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GEOMETRY OF MEYER'S FUNCTION

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0. Introduction

In this short note, we shall give a brief survey on geometric aspects of Meyer's function following the paper [5].

Let $\mathcal{M}_g$ be the mapping class group of a smooth oriented closed surface Σ_g of genus g. Namely it is the group of all isotopy classes of orientation preserving diffeomorphisms of Σ_g. Further let

$$\rho : \mathcal{M}_g \to \mathrm{Sp}(2g; \mathbb{Z})$$

denote the classical representation defined by actions of $\mathcal{M}_g$ on the first integral homology group $H = H_1(\Sigma_g; \mathbb{Z})$.

Our main object here is Meyer's signature cocycle τ (see [4]), which is a group 2-cocycle of the Siegel modular group $\mathrm{Sp}(2g;\mathbb{Z})$. Topologically, this presents the signature of total spaces of surface bundles over surfaces. Of course, if we pull back the cocycle by the representation ρ, then we can regard τ as a 2-cocycle of the mapping class group $\mathcal{M}_g$. But here, let us consider mainly the restriction of τ to a subgroup of $\mathcal{M}_g$, that is, the *hyperelliptic mapping class group* Δ_g which consists of classes commuting with a fixed hyperelliptic involution. As is known, $\Delta_g = \mathcal{M}_g$ if $g = 1, 2$ and $\Delta_g \neq \mathcal{M}_g$ for $g \geq 3$. Then by the fact that the group Δ_g is acyclic over $\mathbb{Q}$, the restricted signature cocycle must be the coboundary of a unique rational 1-cochain $\phi_g : \Delta_g \to \mathbb{Q}$ (*i.e.* $\delta\phi_g = \rho^*\tau|_{\Delta_g}$ holds). In the following, we call it *Meyer's function* of genus g.

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In the case of genus one, geometric meanings of Meyer's function have been studied by Atiyah in [1]. In fact, he related it to many invariants defined for each element of $\Delta_1 = \mathcal{M}_1 \cong SL(2;\mathbb{Z})$ including Hirzebruch's signature defect, the logarithmic monodromy of Quillen's determinant line bundle, the Atiyah-Patodi-Singer η-invariant and its adiabatic limit. This framework suggests the existence of the higher geometry of the Riemann moduli space and of various universal families over it.

In the case where the genus is greater than one, we can also interpret Meyer's function by virtue of other invariants under certain conditions.

1. Eta-invariant

To be brief, the *η-invariant* of the signature operator on a Riemannian 3-manifold measures the extent to which the Hirzebruch signature formula fails for a nonclosed Riemannian 4-manifold whose metric is a product near its boundary.

Let $f \in \mathcal{M}_g$ be of finite order and M_f the mapping torus constructed from f. Namely, it is the identification space $\Sigma_g \times [0,1]/(p,0) \sim (f(p),1)$. We endow M_f with the metric which is induced from the product of the standard metric on S^1 and a metric so that f acts on Σ_g as an isometry.

If we restrict ourselves to the hyperelliptic mapping class group Δ_g, we obtain the following theorem.

Theorem A. *Let Δ_g be the hyperelliptic mapping class group of genus g. Then*

$$\eta(M_f) = \phi_g(f)$$

holds for any element $f \in \Delta_g$ of finite order. In particular, $\eta(M_f)$ is a topological invariant of M_f.

As an application, using the precise definition of Meyer's function (see [5]), we can give a necessary condition that an automorphism of Σ_g to be hyperelliptic.

Corollary. *Let $f \in \mathcal{M}_g$ be of finite order. If $f \in \Delta_g$, namely f commutes with a hyperelliptic involution, then*

$$\eta(M_f) \in \frac{1}{2g+1}\mathbb{Z}$$

holds, where $\frac{1}{2g+1}\mathbb{Z}$ denotes the additive group $\{\frac{n}{2g+1} \in \mathbb{Q} \mid n \in \mathbb{Z}\}$.

Example. Let $f \in \mathcal{M}_3$ be of order 3 so that the quotient orbifold of Σ_3 by its cyclic action is homeomorphic to $S^2(3,3,3,3,3)$. Then direct computations show that the η-invariant of corresponding mapping torus is given by

$$\eta(M_f) = -\frac{2}{3} \notin \frac{1}{7}\mathbb{Z}.$$

Hence the above corollary implies that f cannot be realized as an automorphism of a hyperelliptic Riemann surface.

2. Casson invariant

The *Casson invariant* is an integer valued invariant for oriented homology 3-spheres. Roughly speaking, it counts the number (with signs) of conjugacy classes of irreducible representations of the fundamental group of an oriented homology 3-sphere into the Lie group $SU(2)$.

From the theory of characteristic classes of surface bundles, due to Morita [6], the Casson invariant can be regarded as the secondary characteristic class associated to the first Morita-Mumford class $e_1 \in H^2(\mathcal{M}_g;\mathbb{Q})$ through the correspondence between elements of $\mathcal{M}_g$ and 3-manifolds via the Heegaard splittings. In this point of view, the very core of the Casson invariant is essentially represented by the homomorphism

$$d_0 : \mathcal{K}_g \to \mathbb{Q},$$

which we call *Morita's homomorphism.* Here $\mathcal{K}_g$ is the subgroup of $\mathcal{M}_g$ generated by all the Dehn twists along separating simple closed curves on Σ_g.

If we consider Meyer's function ϕ_g on the subgroup $\mathcal{K}_g$, then we have

Theorem B. *Meyer's function essentially coincides with Morita's homomorphism on $\Delta_g \cap \mathcal{K}_g$. To be more precise, $3\phi_g = d_0$ holds on $\Delta_g \cap \mathcal{K}_g$.*

Remark. The group $\Delta_g \cap \mathcal{K}_g$ coincides with $\Delta_g \cap \mathcal{I}_g$, where $\mathcal{I}_g = \operatorname{Ker} \rho$ is the Torelli group. This fact is shown by the existence of a crossed homomorphism $\tilde{k} : \mathcal{M}_g \to \frac{1}{2}\Lambda^3 H/H$ satisfing $\tilde{k}|_{\Delta_g} = 0$ and the exact sequence $1 \to \mathcal{K}_g \to \mathcal{I}_g \to \Lambda^3 H/H \to 1$, due to Johnson (see [2], [3]). We call this group the *hyperelliptic Torelli group* and denote it by $\mathcal{J}_g$.

Therefore, in principle, we can say that the Casson invariant of homology 3-spheres corresponding to elements of the hyperelliptic Torelli group $\mathcal{J}_g$ is determined by Meyer's function.

By virtue of Theorem B and a formula of Meyer's function (see [5] for details), we can compute explicit values of Morita's homomorphism on interesting elements of $\mathcal{J}_g$. Here, by a *BSCC-map* of genus h, we mean the Dehn twist along a bounding simple closed curve on Σ_g which separate it into two subsurfaces of genera h and $g-h$.

Corollary. *Let $\psi_h \in \mathcal{J}_g$ $(1 \leq h \leq g)$ be a BSCC-map of genus h. Then the value of Morita's homomorphism on ψ_h is given by*

$$d_0(\psi_h) = -\frac{12}{2g+1}h(g-h).$$

Remark. It has been shown by Morita that the above formula actually holds for the whole group $\mathcal{K}_g$ (cf. [7]).

References

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