数理解析研究所講究録
1104 巻 1999 年 91-102

Minsky's pivot theorem and its application to the Earle slice of punctured torus groups

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1 Introduction

In [6], Y.Minsky showed that any marked punctured torus group can be characterized completely by its pair of end invariants, where a punctured torus group is a rank two free Kleinian group whose commutator of generators is parabolic. To prove this result, called the ending lamination theorem, he proved another important result, called the pivot theorem, which controls thin parts of the corresponding hyperbolic manifold from the data of end invariants. One of the applications of these theorems, he showed that the Bers slice and the Maskit slice are Jordan domains.

In this paper we apply his results to the Earle slice which is a holomorphic slice of quasi-fuchsian space representing the Teichmüller space of once-punctured tori. This slice was considered by C.Earle in [1], and its geometric coordinates, named pleating coordinates was studied by C.Series and the author in [3]. By using rational pleating rays, the figure of the Earle slice $\mathcal{E}$ realized in the complex plane C was drawn by P.Liepa (see figure 1). We will show that

1. The boundary of the Earle slice $\mathcal{E}$ is a Jordan curve.

2. There is a right half region which is contained in $\mathcal{E}$.

3. Every pleating ray in $\mathcal{E}$ lands at a unique boundary point.

This paper is organized as follows. Section 1 is dedicated to background material, especially the space of punctured torus groups. We review Minsky's ending lamination theorem and pivot theorem in section 3 and 4. After introducing the Earle slice in section 5, we show the previous claims in section 6, 7 and 8.

The author thank Professor S.Kamiya for his good organization of this conference at RIMS, Kyoto University.

2 Punctured torus groups

Let S be an oriented once-punctured torus and $\pi_1(S)$ be its fundamental group. An ordered pair α, β of generators of $\pi_1(S)$ is called *canonical* if the oriented intersection number $i(\alpha, \beta)$ in S with respect to the given orientation of S is equal to $+1$. The commutator $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$ represents a loop around the puncture.

Define $\mathcal{R}(\pi_1(S))$ to be the set of $PSL_2(\mathbf{C})$-conjugacy classes of representations from $\pi_1(S)$ to $PSL_2(\mathbf{C})$ which take the commutator of generators to a parabolic element. Let $\mathcal{D}(\pi_1(S))$ denote the subset of $\mathcal{R}(\pi_1(S))$ consisting of conjugacy classes of discrete and faithful representations. Any representative of an element of $\mathcal{D}(\pi_1(S))$ is called a *marked punctured torus group.* Let $\mathcal{QF}$ denote the subset of $\mathcal{D}(\pi_1(S))$ consisting of conjugacy classes of representations ρ such that for the action of $\Gamma = \rho(\pi_1(S))$ on the Riemann sphere $\hat{\mathbf{C}}$ the region of discontinuity Ω has exactly two simply connected invariant components $\Omega^{\pm}$. The quotients $\Omega^{\pm}/\Gamma$ are both homeomorphic to S and inherit an orientation induced from the orientation of $\hat{\mathbf{C}}$. We choose the labelling so that Ω^{+} is the component such that the homotopy basis of Ω^{+}/Γ induced by the ordered pair of marked generators $\rho(\alpha), \rho(\beta)$ of Γ is canonical. Any representative of an element of $\mathcal{QF}$ is called a *marked quasifuchsian punctured torus group.* Considering the algebraic topology $\mathcal{D}(\pi_1(S))$ is closed in $\mathcal{R}(\pi_1(S))$, and $\mathcal{QF}$ is open in $\mathcal{D}(\pi_1(S))$ (see [4]). A quasifuchsian group Γ is called *Fuchsian* if the components $\Omega^{\pm}$ are round discs.

Recall that the set of measured geodesic laminations on a hyperbolic surface is independent of the hyperbolic structure. Denote by $PML(S)$ the set of projective measured laminations on S. Let $\mathcal{C}(S)$ denote the set of free homotopy classes of unoriented simple non-peripheral curves on S. There are in one-to-one correspondence with $\hat{\mathbf{Q}} \equiv \mathbf{Q} \cup \{\infty\}$, after choosing an canonical basis (α, β) for $\pi_1(S)$ as follows; Any element of $H_1(S)$ can be written as $(p, q) = p[\alpha] + q[\beta]$ in the basis $([\alpha], [\beta])$ for $H_1(S)$, and we associate to this the slope $-p/q \in \hat{\mathbf{Q}}$ which describes an element of $\mathcal{C}(S)$. Cosidering projective classes of weighted counting measures, we can identify $\mathcal{C}(S)$ with the set of projective rational raminations. Recall that $PML(S)$

may be identified with $\hat{\mathbf{R}}$, in such a way that rational laminations correspond to $\hat{\mathbf{Q}}$.

