$, $\mathcal{N}/\Delta \times S^1$, and $\tilde{\mathbb{H}}^{1,2}/\Gamma \times S^1$ are l.c. Kähler manifolds whose Hermitian metrics are invariant under the automorphism group. On the other hand, Wall noticed that the compact complex surfaces modeled on the above solvable Lie groups are Inoue surfaces. (See [BPV].) Tricerri [TR] has proved that the Inoue surfaces modeled on Sol_0^4, Sol_1^4 are l.c. Kähler manifolds. In this case, Vaisman [VA3] has proved that the canonical Hermitian metrics are invariant under Sol_0^4, Sol_1^4, while he showed the Inoue surface modeled on the solvable Lie group $Sol_1^{4'}$ (which is Sol_1^4 with another complex structure) cannot admit any l.c. Kähler structure whose Hermitian metric is invariant under $Sol_1^{4'}$.

We observe the necesarry conditions when the geometric complex manifold $\Gamma\backslash X$ $(=\Gamma\backslash G/K)$ will be a l.c. Kähler manifold. Let (G, X) be a conformal Kähler homogeneous geometry. As G consists of finitely many components, there exists a 1-parameter subgroup R from G such that $\rho(R) = R^+$. Thus $G = H \rtimes R$ where $H = \text{Ker } \rho$. In summary, (G, X) has the following properties:

1. (G, X) is a 4-dimensional Riemannian homogeneous geometry.
2. X supports a complex structure compatible with the automorphism group G.
3. There exists a cofinite discrete subgroup Γ in G. (*That is,* G/Γ is of finite volume.)
4. G is the semidirect product $H \rtimes R$.

We have alreday treated the case $G = H \times R$ in [KA]. (Compare [VA1],[VA2].) So we study the semidiect case.

Semidirect product $\mathbf{H} \rtimes \mathbf{R}^+$. We shall construct a 4-dimensional conformal Kähler homogeneous geometry (G, X) when G is the semidirect product $H \rtimes R^+$. Consider the solvable Lie groups Sol_0^4, Sol_1^4 characterized by Wall [WA]; they act on the domain of the complex affine space $\mathbb{C}^2$ by holomorphic affinely flat transformations.

Let $A_{\mathbb{C}}(2) = \mathbb{C}^2 \rtimes \mathrm{GL}(2, \mathbb{C})$ be the 2-dimensional complex affine group acting on the complex number space $\mathbb{C}^2$. Choose the upper half plane $\mathbb{H}$ from $\mathbb{C}$ so that $\mathbb{C} \times \mathbb{H}$ is a domain of $\mathbb{C}^2$.

Case 1 (Tricerri [TR]). Let G be the subgroup of $A_{\mathbb{C}}(2)$ generated by the elements:

$$\left\{h=\left(\begin{pmatrix} a \\ b \end{pmatrix},\begin{pmatrix} \lambda & 0 \\ 0 & |\lambda|^{-2} \end{pmatrix}\right) \mid a\in\mathbb{C},\ b\in\mathbb{R},\ \lambda\in C^*\right\}.$$

Put $Sol_0^4=\{h\in G\mid \lambda\in R^+\}$. Then, $G=Sol_0^4\rtimes \mathrm{U}(1)$. Each element h leaves $\mathbb{C}\times\mathbb{H}$ invariant. Thus G is the transitive subgroup of holomorphic transformations of $\mathbb{C}\times\mathbb{H}$ with respect to the restricted complex structure. The stabilizer at $(0,i)$ is isomorphic to the circle $\mathrm{U}(1)$.
If we assign to each h the positive number $|\lambda|^2$, then G splits as the semidirect product $H\rtimes R^+$ where

$$H=\left\{\left(\begin{pmatrix} a \\ b \end{pmatrix},\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}\right) \mid a\in\mathbb{C},\ b\in\mathbb{R},\ \lambda\in\mathrm{U}(1)\right\}.$$

Note that H is the product $\mathrm{E}_{\mathbb{C}}(1)\times\mathbb{R}$ where the complex euclidean group $\mathrm{E}_{\mathbb{C}}(1)=\mathbb{C}\rtimes\mathrm{U}(1)$.

