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# On the Ford domains of once-punctured torus groups

Hirotaka Akiyoshi*

秋吉 宏尚 (九大 数理)

## 1 Introduction

In [1], T. Jorgensen studies the Ford domains of the quasi-Fuchsian groups obtained from a hyperbolic once-punctured torus. Eventhough [1] is not finished nor easy to read, the characterization and the idea of proof given there seem to be efficient to understand the quasi-Fuchsian punctured torus groups. In the joint work with M. Sakuma, M. Wada and Y. Yamashita, we can fill most of the statements given in [1]. Moreover, we can see that some techniques used there are applicable to the groups in the boundary of the quasi-Fuchsian space.

Let $T$ be a hyperbolic once-punctured torus and $\rho_0 : \pi_1(T) \to PSL(2,\mathbb{R}) \subset PSL(2,\mathbb{C})$ the holonomy representation. We define the representation space $\mathcal{R} = \{\rho : \pi_1(T) \to PSL(2,\mathbb{C}) \mid \rho(g)$ is parabolic if $\rho_0(g)$ is parabolic$\}/\sim$, where $\sim$ is the equivalence relation defined by the conjugation in $PSL(2,\mathbb{C})$. The quasi-Fucsian space is the subspace $QF \subset \mathcal{R}$ consisting of the quasi-conformal deformations of $\rho_0$. We will denote by $\overline{QF}$ the closure of $QF$ in $\mathcal{R}$.

In [2], Y. N. Minsky studies the once-punctured torus groups, where a *once-punctured torus group* is the image of an injective representation in $\mathcal{R}$. By the result of [2], all once-punctured torus groups are contained in $\overline{QF}$ and are classified by the ending lamination $\nu^M(\rho) \in \overline{\mathbb{H}^2} \times \overline{\mathbb{H}^2} - \Delta$, where $\Delta$ is the diagonal set of $\partial\mathbb{H}^2 \times \partial\mathbb{H}^2$.

In Section 3, we give Condition (J) which gives a characterization of the ford domain of $\operatorname{Im}\rho$ ($\rho \in \overline{QF}$). Then we have the following theorem.

*Graduate School of Mathematics, Kyushu University 33, Fukuoka 812-8581.
e-mail: akiyoshi@math.kyushu-u.ac.jp

**Theorem 1.1.** (1) *There is a continuous map* $\nu = (\nu_+, \nu_-) : QF \longrightarrow \mathbb{H}^2 \times \mathbb{H}^2$ *such that the Ford domain of* $\mathrm{Im}\,\rho$ *satisfies Condition* (J) *with respect to* $\nu(\rho)$ $(\rho \in QF)$.

(2) *The map* $\nu$ *in* (1) *can be extended to* $\nu = (\nu_+, \nu_-) : \overline{QF} \longrightarrow \overline{\mathbb{H}^2} \times \overline{\mathbb{H}^2} - \Delta$ *such that the Ford domain of* $\mathrm{Im}\,\rho$ *satisfies Condition* (J) *with respect to* $\nu(\rho)$ $(\rho \in \overline{QF})$. *Moreover,* $\nu_\epsilon(\rho) \in \partial\mathbb{H}^2$ *if and only if* $\nu_\epsilon^M(\rho) \in \partial\mathbb{H}^2$ *and the equation* $\nu_\epsilon(\rho) = \nu_\epsilon^M(\rho)$ *holds.*

**Remark 1.2.** (1) Recently, the map $\nu : \overline{QF} \longrightarrow \overline{\mathbb{H}^2} \times \overline{\mathbb{H}^2} - \Delta$ is shown to be surjective. (Jorgensen conjectures in [1] that $\nu$ is also injective.)

(2) By Theorem 1.1 and the Minsky's result in [2], we can characterize the Ford domains of the once-punctured torus groups.

(3) There is a fine computer program OPTi by M. Wada [3]. It would help to understand the phenomenon.

# 2 Elliptic generators

It is desirable to describe the representations of $\pi_1(T)$ before moving on to the study on the Ford domains of their images. As is well known, the once-punctured torus $T$ has a two-fold symmetry and the quotient space by the symmetry is the obifold $\mathcal{O}$ with base manifold $S^2$ and $(\infty, 2, 2, 2)$-type singularity. Then the fundamental group $\pi_1(T)$ is naturally identified with an index two normal subgroup of $\pi_1^{orb}(\mathcal{O})$ as follows;

$$\begin{aligned} \pi_1^{orb}(\mathcal{O}) &= \langle P_0, Q_0, R_0 \mid P_0^2 = Q_0^2 = R_0^2 = 1 \rangle, \\ K &= R_0 Q_0 P_0, \\ \pi_1(T) &= \langle A_0, B_0 \rangle, \\ A_0 &= K P_0,\ B_0 = K^{-1} R_0,\ A_0 B_0 A_0^{-1} B_0^{-1} = K^2. \end{aligned}$$

(See Figure 1.) The following proposition holds. (It is not mentioned obviously in [1], however, one of the important ideas of the paper is the following proposition.)