3 Minsky's ending lamination theorem

We associate to a punctured torus group an ordered pair of "end invariants" (ν_-, ν_+), each lying in $\overline{\mathbf{H}}^2 \equiv \mathbf{H}^2 \cup \hat{\mathbf{R}}$. Let $\rho : \pi_1(S) \to PSL_2(\mathbf{C})$ denote a marked punctured torus group and $N = \mathbf{H}^3/\rho(\pi_1(S))$ its associated manifold. Then by Bonahon's theorem of geometric tameness (see [4]), N is homeomorphic to $S \times \mathbf{R}$. Let us name the ends e_+ and e_-. We choose the labelling as follows; Let the orietation $S \times \{1\}$ agree with the orientation of S. Orient $S \times (-1, 1)$ by the orientation of $S \times \{1\}$ and its inward-pointing vector. The orientation of $\mathbf{H}^3$ induces the orientation of N. Then up to homotopy there exists uniquely an orientation preserving homeomorphism between N and $S \times (-1, 1)$ which induces the representation ρ. Let e_+ be the end of N whose neighborhoods are neighborhoods of $S \times \{1\}$ under this identification. Let Ω denote the (possibly empty) domain of discontinuity of $\Gamma = \rho(\pi_1(S))$ and $\overline{N}$ denote the quotient $\mathbf{H}^3 \cup \Omega/\Gamma$. Any component of the boundary Ω/Γ is reached by going to one of the ends e_+ or e_-, and this divides it into two disjoint pieces Ω_+/Γ and Ω_-/Γ. There are three possibilities for each of these boundaries, corresponding to three types of end invariants (here let s denote either $+$ or $-$):

1. Ω_s is a topological disc; In this case Ω_s/Γ is a marked punctured torus. Then there are $\nu_+, \nu_- \in \mathbf{H}^2$ uniquely such that marked flat tori $\mathbf{C}/\mathbf{Z} \cdot 1 + \mathbf{Z} \cdot \nu_+$ and $\mathbf{C}/\mathbf{Z} \cdot \overline{\nu}_- + \mathbf{Z} \cdot 1$ are equivalent to the compactifications of Ω_+/Γ and Ω_-/Γ respectively as marked Riemann surfaces. In particular, $\nu_+ = \nu_-$ if and only if Γ is a Fuchsian group.

2. Ω_s is an infinite union of round discs; In this case Ω_s/Γ is a thrice-punctured sphere, obtained from the corresponding boundary of $S \times \mathbf{R}$ by deleting a simple closed curve γ_s. In this case $\nu_s \in \hat{\mathbf{Q}}$ denotes the slope of γ_s. The conjugacy class of γ_s in Γ is parabolic.

3. Ω_s is empty; In this case we can find a sequence of simple closed curves $\{\gamma_n\}$ in S whose geodesic representative γ_n^* eventually contained in any neighborhood of e_s ("exits the end"), and the slopes of γ_n converge in $\mathbf{R}$ to a unique irrational number. We denote ν_s to be this limitting irrational slope which is called an *ending lamination*.

To a marked punctured torus group $\rho : \pi_1(S) \to PSL_2(\mathbf{C})$ one may associate an ordered pair of end invariants (ν_-, ν_+) lying in $\overline{\mathbf{H}}^2 \times \overline{\mathbf{H}}^2 \setminus \Delta$, where Δ denote the diagonal of $\hat{\mathbf{R}} \times \hat{\mathbf{R}}$. Minsky's ending lamination theorem is

Theorem 3.1. *The map*

$$\nu : \mathcal{D}(\pi_1(S)) \to \overline{\mathbf{H}}^2 \times \overline{\mathbf{H}}^2 \setminus \Delta$$

defined by $\rho \mapsto (\nu_-, \nu_+)$ is bijective. ν is not continuous but its inverse ν^{-1} is continuous.

Proof: See theorems A and B in [6].

4 Minsky's pivot theorem

Next we review Minsky's pivot theorem which is a key idea to prove the ending lamination theorem 3.1, and is also a main idea to prove our results in this paper.

First we define the *Farey triangulation* of the upper half plane $\mathbf{H}^2$ as follows. For any two rational numbers written in lowest terms as p/q and r/s, say they are *neighbors* if $|ps - qr| = 1$. Allow also the case $\infty = 1/0$. Joining any two neighbors by a hyperbolic geodesic, we obtain a triangulation invariant under the natural action of $PSL_2(\mathbf{Z})$.