We give a Kähler structure on the domain $\mathbb{C}\times\mathbb{H}$ on which G acts as homothetic transformations. Choose the coordinates θ and $t>0$ with $\theta+it\in\mathbb{H}$. Put

$$\Omega=\frac{-i}{2}d\bar{z}\wedge dz+t^{-3}dt\wedge d\theta \ (=dx\wedge dy+t^{-3}dt\wedge d\theta).$$

Then $\Omega^2=-it^{-3}d\bar{z}\wedge dz\wedge dt\wedge d\theta\neq 0$ and $d\Omega=0$. Moreover, if J is the canonical complex structure on $\mathbb{C}\times\mathbb{H}$, then Ω is invariant under J and $g(X,JY)=\Omega(X,Y)$ is positive definite. Hence Ω is a Kähler structure on $\mathbb{C}\times\mathbb{H}$. Let $h\in G$ so that

$$h\begin{pmatrix} z \\ x \end{pmatrix}=\begin{pmatrix} a \\ b \end{pmatrix}+\begin{pmatrix} \lambda z \\ |\lambda|^{-2}x \end{pmatrix}.$$

Then it is easy to see that $h^*\Omega=|\lambda|^2\Omega$. Therefore,

$$\mathbb{R}^+=\left\{\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix},\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-2} \end{pmatrix}\right) \mid \lambda\in\mathbb{R}^+\right\}$$

acts as homothetic transformations of Ω.

Case 2 (Tricerri [TR]). Let G_1 be the subgroup of $A_{\mathbb{C}}(2)$ generated by the elements:

$$\left\{\left(\begin{pmatrix} c \\ a \end{pmatrix},\begin{pmatrix} \epsilon & b \\ 0 & \alpha \end{pmatrix}\right) \mid a,b,c\in\mathbb{R},\ \alpha>0,\epsilon=\pm 1\right\}.$$

Put $Sol_1^4 = \{h \in G_1 \mid \epsilon = 1\}$. The group Sol_1^4 acts transitively on the domain $\mathbb{C} \times \mathbb{H}$ with trivial stabilizer. Moreover, $G_1 = Sol_1^4 \rtimes \mathbb{Z}/2$ which is the full group leaving invariant $\mathbb{C} \times \mathbb{H}$ whose stabilizer at $(0, i)$ is isomorphic to $\mathbb{Z}/2$.

We give a Kähler structure on $\mathbb{C}\times\mathbb{H} = \{(z, w) \mid z = x+yi,\ w = \theta+ti,\ t > 0\}$ for which G_1 acts as homothetic transformations. Put

$$\Omega = \frac{-2}{t}\left(\frac{1+y^2}{t^2}dt \wedge d\theta - \frac{y}{t}(dt \wedge dx + dy \wedge d\theta) + dy \wedge dx\right).$$

Then $\Omega^2 = \dfrac{8(1+y^2)}{t^4}dt \wedge d\theta \wedge dy \wedge dx \neq 0$ and $d\Omega = 0$. Obviously Ω is invariant under J. Since $g(X, JY) = \Omega(X, Y)$ is positive definite, Ω is a Kähler structure on $\mathbb{C} \times \mathbb{H}$. If $h \in G_1$, then

$$h\begin{pmatrix} x+yi \\ \theta+ti \end{pmatrix} = \begin{pmatrix} (c+\epsilon x + b\theta) + (\epsilon y + bt)i \\ (a+\alpha\theta) + \alpha ti \end{pmatrix}.$$