**Proposition 2.1.** *Let* $\rho$ *be a representation in* $\overline{QF}$. *Then there is a unique representation* $\hat{\rho} : \pi_1^{orb}(\mathcal{O}) \longrightarrow PSL(2, \mathbb{C})$ *such that the restriction of* $\hat{\rho}$ *to* $\pi_1(T)$ *is equal to* $\rho$.

To simplify the notation, we will denote $\hat{\rho}$ in Proposition 2.1 by $\rho$ and regard $\overline{QF}$ a set of representations of $\pi_1^{orb}(\mathcal{O})$ rather than $\pi_1(T)$.

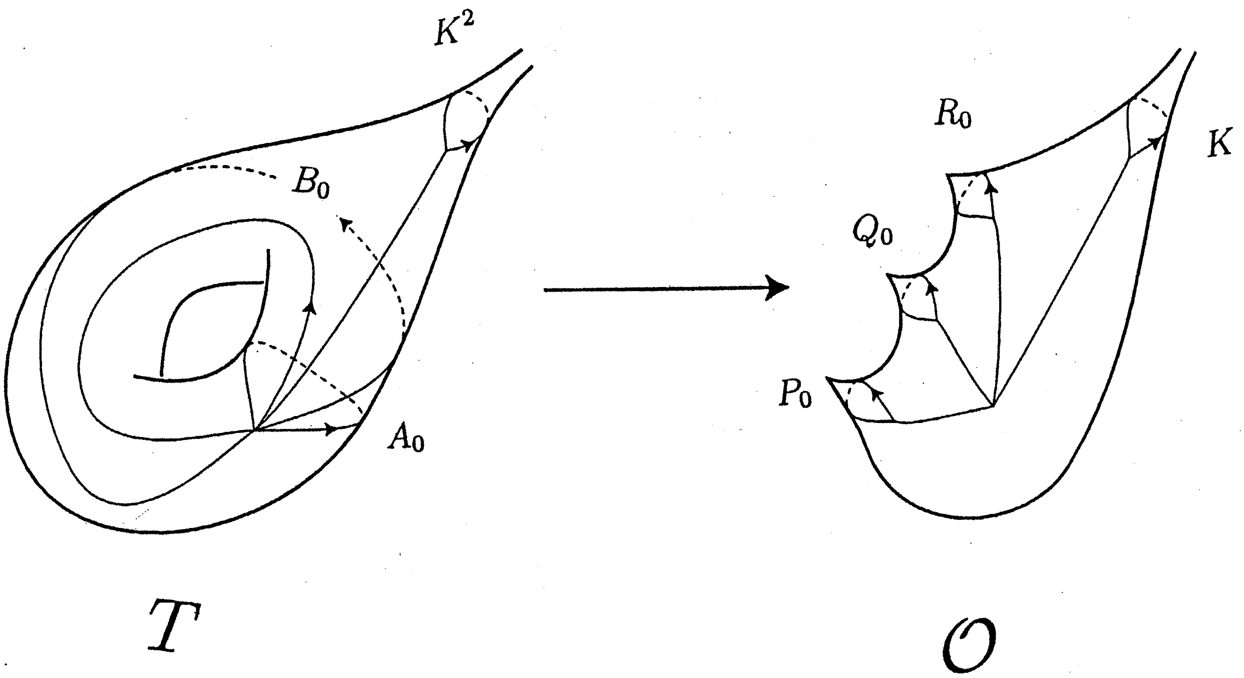


Figure 1

By the above observation, it seems reasonable to study the structure of the group $\pi_1^{orb}(\mathcal{O})$. We define the elliptic generators, which plays the key role of our study, as follows.

**Definition 2.2.** (1) An *elliptic generator triple* $(P, Q, R)$ is a triple of elements in $\pi_1^{orb}(\mathcal{O})$ which satisfies

$$\pi_1^{orb}(\mathcal{O}) = \langle P, Q, R \rangle,\ P^2 = Q^2 = R^2 = 1,\ RQP = K.$$

(2) An element $P \in \pi_1^{orb}(\mathcal{O})$ is an *elliptic generator* if there are $Q, R \in \pi_1^{orb}(\mathcal{O})$ such that $(P, Q, R)$ is an elliptic generator triple.