Next we recall the notion of *pivots* for marked punctured torus groups. Let (ν_-, ν_+) be the end-invariant pair of a marked punctured torus group $\rho : \pi_1(S) \to PSL_2(\mathbf{C})$. Letting s denote $+$ or $-$, define a point $\alpha_s \in \hat{\mathbf{R}}$ to be closest to ν_s in the following sense: If $\nu_s \in \hat{\mathbf{R}}$ let $\alpha_s = \nu_s$. If $\nu_s \in \mathbf{H}^2$, let $\alpha_s \in \mathcal{C}(S)$ represent a geodesic of shortest length in the hyperbolic structure corresponding to ν_s. More pricisely, if ν_s is contained in a Farey triangle Δ, we divide up Δ into six regions by the axes of its reflection symmetries, and then each vertex $u \in \mathcal{C}(S)$ has minimal hyperbolic length in the hyperbolic structure corresponding to ν_s when ν_s is in the pair of regions that meet u. Now define $E = E(\alpha_-, \alpha_+)$ to be the set of edges of the Farey graph which separate α_- from α_+ in $\mathbf{H}^2$. Let P_0 denote the set of vertices of $\mathcal{C}(S)$ which belong to at least 2 edges in E. We call thse vertices *internal pivots* of ρ. The edges of E admit a natural order where $e < f$ if e separates the interior of f from α_-, and this induces an ordering on P_0. The full pivot sequence P of ρ is obtained by appending to the beginning of P_0 the vertex α_- if $\alpha_- \in \mathcal{C}(S)$, and appending to the end of P_0 the vertex α_+ if $\alpha_+ \in \mathcal{C}(S)$.

Finally we review the *complex translation length* for a loxiodromic element γ of $SL_2(\mathbf{C})$. Let $\lambda(\gamma) = l + i\theta$ denote its complex translation length; geometrically, $l > 0$ gives the translation length of γ along its axis, and θ $(mod\ 2\pi)$ gives the rotation. It is determined by the identity $\mathrm{Tr}\,\gamma = 2\cosh\frac{\lambda}{2}$. Thus, fixing a marked punctured torus group $\rho : \pi_1(S) \to PSL_2(\mathbf{C})$, we obtain a function on $\mathcal{C}(S)$ which we write $\lambda_\rho(\alpha) \equiv \lambda(\rho(\alpha))$.

Now we can state the pivot theorem. For each $\beta \in \mathcal{C}(S)$ fix an element of $PSL_2(\mathbf{Z})$ such that β is taken to ∞. Then the set of neighbors of β go to $\mathbf{Z}$. Such a transformation is unique up to integer translation. Let $\nu_+(\beta)$ and $\nu_-(\beta)$ denote the points of $\overline{\mathbf{H}}^2$ to which $\nu_\pm \in \overline{\mathbf{H}}^2$ are taken by this transformation. Minsky's pivot theorem is

Theorem 4.1. *There exist positive constants ϵ, c_1 such that, if ρ is a marked punctured torus group,*

1. *If $l_\rho(\beta) \leq \epsilon$ then β is a pivot of ρ.*
2. *Let α be a pivot of ρ. If we take a branch of $\lambda_\rho(\alpha)$ satisfying $|Im\ \lambda_\rho(\alpha)| < \pi$, then*

$$d_{\mathbf{H}^2}(\frac{2\pi i}{\lambda_\rho(\alpha)}, \nu_+(\alpha) - \overline{\nu_-(\alpha)} + i) < c_1$$

where $d_{\mathbf{H}^2}(\cdot,\cdot)$ denotes the hyperbolic metric on $\mathbf{H}^2$.

Proof: See theorem 4.1 in [6].

5 The Eale slice of punctured torus groups

The following theorem defines a holomophic embedding of the Teichmüller space Teich(S) of once-punctured tori into $\mathcal{QF}$.

Theorem 5.1. *Let (α, β) be a canonical homotopy basis of $\pi_1(\mathcal{T}_1)$ where $\mathcal{T}_1$ is an analytically finite Riemann surface homeomorphic to S. Let θ be an involution of $\pi_1(\mathcal{T}_1)$ defined by $\theta(\alpha) = \beta$. Then, up to conjugation in $PSL_2(\mathbf{C})$, there exists a unique marked quasifuchsian group $\rho : \pi_1(\mathcal{T}_1) \to \Gamma$, such that:*

1. *There is a conformal map $\mathcal{T}_1 \to \Omega^+/\Gamma$ inducing the representation ρ.*
2. *There is a Möbius transformation $\Theta \in PSL_2(\mathbf{C})$ of order two which induces a conformal homeomorphism $\Omega^+ \to \Omega^-$ such that $\Theta(\gamma z) = \theta(\gamma)\Theta(z)$ for all $\gamma \in \Gamma$ and $z \in \Omega^+$.*

Proof: See [1] and theorem 2.1 in [3].