Then it is easy to see that $h^*\Omega = \alpha^{-1} \cdot \Omega$. So the group

$$\mathbb{R}^+ \rtimes \mathbb{Z}/2 = \left\{\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \epsilon & 0 \\ 0 & \alpha \end{pmatrix}\right) \mid \alpha \in \mathbb{R}^+, \epsilon = \pm 1\right\}$$

acts as homothetic transformations of Ω. It is easy to see that G_1 is isomorphic to the semidirect product $(\mathcal{N} \rtimes \mathbb{Z}/2) \rtimes R^+$ where $\mathcal{N}$ is the 3-dimensional nilpotent Lie group consisting of the elements

$$\left\{\left(\begin{pmatrix} c \\ a \end{pmatrix}, \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) \mid a, b, c \in \mathbb{R}\right\}.$$

Case 3. We have another isomorphism class of complex structures on Sol_1^4. Denote by $Sol_1^{4'}$ the holomorphic action of Sol_1^4 on the domain $\mathbb{C} \times \mathbb{H}$. By the result of Wall [WA], $Sol_1^{4'}$ is a subgroup of $A_{\mathbb{C}}(2)$ represented by the elements:

$$\left\{h = \left(\begin{pmatrix} c + i\log\alpha \\ a \end{pmatrix}, \begin{pmatrix} 1 & b \\ 0 & \alpha \end{pmatrix}\right) \mid a,\ b, c \in \mathbb{R},\ \alpha > 0\right\}.$$

As is remarked before (cf. [VA3]), there exists no l.c. Kähler structure on $\mathbb{C}\times\mathbb{H}$ whose Hermitian metric is invariant under $Sol_1^{4'}$. In summary, we have obtained the following.

Theorem 4.1 *Every compact geometric complex surface $\Gamma\backslash X$ except for the Inoue surface $\Gamma\backslash Sol_1^{4'}$, admits a l.c. Kähler structure compatible with the homogeneous structure. Among them, non-Kähler manifolds are one of the following types. It is unique up to holomorphically conformal diffeomorphism:*

(i) *An infra-Hopf manifold $S^3 \underset{\Gamma}{\times} R^+$. (Some finite covering is homeomorphic to a Hopf manifold $S^3 \times S^1$. $H_1(X/\Gamma) = \mathbb{Z} + \{torsion\}$.)*

(ii) *An infranilmanifold $\mathcal{N} \underset{\Gamma}{\times} R^+$. (Some finite covering is a T^2-bundle over a torus T^2.*
$H_1(X/\Gamma) = \mathbb{Z}^3 + \{torsion\}$ if $\Gamma \subset \mathcal{N} \times R^+$, or $H_1(X/\Gamma) = \mathbb{Z} + \{torsion\}$ if Γ has a nontrivial summand in $U(1)$, which lies in $\mathbb{Z}/4$ at most.)

(iii) *A Lorentz space form $\tilde{\mathbb{H}}^{1,2} \underset{\Gamma}{\times} R^+$. (Some finite covering is a T^2-bundle over a closed orientable surface Σ_g. $H_1(X/\Gamma) = \mathbb{Z}^{2g+1} + \{torsion\}$ $(g \geqq 2)$.)*

(iv) *An generalized solvmanifold $\Gamma\backslash Sol_0^4/\mathrm{U}(1)$. (Some finite covering is a T^3-bundle over S^1.*
$H_1(X/\Gamma) = \mathbb{Z} + \{torsion\}$.)

(v) *Solvmanifolds $\Gamma\backslash Sol_1^4$. (Some finite covering is a fiber space over S^1 with fiber a nilmanifold $\Delta\backslash\mathcal{N}$. $H_1(X/\Gamma) = \mathbb{Z} + \{torsion\}$.)*

Remark 4.2 *Note that a more refined fiber space structure for X/Γ can be described in terms of the injective Seifert fibering with fiber a nilmanifold. (Compare [KLR].)*

Recently, Belgun [BE] has shown that there is no l.c. Kähler structure on the Inoue surface $\Gamma\backslash Sol_1^{4'}$. As a consequence, the existence of locally conformal Kähler structure on locally homogeneous complex surfaces has been done. Namely, among all compact geometric complex non-Kähler surfaces, the geometric complex surfaces of the above 5 types can only admit a locally conformal Kähler structure.