The elliptic generators satisfy the following proposition.

**Proposition 2.3.** *Let $(P, Q, R)$ be an elliptic generator triple. Then the following holds.*

(1) *Each $(R^{K^{-1}}, P, Q)$ and $(Q, R, P^K)$ is an elliptic generator triple, where $X^Y$ denotes $YXY^{-1}$.*

(2) *The triple $(P, R, Q^R)$ is an elliptic generator triple.*

(3) *Any elliptic generator triple is obtained from $(P_0, Q_0, R_0)$ by successively applying the operations in (1) and (2).*

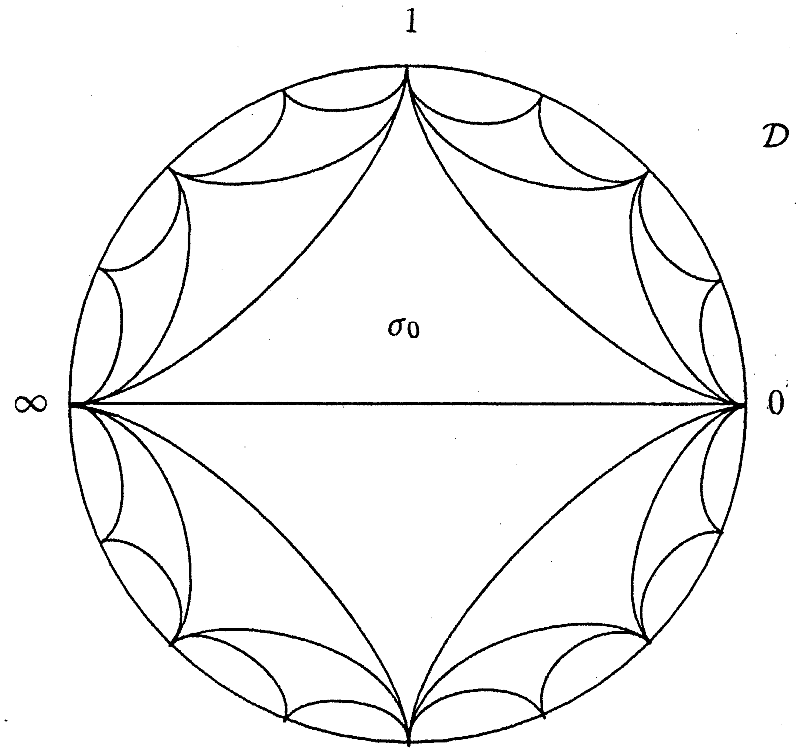


Figure 2

The operation in Proposition 2.3(2) corresponds to applying a Dehn-twist on $\mathcal{O}$. Geometrically, we introduce the notion of slopes of elliptic generators.

**Definition 2.4.** The isotopy classes of the essential loops in $T$ is in one-to-one correspondence with $\mathbb{Q} \cup \{\infty\}$. We call the isotopy class of an essential loop $\gamma$ a *slope* of $\gamma$. The *slope* $s(P)$ of an elliptic generator $P$ is the slope of an essential loop which represents $KP \in \pi_1(T)$. (We identify the slopes with $\mathbb{Q} \cup \{\infty\}$ so that $(s(P_0), s(Q_0), s(R_0)) = (\infty, 0, 1)$.)

We define the *Farey triangulation* $\mathcal{D}$ of $\mathbb{H}^2$, which is helpful to study the elliptic generators, as follows. The set of slopes $\mathbb{Q} \cup \{\infty\}$ is naturally identified with a subset of $\partial\mathbb{H}^2$. Put

$$\mathcal{D} = \{X\sigma_0 \,|\, \sigma_0 = (\infty, 0, 1),\, X \in SL(2, \mathbb{Z})\}.$$

(See Figure 2). Then the following proposition holds.

**Proposition 2.5.** (1) *For two elliptic generators $P$ and $P'$, $s(P) = s(P')$ if and only if $P' = P^{K^n}$ for some $n \in \mathbb{Z}$.*

(2) *For any elliptic generator $(P, Q, R)$, the slopes $s(P), s(Q), s(R)$ spans a triangle in $\mathcal{D}$. Conversely, any triangle in $\mathcal{D}$ is spaned by the slopes of an elliptic generator triple.*

# 3 Characterization of the Ford domains

We prepare several notations.

**Definition 3.1.** • The *isometric circle* $I(A)$ of an isometry $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL(2,\mathbb{C})$ is the circle in $\mathbb{C}$ with center $-d/c$ and radius $1/|c|$.