Theorem 5.1 implies that for any marked Riemann surface $(\mathcal{T}_1; \alpha, \beta)$ which is analytically finite and homeomorphic to S, there is a marked quasi-fuchsian group $\Gamma = \langle A, B \rangle$ such that as a marked Riemann surface, $(\mathcal{T}_1; \alpha, \beta)$ is equivalent to $(\Omega_+/\Gamma; A, B)$ and $(\Omega_-/\Gamma; B, A)$. The embedding of Teich(S) depends only on the choice of the involution θ of $\pi_1(\mathcal{T}_1)$; in fact we can take any involution of $\pi_1(\mathcal{T}_1)$ which is induced from an orientation reversing diffeomorphism of $\mathcal{T}_1$ (see [1]). We call the image of Teich(S) in $\mathcal{QF}$, the *Earle slice* of $\mathcal{QF}$. This slice can be thought of as a holomorphic extension of the rhombus line in the *Fuchsian locus* $\mathcal{F}$ into $\mathcal{QF}$ (see [3]).

Next we show how to realise the Earle slice in $\mathbf{C}$.

Theorem 5.2. *Let $\rho : \pi_1(\mathcal{T}_1) \to PSL_2(\mathbf{C})$ be a marked quasifuchsian punctured torus group in the Earle slice. Then, after conjugation by $PSL_2(\mathbf{C})$ if necessary, we can take representatives of $A = \rho(\alpha), B = \rho(\beta)$ in $SL(2, \mathbf{C})$ of the form $A = A_d, B = B_d, d \in \mathbf{C} - \{0\}$, where*

$$A_d = \begin{pmatrix} \frac{d^2+1}{d} & \frac{d^3}{2d^2+1} \\ \frac{2d^2+1}{d} & d \end{pmatrix}, B_d = \begin{pmatrix} \frac{d^2+1}{d} & -\frac{d^3}{2d^2+1} \\ -\frac{2d^2+1}{d} & d \end{pmatrix}.$$

The parameter d^2 is uniquely determined by the conjugacy class of ρ.

Proof: See theorem 3.1 in [3].

Let $\mathbf{C}^+$ denote the right half d-plane $\{d \in \mathbf{C} | Re\ d > 0\}$. Then the map

$$\varphi : \mathbf{C}^+ \to \mathcal{R}(\pi_1(S))$$

defined by $d \mapsto (A_d, B_d)$ is a holomorphic injection and we can realize the Earle slice in $\mathbf{C}^+$. Define $\mathcal{E}$ to be the corresponding region in $\mathbf{C}^+$. Then the positive real line $\mathbf{R}^+$ corresponds to the Fuchsian locus of $\mathcal{E}$, the rhombus line. Moreover there exist two involutions of $\mathcal{E}$: a holomorphic involution $\sigma(d) = 1/2d$ and an anti-holomorphic involution $\iota(d) = \bar{d}$ where $\bar{d}$ denotes the complex cojugation of d.

Next we consider the relation between the closure of the Earle slice in $\mathcal{QF}$ and the closure of $\mathcal{E}$ in the d-plane.

Lemma 5.3. *If non-zero $d \in \mathbf{C}$ is on the imaginary axis of the d-plane, A_dB_d or $A_dB_d^{-1}$ is elliptic.*

Proof: From the trace equations $\mathrm{Tr}\, A_dB_d = 2 + \frac{1}{d^2}$ and $\mathrm{Tr}\, A_dB_d^{-1} = 2(2d^2 + 1)$, we can check the claim.

Proposition 5.4. *1. The closure $\bar{\mathcal{E}}$ of $\mathcal{E}$ in $\mathbf{C}^+$ is homeomorphic to the closure $\overline{\varphi(\mathcal{E})}$ of $\varphi(\mathcal{E})$ in $\mathcal{D}(\pi_1(S))$ under φ.*

2. The closure of $\mathcal{E}$ in $\hat{\mathbf{C}}$ is equal to $\bar{\mathcal{E}} \cup \{0, \infty\}$.

Proof:

1. φ is a homeomorphism from $\mathbf{C}^+$ to its image under φ, and $\varphi(\mathbf{C}^+) \cap \mathcal{D}(\pi_1(S))$ is closed in $\mathcal{D}(\pi_1(S))$ by the above lemma 5.3.