5 Invariant l.c Kähler structure on $\Gamma\backslash Sol_1^4$

As an application to the above results, we shall prove that the locally conformal Kähler solvmanifold obtained by Andres, Fernandez, Mencia and

Cordero ([ACFM]) coincides with the locally conformal Kähler structure on the Inoue surface $\Gamma\backslash Sol_1^4$ constructed by Tricerri [TR].
As in Case 2, the group Sol_1^4 acts on $\mathbb{C}\times\mathbb{H}$ as a group of homothetic transformations with respect to Ω, *i.e.*, $h^*\Omega = \alpha^{-1}\cdot\Omega$ for the element $h \in Sol_1^4 \rtimes Z/2$. If we set $\Theta = t\cdot\Omega$, then $d\Theta = d\log t\wedge\Theta$ so that $h^*\Theta = \Theta$. Letting $g(X, JY) = \Theta(X, Y)$, g is a left invariant l.c. Kähler metric on $\mathbb{C}\times\mathbb{H}$ and $(Sol_1^4, \mathbb{C}\times\mathbb{H}, g)$ is a left invariant homogeneous l.c. Kähler space. As the orbit space $Sol_1^4\cdot(0,i) = \mathbb{C}\times\mathbb{H}$, Sol_1^4 is viewed as the space. We show that Sol_1^4 admits also a right invariant l.c. Kähler metric. Moreover, it is indeed the l.c. Kähler metric on the solvmanifold obtained in [ACFM]. To see this, let $\mathcal{N}$ be the space $\mathbb{R}^3$ with group law;

$$\begin{pmatrix} x \\ y \\ \theta \end{pmatrix}\cdot\begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix} = \begin{pmatrix} x+x'-\theta\cdot y' \\ y+y' \\ \theta+\theta' \end{pmatrix}.$$

Then, note that $\mathcal{N}$ is isomorphic to the 3-dimensional Heisenberg Lie group consisting of unipotent matrices

$$\left\{\begin{pmatrix} 1 & \theta i & x \\ 0 & 1 & yi \\ 0 & 0 & 1 \end{pmatrix} \mid x,\ y, \theta \in \mathbb{R}\right\}.$$

Form the 4-dimensional Lie group $G(k,1) = \mathcal{N}\rtimes R^+$ with group law:

$$\left(\begin{pmatrix} x \\ y \\ \theta \end{pmatrix}, t\right)\cdot\left(\begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix}, t'\right) = \left(\begin{pmatrix} x+x'-\theta\cdot t^k\cdot y' \\ y+t^k\cdot y' \\ \theta+t^{-k}\cdot\theta' \end{pmatrix}, tt'\right).$$

Here k is a real number such that $\mathrm{e}^k+\mathrm{e}^{-k}$ is an integer but not 2. The group $G(k,1)$ is the solvable Lie group $G(k,n)$ in [ACFM] when $n=1$. (Note that $G(k,n)$ has been introduced in [ACFM], however we work with the universal covering space Y and so $n=1$ is sufficient.) $G(k,1)$ has a central group extension $1\rightarrow R\rightarrow G(k,n)\longrightarrow Sol^3\rightarrow 1$ where $Sol^3 = \mathbb{R}^2\rtimes R^+ = \left\{\left(\begin{pmatrix} 0 \\ y \\ \theta \end{pmatrix}, t\right)\right\}$ is the 3-dimensional solvable Lie group.