- The *isometric hemisphere* $Ih(A)$ of an isometry $A \in PSL(2,\mathbb{C})$ is the totally geodesic plane in $\mathbb{H}^3$ (identified with the upper half space) with $\partial Ih(A) = I(A)$.
- For $\rho : \pi_1^{orb}(\mathcal{O}) \to PSL(2,\mathbb{C})$, we denote by $c(\rho, X)$ (resp. $r(\rho, X)$) the center (resp. the radius) of $I(\rho(X))$ for any $X \in \pi_1^{orb}(\mathcal{O})$.

We shall use the Jorgensen's cross section to canonize the Ford domains. For any element of $\overline{QF}$, we can find a unique representative $\rho : \pi_1^{orb}(\mathcal{O}) \to PSL(2,\mathbb{C})$ such that $\rho(K) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $c(\rho, Q_0) = 0$. From now on, we regard $\overline{QF}$ the set of such representations. Now we define the Ford domains.

**Definition 3.2.** We define the (extended) *Ford domain* $Ph(\rho)$ of $\rho \in \overline{QF}$ by

$$Ph(\rho) = \cap\{Ext(Ih(\rho(X))) \,|\, X \in \pi_1^{orb}(\mathcal{O}),\ X(\infty) \neq \infty\}.$$

Note that the (extended) Ford domain $Ph(\rho)$ is *not* a fundamental domain for the action of $\mathrm{Im}\,\rho$ on $\mathbb{H}^3$ unless it is quotiented by the group $\langle\rho(K)\rangle$. However, to simplify the notation, we define as above. (It is rather obvious to quotient by $\rho(K)$, since it is normalized to be the parallel translation by 1.) Note also that $Ph(\rho)$ is, by definition, *not* the Ford domain of $\rho(\pi_1(T))$. However, it is easy to observe that the combinatorial structure of the Ford domain of $\rho(\pi_1(T))$ coincides with the one of $Ph(\rho)$ if it satisfies Condition (J) introduced below. (The glueing pattern is a bit different.)

## 3.1 Condition (J)

For $\rho \in \overline{QF}$ and $\nu = (\nu_+, \nu_-) \in \overline{\mathbb{H}^2} \times \overline{\mathbb{H}^2} - \Delta$, we make the following operation.
**Step 1:** Since $\nu \notin \Delta$, there is the geodesic segment $l$ in $\mathbb{H}^2$ which connects $\nu_+$ and $\nu_-$. (It might be a single point in $\mathbb{H}^2$.)
**Step 2:** Put $\Sigma = \{\sigma \,|\, \sigma \text{ is a simplex in } \mathcal{D} \text{ with } \mathrm{int}\sigma \cap l \neq \emptyset\}$.