2. From the above lemma 5.3 and the fact that $\mathcal{E}$ contains the positive real line $\mathbf{R}^+$, we can check the claim.

Now we have a following diagram:

$$\begin{tikzcd} \mathbf{C}^+ \arrow[r, "\varphi"] & \mathcal{R}(\pi_1(S)) & \\ \mathcal{E} \arrow[u] \arrow[r, "\varphi"] & \mathcal{D}(\pi_1(S)) \arrow[u] \arrow[r, "\nu"] & \overline{\mathbf{H}}^2 \times \overline{\mathbf{H}}^2 \setminus \Delta \end{tikzcd}$$

By the restriction of ν to the Earle slice $\varphi(\mathcal{E})$ in $\mathcal{QF}$, We have

Proposition 5.5. $\nu \circ \varphi(\mathcal{E}) = \{(\nu_-, \nu_+) \in \mathbf{H}^2 \times \mathbf{H}^2 | \nu_- \overline{\nu_+} = 1\}$

Proof: $\mathbf{C}/\mathbf{Z} \cdot 1 + \mathbf{Z} \cdot \tau$ is conformal to $\mathbf{C}/\mathbf{Z} \cdot \frac{1}{\tau} + \mathbf{Z} \cdot 1$.

Therefore its closure in $\overline{\mathbf{H}}^2 \times \overline{\mathbf{H}}^2 \setminus \Delta$ can be written as

Corollary 5.6. $\overline{\nu \circ \varphi(\mathcal{E})} = \{(\nu_-, \nu_+) \in \overline{\mathbf{H}}^2 \times \overline{\mathbf{H}}^2 \setminus \Delta | \nu_- \overline{\nu_+} = 1\}$

Finally we review the notion of pleating rays (see [2, 3]). For a quasi-fuchsian punctured torus group Γ, let $\mathcal{C}/\Gamma$ be the convex core of $\mathbf{H}^3/\Gamma$; equivalently $\mathcal{C}$ is the hyperbolic convex hull of the limit set Λ of Γ. The boundary $\partial\mathcal{C}/\Gamma$ of $\mathcal{C}/\Gamma$ has two connected components $\partial\mathcal{C}^{\pm}/\Gamma$, each homeomorphic to S. These components are each pleated surfaces whose pleating loci carry the bending measure whose projective classes we denote $pl^{\pm}(\Gamma)$.

For $x, y \in PML(S) = \hat{\mathbf{R}}$, The (x, y)-*pleating rays* in $\mathcal{E}$ is the set defined by $\mathcal{P}(x, y) = \{d \in \mathcal{E} : pl^+(d) = x, pl^-(d) = y\}$. Since the boundary components $\partial\mathcal{C}^{\pm}$ are conjugate under the involution for groups in $\mathcal{E}$, we have that $\mathcal{P}(x, 1/x) \neq \emptyset$ provided $x \neq \pm 1$, and $\mathcal{P}(x, y) = \emptyset$ otherwise. In particular, the set of rational pleating rays $\mathcal{P}(x, 1/x)$ $(x \in \hat{\mathbf{Q}} \setminus \{\pm 1\})$ are dense in $\mathcal{E}$ (see [3]). This allows us to draw the picture shown in figure 1. The positive real axis represents Fuchsian groups with the rhombic symmetry, and only the upper half of the Earle slice is shown, the picture being symmetrical under reflection in the real axis.