Viewed $G(k,1)$ as the space Y, $G(k,1)$ acts on Y as translations from the right. In fact, let $p = \left(\begin{pmatrix} x \\ y \\ \theta \end{pmatrix}, t\right) \in Y$, and $h = \left(\begin{pmatrix} c \\ b \\ a \end{pmatrix}, \alpha\right) \in G(k,1)$,

$G(k,1)$ acts on Y as

$$R_h \cdot p = p \cdot h = \left(\begin{pmatrix} x + c - \theta t^k b \\ y + t^k b \\ \theta + t^{-k} a \end{pmatrix}, t\alpha \right).$$

Choose the coordinates x, y, θ, t in Y. Put $\alpha' = dy - \frac{ky}{t}dt$, $\beta' = d\theta + \frac{k\theta}{t}dt$, $\gamma' = \frac{dt}{t}dt$ and $\eta' = dx + yd\theta + \frac{ky\theta}{t}dt$. Then they are right invariant 1-forms on Y, *i.e.*, $R_h^*\alpha' = \alpha'$ for $h \in G(k,1)$, *etc.* It is easy to check that $\alpha' \wedge \beta' = dy \wedge d\theta + \frac{k}{t}d(y\theta) \wedge dt = d\eta'$. So the 1-form η' is viewed as a connection form (up to a scale factor) on the principal bundle: $R \rightarrow Y \rightarrow Sol^3$. Put

$$\Omega' = \tfrac{-2}{k \cdot t^k}(\alpha' \wedge \eta' + k \cdot \beta' \wedge \gamma') =$$
$$\tfrac{-2}{k \cdot t^k}\left(-\tfrac{k(1+y^2)}{t} dt \wedge d\theta + y dy \wedge d\theta + \tfrac{ky\theta}{t} dy \wedge dt - \tfrac{ky}{t} dt \wedge dx + dy \wedge dx \right).$$

Then we can check that $d\Omega' = 0$ and so Ω' is a Kähler form on Y. A calculation shows that $R_h^*\Omega' = \frac{1}{\alpha^k}\Omega'$, *i.e.*, $G(k,1)$ acts as a group of homothetic transformations with respect to Ω'. Define a 2-form Θ' to be $t^k \cdot \Omega'$ on Y. Then we see that $d\Theta' = k \cdot d\log t \wedge \Theta'$. Since $\Theta' = -2(\frac{1}{k}\alpha' \wedge \eta' + \beta' \wedge \gamma')$, and $\alpha', \beta', \gamma', \eta'$ are all right invariant, Θ' is also a right invariant l.c. Kähler metric on Y.

We define an equivariant map

$$(\Psi, \Phi) : (Sol_1^4, \mathbb{C} \times \mathbb{H}) \longrightarrow (G(k,1), Y).$$

by setting

$$\Phi(x + iy, \theta + it) = \left(\begin{pmatrix} \frac{1}{k}x \\ y \\ -\frac{\theta}{kt} \end{pmatrix}, t^{\frac{1}{k}} \right)$$

$$\Psi\left(\left(\begin{pmatrix} c \\ a \end{pmatrix} \right), \begin{pmatrix} 1 & b \\ 0 & \alpha \end{pmatrix} \right) = \left(\begin{pmatrix} \frac{c}{k} \\ b \\ -\frac{a}{k\alpha} \end{pmatrix}, \alpha^{\frac{1}{k}} \right).$$

It is easy to see that Φ is a diffeomorphism between $\mathbb{C} \times \mathbb{H}$ and Y, and $\Psi(g \cdot h) = \Psi(h) \cdot \Psi(g)$ for $g, h \in Sol_1^4$, *i.e.*, Ψ is an anti-isomorphism between Sol_1^4 and $G(k,1)$. Moreover we can check that for $h \in Sol_1^4$,

$$\Phi(h \cdot \begin{pmatrix} x+iy \\ \theta+it \end{pmatrix}) = \Phi(x+iy, \theta+it) \cdot \Psi(h) = R_{\Psi(h)}\Phi(\begin{pmatrix} x+iy \\ \theta+it \end{pmatrix}).$$