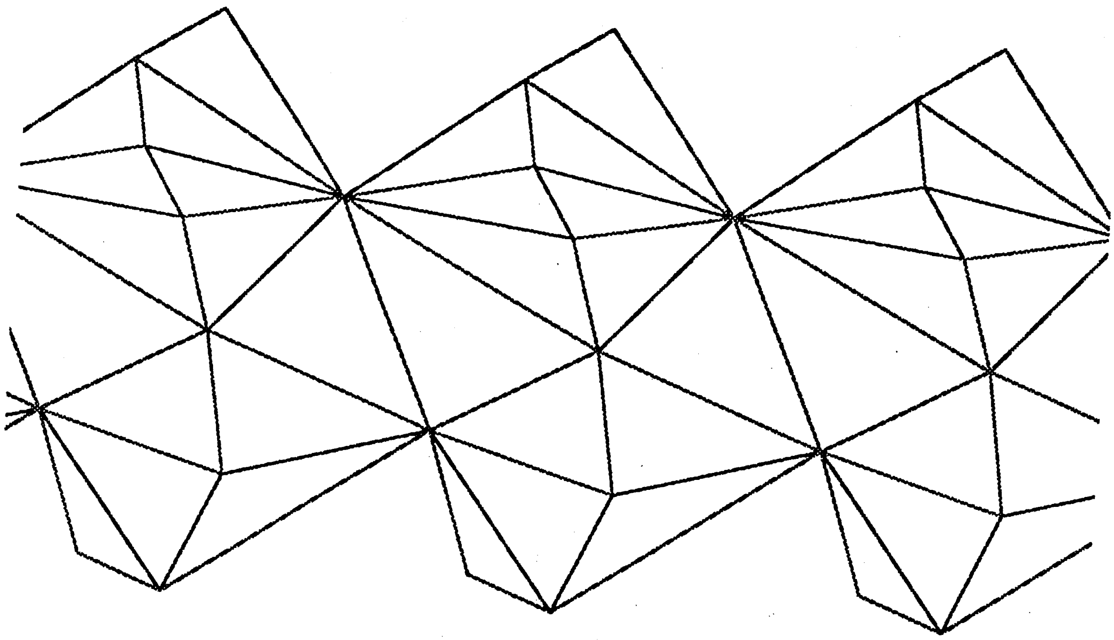

Figure 3

**Step 3:** For each triangle $\sigma$ in $\Sigma$, take an elliptic generator triple $(P, Q, R)$ such that $s(P), s(Q), s(R)$ spans $\sigma$. Then draw segments in $\mathbb{C}$ which successively connect the points

$$\cdots, c(\rho, R^{K^{-1}}), c(\rho, P), c(\rho, Q), c(\rho, R), c(\rho, P^K), \cdots.$$

We shall say that $\rho$ *satisfies Condition* (J) if the pattern drawn in Step 3 is nonsingular and $\partial Ph(\rho)$ is dual to the pattern. (See Figures 3 and 4.) When $\rho$ satisfies Condition (J), $\partial Ph(\rho) \cap \mathbb{C}$ consists of some part of $I(\rho(P))$ for elliptic generators $P$ such that $s(P)$ is a vertex of an end triangle. Then we define the angle parameter $\theta_P$ to be the half the visible angle of $I(\rho(P))$. (See Figure 5.) By the symmetry with respect to $\rho(K)$, the angle parameter $\theta_P$ is well defined by the slope $s(P)$. It is also possible to observe that $\theta_P + \theta_Q + \theta_R = \pi/2$ for an end triangle spanned by $s(P), s(Q), s(R)$. If for each end triangle $\sigma_\pm$ (if exist), the barycentric coordinate of $\nu_\pm$ is equal to the angle parameter, we shall say that $\rho$ *satisfies Condition* (J) *with respect to* $\nu$.

Supporse that $\rho \in \overline{QF}$ satisfies Condition (J). Since $\partial Ph(\rho)$ consists of the isometric hemispheres of elliptic generators and $Ih(\rho(P)) = Ih(\rho(KP))$ for each elliptic generator $P$, we can see that the ford domain of $\rho(\pi_1(T))$ coincides with $Ph(\rho)$.