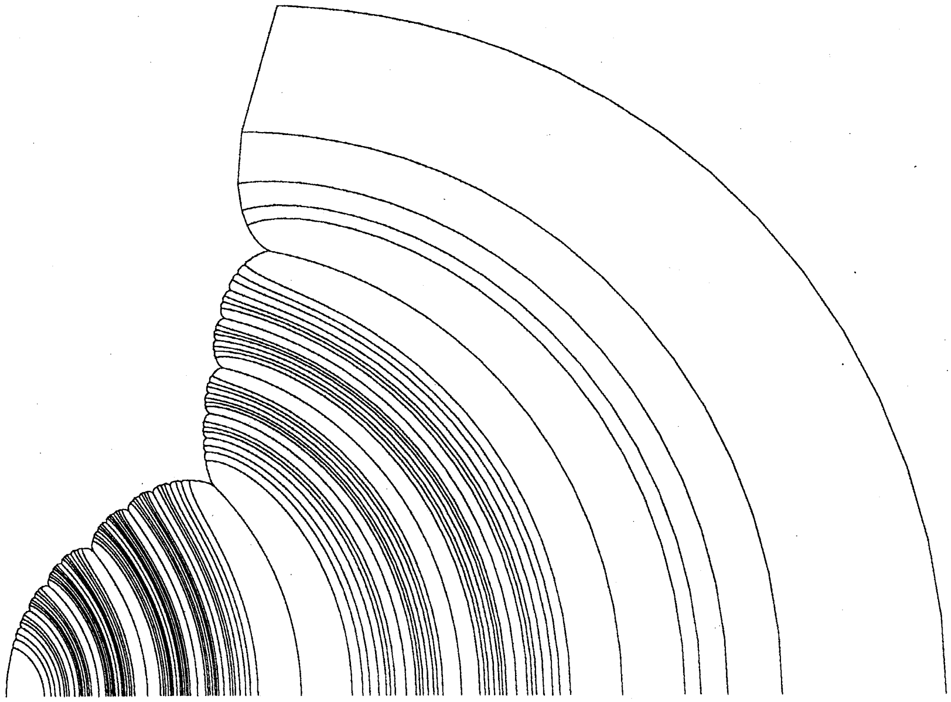

Figure 1: The upper half of the Earle Slice.

6 $\mathcal{E}$ is a Jordan domain

In this section we show that $\mathcal{E}$ is a Jordan domain by using the pivot theorem 4.1.

Proposition 6.1. *If a sequence of points* (ν_-^i, ν_+^i) *in* $\overline{\nu \circ \varphi(\mathcal{E})}$ *goes to the point* $(1,1)$ *in* $\hat{\mathbf{R}} \times \hat{\mathbf{R}}$, *then* $d_i = (\nu \circ \varphi)^{-1}((\nu_-^i, \nu_+^i))$ *converges to* 0 *in the d-plane. Similarly if* (ν_-^i, ν_+^i) *goes to* $(-1,-1)$, *then* d_i *diverges to infinity.*

Proof: Suppose first that $(\nu_-^i, \nu_+^i) \to (1,1)$. There is a unique element $A \in PSL_2(\mathbf{Z})$ satisfying $A(1) = \infty$ and $A(-1) = 1/2$. Let $\nu_\pm^i(1)$ denote the points of $\overline{\mathbf{H}}^2$ to which $\nu_\pm^i$ are taken by A. $\nu_+^i(1)$ and $\nu_-^i(1)$ are related by $\nu_-^i(1) = 1 - \overline{\nu_+^i(1)}$ from the relation in corollary 5.6.

First we show that for a sufficiently large i, $1 \in \hat{\mathbf{Q}}$ becomes a pivot for the representation ρ_i whose pair of end invariants is (ν_-^i, ν_+^i). When $Im\ \nu_+^i(1) \to \infty$, then $Im\ \nu_-^i(1) \to \infty$ by the relation $\nu_-^i(1) = 1 - \overline{\nu_+^i(1)}$. From a well-known comparison of extremal and hyperbolic length (see [5]), the length $l_\pm^i(1)$ of the geodesic corresponding to the slope $1 \in \hat{\mathbf{Q}}$ becomes short in the boundary torus $\Omega_\pm/\rho(\pi_1(S))$. Then by Bers' inequality $1/l^i(1) \geq \frac{1}{2}(1/l_+^i(1) + 1/l_-^i(1))$, the length $l^i(1)$ of the geodesic $\gamma(1)$ in $\mathbf{H}^3/\rho_i(\pi_1(S))$ corresponding to $1 \in \hat{\mathbf{Q}}$ is also short, hence by the pivot theorem 4.1(1), $1 \in \hat{\mathbf{Q}}$ is a pivot for ρ_i. When $Im\ \nu^i_+(1)$ remains bounded and hence $Re\ \nu^i_+(1) \to \pm\infty$, then $Re\ \nu^i_-(1) \to \mp\infty$ and in this case, by definition, $1 \in \hat{\mathbf{Q}}$ is also a pivot for ρ_i (see figure 2).