Thus Φ is Ψ-equivariant. Using this map, we can define a complex structure J' on Y by setting $J' \circ \Phi_* = \Phi_* \circ J : T_z(\mathbb{C} \times \mathbb{H}) \to T_{\Phi(z)}Y$ for each $z \in Y$. If we put $\begin{pmatrix} 0 \\ i \end{pmatrix} = O$ and $e = (\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, 1)$, then $\Phi(O) = e$. Moreover, a calculation shows that $\Phi_*(\left(\frac{d}{dx}\right)_O) = \frac{1}{k}\left(\frac{d}{dx}\right)_e$, $\Phi_*(\left(\frac{d}{dy}\right)_O) = \left(\frac{d}{dy}\right)_e$, $\Phi_*(\left(\frac{d}{d\theta}\right)_O) = -\frac{1}{k}\left(\frac{d}{d\theta}\right)_e$, $\Phi_*(\left(\frac{d}{dt}\right)_O) = \frac{1}{k}\left(\frac{d}{dt}\right)_e$. As the tangent space T_eY is identified with the Lie algebra $\mathcal{G}(k,1)$, we have the right invariant vector fields on $G(k,1)$, $T' = dR_h\left(\frac{d}{dx}\right)_e$, $X' = dR_h\left(\frac{d}{dy}\right)_e$, $Y' = dR_h\left(\frac{d}{d\theta}\right)_e$, $Z' = dR_h\left(\frac{d}{dt}\right)_e$. The right invariance of the form η' implies that $\eta'(T') = R_h^*\eta'(\left(\frac{d}{dx}\right)_e) = \eta'(\left(\frac{d}{dx}\right)_e) = 1$, and $\eta'(X') = \eta'(Y') = \eta'(Z') = 0$, similarly for α', β', γ', *i.e.*, $\alpha'(X') = 1$, $\beta'(Y') = 1$, $\gamma'(Z') = 1$, and so on. Since the (left invariant) complex structure J on $\mathbb{C} \times \mathbb{H}$ satisfies that $J(\frac{d}{dx}) = -\frac{d}{dy}$, $J(\frac{d}{dy}) = \frac{d}{dx}$, $J(\frac{d}{d\theta}) = -\frac{d}{dt}$, $J(\frac{d}{dt}) = \frac{d}{d\theta}$, we obtain that $J'T' = -kX'$, $J'X' = \frac{1}{k}T'$, $J'Y' = Z'$, $J'Z' = -Y'$. When we look at *p.* 230 of [ACFM], this implies that

Proposition 5.1 *The complex structure J' on $Y = G(k,1)$ coincides with one defined in [ACFM].*

Theorem 5.2 *The pair (Φ, Ψ) induces a holomorphically homothetic transformation between the locally conformal Kähler structure on $Y = G(k,1)$ by Andres, Fernandez, Mencia and Cordero and the locally conformal Kähler structure on $\mathbb{C} \times \mathbb{H}$ by Tricerri.*

P roof. By the construction of complex structure on Y, we have alreday shown that $(\Psi, \Phi) : (Sol_1^4, \mathbb{C} \times \mathbb{H}, J) \longrightarrow (G(k,1), Y, J')$ is a holomorphic diffeomorphism. It has only to prove that Φ is homothetic with respect to Ω and Ω'. When we recall the 2-forms

$$\Theta = t \cdot \Omega = -2\left(\frac{1+y^2}{t^2} dt \wedge d\theta - \frac{y}{t}(dt \wedge dx + dy \wedge d\theta) + dy \wedge dx\right) \text{ on } \mathbb{C} \times \mathbb{H}$$

from **Case 2** and $\Theta' = t^k \cdot \Omega' = -2(\frac{1}{k}\alpha' \wedge \eta' + \beta \wedge \gamma')$ on Y, we can show that $\Phi^*\Theta' = \frac{1}{k^2}\Theta$. Thus Φ is homothetic. Similary we have $\Phi^*\Omega' = \frac{1}{k^2}\Omega$.

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