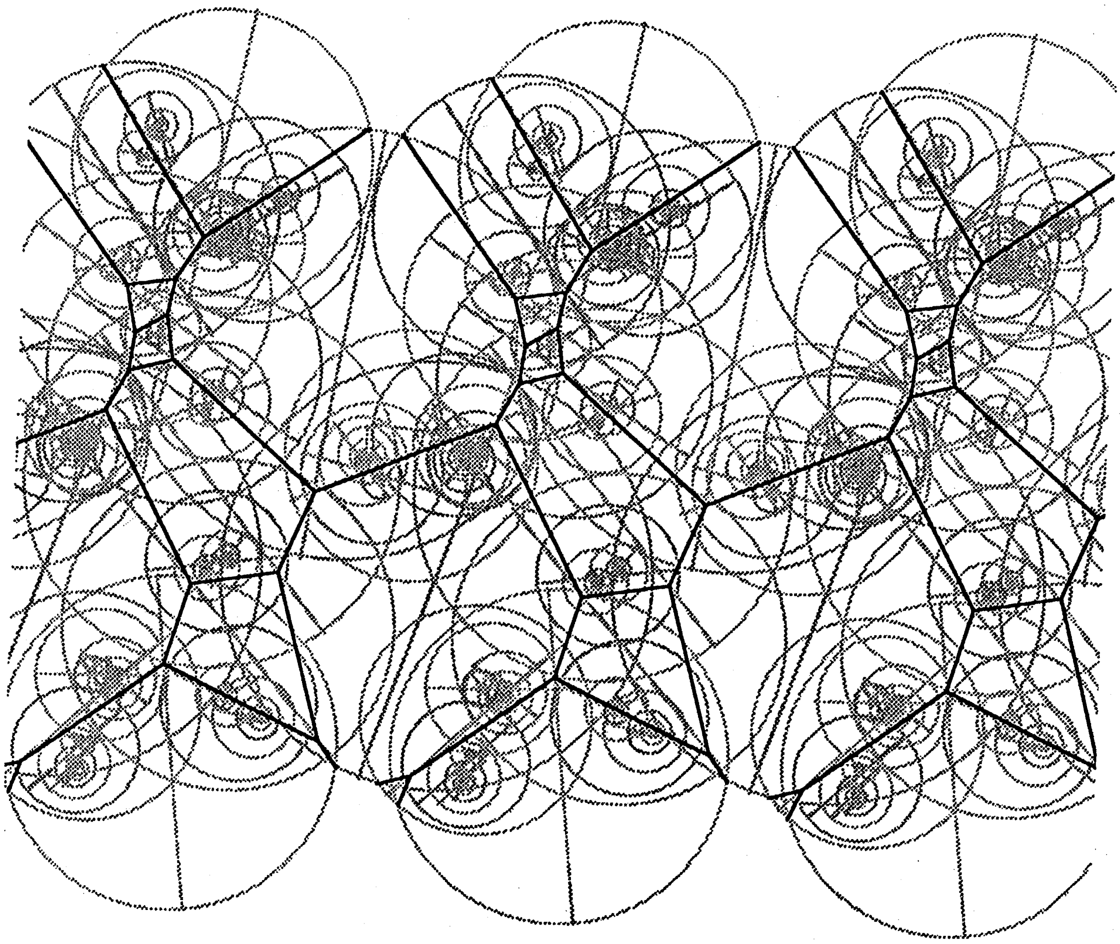

Figure 4

# 4 Proof of Theorem 1.1

## 4.1 Proof of Theorem 1.1(1)

The proof uses the argument of *geometric continuity*. Let $\mathcal{J}$ be the subset of $QF$ consisting of the representations which satisfies Condition (J). Since $QF$ is connected, to show that $\mathcal{J} = QF$, we only need to prove that (i) $\mathcal{J} \neq \emptyset$, (ii) $\mathcal{J}$ is open in $QF$ and (iii) $\mathcal{J}$ is closed in $QF$.

### 4.1.1 Proof of (i)

Since $\rho_0 \in \mathcal{J}$, $\mathcal{J} \neq \emptyset$.

### 4.1.2 Proof of (ii)

For simplicity, we only consider the generic situation. Supporse that $\rho \in \mathcal{J}$. Then $\partial Ph(\rho)$ consists of finite number of isometric hemispheres $Ih(\rho(P_1))$, $\ldots$, $Ih(\rho(P_r))$ (modulo the action of $\rho(K)$). Since we are considering the generic situation, the combinatorial type which is formed by $Ih(\rho(P_1)), \ldots, Ih(\rho(P_r))$ is unchanged after a slight deformation of $\rho$. Let $\rho' \in QF$ be such representation. Since the combinatorial type which is formed by $Ih(\rho'(P_1)), \ldots, Ih(\rho'(P_r))$

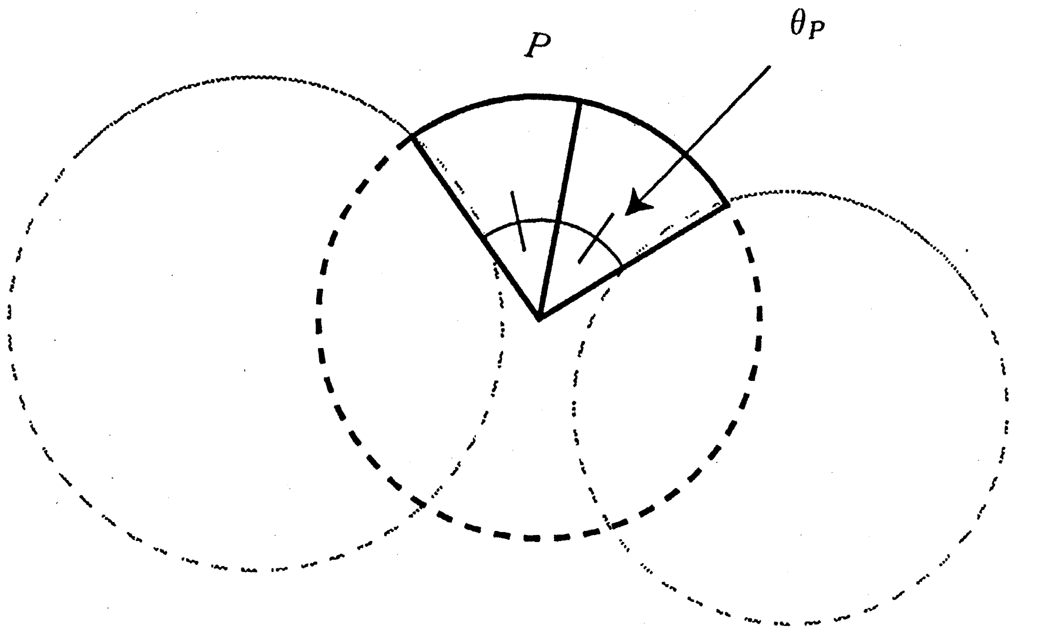


Figure 5

is equal to that of $\partial Ph(\rho)$, we can see that the set

$$P = \cap\{Ext(Ih(\rho'(P_1^{K^n}))), \ldots, Ext(Ih(\rho'(P_r^{K^n}))) \mid n \in \mathbb{Z}\}$$