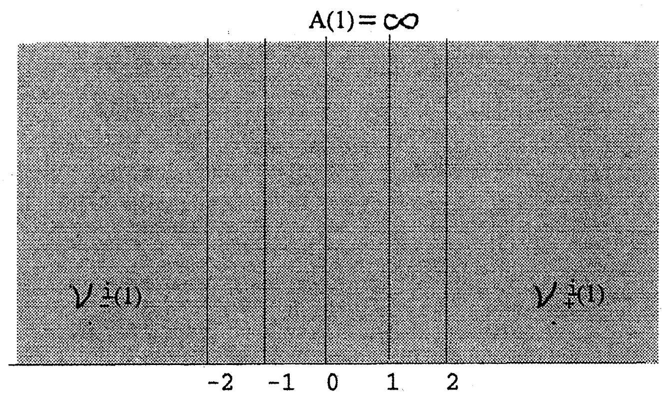

Figure 2:

Hence by the pivot theorem 4.1(2), the complex translation length $\lambda_{\rho_i}(1)$ satisfying $|Im\ \lambda_{\rho_i}(1)| < \pi$ goes to 0. This implies that $\mathrm{Tr}\,\gamma(1)$ goes to 2. From the equality $\mathrm{Tr}\,\gamma(1) = \mathrm{Tr}\,A_{d_i}B_{d_i}^{-1} = 2(2{d_i}^2 + 1)$, d_i goes to 0.

The remaining case that $(\nu^i_-, \nu^i_+) \to (-1, -1)$ can be proved by the same argument.

Theorem 6.2. *The restriction of ν^{-1} to $\overline{\nu \circ \varphi(\mathcal{E})}$ is a homeomorphism from $\overline{\nu \circ \varphi(\mathcal{E})}$ to $\overline{\varphi(\mathcal{E})}$.*

Proof: Because $\nu^{-1}(\overline{\nu \circ \varphi(\mathcal{E})})$ is closed by the above proposition 6.1, it must be the closure $\overline{\varphi(\mathcal{E})}$ of $\varphi(\mathcal{E})$ in $\mathcal{D}(\pi_1(S))$. From the same reason $\nu^{-1}|_{\overline{\nu\circ\varphi(\mathcal{E})}}$ the restriction of ν^{-1} to $\overline{\nu \circ \varphi(\mathcal{E})}$ is a homeomorphism.

Next result is a corollary of theorem 6.2 and proposition 5.4.

Corollary 6.3. *1. The boundary of $\mathcal{E}$ in $\mathbf{C}^+$ consisits of two open Jordan arcs terminating* 0 *and* ∞.

2. The boundary of $\mathcal{E}$ in $\hat{\mathbf{C}}$ is a Jordan curve. Therefore $\mathcal{E}$ is a Jordan domain.

7 Asymptotic bihaviour of the boundary $\partial\mathcal{E}$

Theorem 7.1. *In the d-plane, there exist two open round discs B in $\mathcal{E}$ and $-B$, symmetric with respect to the imaginary axis whose closures are tangent at* 0.

Proof: First we fix a branch of the complex length function $\lambda_d(1)$ on $\mathcal{E}$ by the condition that it is real valued on the positive real line $\mathbf{R}^+$. We remark that $Re\ \lambda_d(1) = l_d(1) > 0$ on $\mathcal{E}$, hence $\lambda(\mathcal{E}) := \{\lambda(d) \in \mathbf{C} | d \in \mathcal{E}\}$ is contained in the right half λ-plane $\mathbf{C}^+$.

Next we extend this branch to a neighborhood of 0 in the d-plane. The equality $\operatorname{Tr} A_d B_d^{-1} = 2\cosh \frac{\lambda_d(1)}{2} = 2(2d^2+1)$ implies that $d = \sinh \frac{\lambda_d(1)}{4}$, hence the branch $\lambda_d(1)$ can be extended conformally in a a neighborhood U of 0 in $\mathbf{C}$ (see figure 3). Especially by taking U sufficiently small, we may assume that $|Re\ \lambda_d(1)|$ and $|Im\ \lambda_d(1)|$ are both small. Then by the pivot theorem 4.1(1), $1 \in \hat{\mathbf{Q}}$ is a pivot for any points in $U \cap \mathcal{E}$.

Now take a horizontal line $L_k = Im\ \nu_+(1) = k$ $(k > 0)$ in $\mathbf{H}^2$ parametrized by the real part of $Re\ \nu_+(1)$, i.e., $L_k = \{\sigma(s) | s = Re\ \nu_+(1) \in \mathbf{R}\}$. From a well-known comparison of extremal and hyperbolic length (see [5]), $\nu_+^{-1}(\sigma(s))$ goes to 0 as $|s| \to \pm\infty$. In particular, there exists $r_1 > 0$ such that $\nu_+^{-1}(\sigma(s)) \in U \cap \mathcal{E}$ for $|s| > r_1$.

On the other hand, by the pivot theorem 4.1(2),

$$d_{\mathbf{H}^2}\left(\frac{2\pi i}{\lambda_{\nu_+^{-1}(\sigma(s))}(1)}, 2s-1+i(2k+1)\right) < c_1$$

for $|s| > r_1$ which implies that the curve $\{\lambda_s(1)\}_{s\in\mathbf{R}}$ is tangent at 0. Therefore in $\lambda(U \cap \mathcal{E})$, we can take a small open round disc tangent to the imaginary axis at 0. Take B as the image of this disc under the conformal map $d = \sinh(\frac{\lambda}{4})$ around 0 (see figure 4).

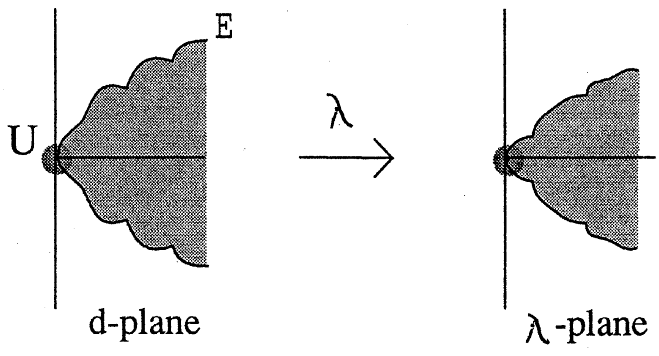

Figure 3:

Now we have the follwing result for the asymptotic behaviour of the boundary $\partial\mathcal{E}$.

Corollary 7.2. *In the d-plane there exists a right half region contained in $\mathcal{E}$.*

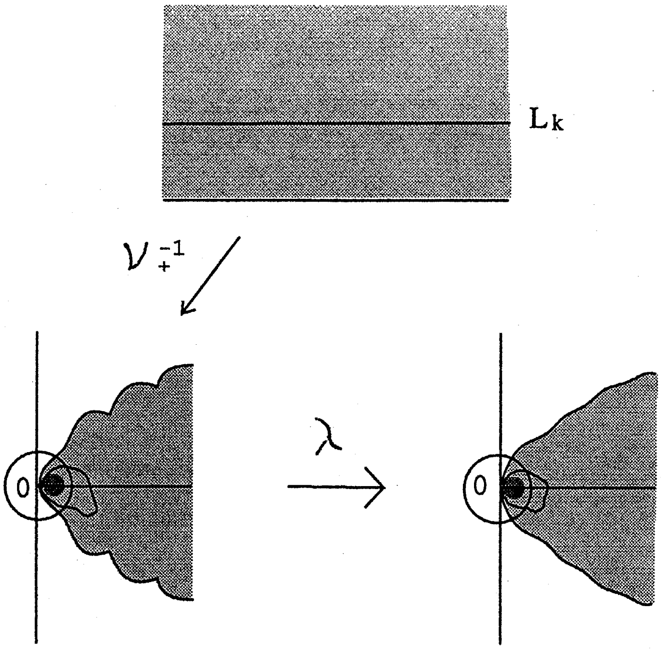

Figure 4:

Proof: Take the image of the round disc B in the previous theorem 7.1 under the conformal involution $\sigma(d) = 1/2d$ of $\mathcal{E}$.

Remark 7.3. *By using the pivot theorem 4.1, we can show that $\mathcal{E}$ is not a quasi-disc (see [7]). Miyachi recently announced a more strong result; for the case of the Maskit slice and the Earle slice of punctured torus groups, every boundary point corresponding to a cusp group is a inward-pointing cusp.*

8 End invariants and pleating invariants

In [3], we showed that any rational pleating ray $\mathcal{P}(x, 1/x)$ ($x \in \hat{\mathbf{Q}} \setminus \{\pm 1\}$) lands at a point $c_x \in \partial\mathcal{E}$ representing a cusp group at which $|\operatorname{Tr}\gamma(x)| = 2$. Therefore c_x is obtained from the corresponding boundary of $S \times \mathbf{R}$ by deleting a simple closed curve corresponding to $x \in \hat{\mathbf{Q}}$. This implies that its pair of end invarinats is $(1/x, x)$. Since $\partial\mathcal{E}$ and $\hat{\mathbf{R}} \setminus \{\pm 1\}$ are identified under the map $\nu_+ \circ \varphi$, we have

Theorem 8.1. *Every pleating ray lands at the boundary of $\mathcal{E}$; rational pleating ray lands at doublly cusped group, while irrational pleating ray lands at doublly degenerate group. In particular, $\mathcal{P}(x, 1/x)$ lands to the boundary group whose end invariant pair is $(1/x, x)$.*

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