satisfies the condition that is required by Poincare's theorem. Thus $P$ is the Ford domain of $\operatorname{Im}\rho'$, and $\rho' \in \mathcal{J}$. Note that the combinatorial types of $\partial Ph$ changes at the non-generic representations. (i.e. Either $\nu_+$ or $\nu_-$ lies on an edge of $\mathcal{D}$.) In that case, we need more careful observation.

### 4.1.3 Proof of (iii)

Let $\{\rho_n\} \subset \mathcal{J}$ be a sequence which converges to $\rho_\infty \in QF$. Then it is known that $\{\rho_n\}$ converges strongly to $\rho_\infty$. Thus $\rho_\infty \in \mathcal{J}$ by Proposition 4.1 below.

## 4.2 Proof of Theorem 1.1(2)

Let $\rho_\infty \in \partial QF$. It is known that there is a sequence $\{\rho_n\} \subset QF$ which converges strongly to $\rho_\infty$. Thus we only need to prove the following proposition.

**Proposition 4.1.** *Let $\{\rho_n\} \subset \mathcal{J}$ be a sequence which converges strongly to $\rho_\infty \in \overline{QF}$. Then the following holds.*

(1) *$\rho_\infty$ satisfies Condition* (J).

(2) *$\nu_\epsilon(\rho) \in \partial\mathbb{H}^2$ if and only if $\nu_\epsilon^M(\rho) \in \partial\mathbb{H}^2$ and the equation $\nu_\epsilon(\rho) = \nu_\epsilon^M(\rho)$ holds.*

The proof is divided into several steps.

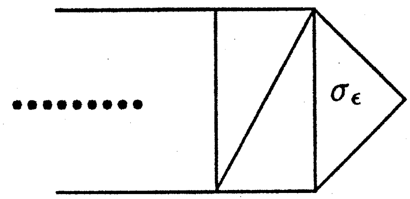


Figure 6

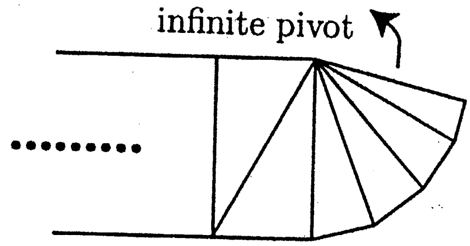


Figure 7

### 4.2.1 Upper bound for the visible isometric hemispheres

Since $\{\rho_n\}$ converges strongly to $\rho_\infty$, by taking a subsequence, we can find an ascending sequence of chains of triangles in $\mathcal{D}$

$$\Sigma_1 \subset \Sigma_2 \subset \Sigma_3 \subset \cdots \to \Sigma_\infty$$

such that (i) each $\Sigma_n$ is a subchain of $\Sigma(\rho_n)$, (ii) if $\Sigma_\infty$ has an end triangle $\sigma_\epsilon$, $\sigma_\epsilon$ is an end triangle of each $\Sigma_n$ and $\Sigma(\rho_n)$ ($n \in \mathbb{N}$), and (iii) $\partial Ph(\rho_\infty)$ consists of the faces supported by the isometric hemispheres of elliptic generators with slope in $\Sigma_\infty$. (Several faces might be degenerate to points.)

### 4.2.2 Limit of end invariants

The two ends of $\Sigma_\infty$ is one of the following three types.

(i) The end contains at most finite number of triangles, thus contains an end triangle $\sigma_\epsilon$. (See Figure 6.)

(ii) The end contains infinite number of triangles and finite number of pivots, thus contains an infinite pivot. (See Figure 7.)

(iii) The end contains infinite number of pivots.

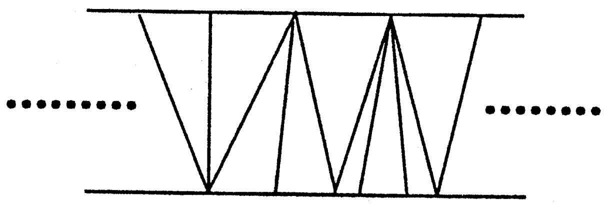

Figure 8

In each case, the end invariant $\{\nu_\epsilon(\rho_n)\}$ converges to a point $\nu_{\infty,\epsilon} \in \overline{\mathbb{H}^2}$. We shall observe that $\rho_\infty$ satisfies Condition (J) and $\nu(\rho_\infty) = \nu_\infty = (\nu_{\infty,+}, \nu_{\infty,-})$. (See Figure 8.)

### 4.2.3 Proof of Proposition 4.1(1)

The core of the proof is the observation that the local degeneration of the cells of $\partial Ph(\rho_n)$ is determined by $\nu(\rho_n) \to \nu_\infty$. (The local structures are assembled into the global structure.) By 4.2.1, we may assume that (locally) the combinatorial structure is unchanged for sufficiently large $n$. (To be precise, the face which is dual to an infinite pivot of the type (ii) end changes eternally, and we should be careful for it.) Thus we only need to see that the degeneration of the cells of $\partial Ph(\rho_n)$ with stable combinatorial structure is determined by $\nu(\rho_n) \to \nu_\infty$.

Since the full proof is elementary but too long, we shall make only one typical observation here. We call a vertex of $\partial Ph$ an *inner vertex* if it is contained in $\mathbb{H}^3$ and an edge is an *inner edge* if both of its endpoints are inner vertices. By 3.1, every inner edge is dual to a segment which corresponds to a triangle in $\Sigma(\rho)$. (See 3.1-Step 3.)

**Observation 4.2.** *If an inner edge $e \subset \partial Ph$ shrinks to a point as $n \to \infty$, then $e$ is dual to a segment corresponding to a triangle which contains end pivot $s(P)$ of $\Sigma_\infty$. Moreover, we can see that $\nu_{\infty,\epsilon} = s(P)$ and $\rho_\infty(KP)$ is an accidental parabolic.*

*Proof.* Since $e$ is an inner edge, $e$ corresponds to the triangle as depicted in Figure 9. Then $e$ has a neighborhood as depicted in 10. By the symmetry with respect to $Q$ and $R$, we can see that the both Euclidean and hyperbolic length of $e_1$, $e_2$ and $e_3$ coincide. Thus they all shrink to the same point. Then we can see that $Axis(\rho_\infty(Q)) \cap Axis(\rho_\infty(R)) \neq \emptyset$ and thus $\rho_\infty(RQ) = \rho_\infty(KP)$ has a fixed point in $\overline{\mathbb{H}^3}$. (Remind that $P$, $Q$ and $R$ are order 2

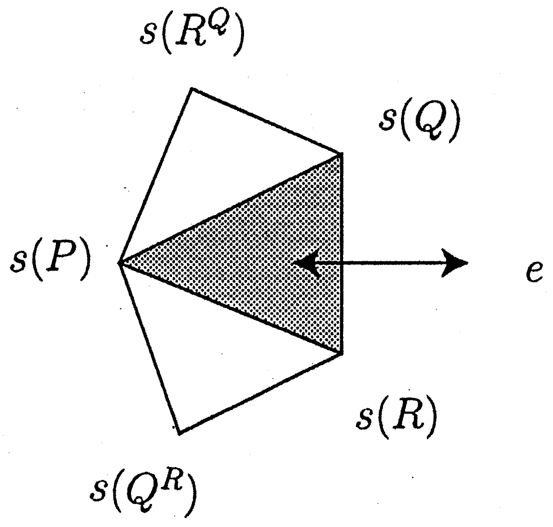


Figure 9

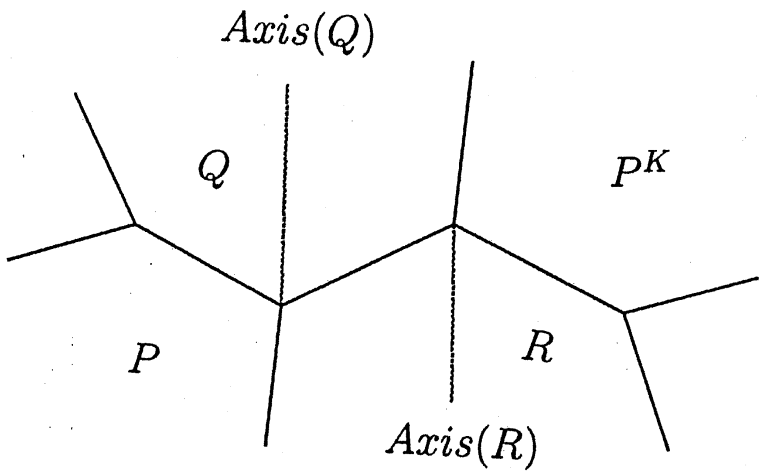


Figure 10

elements.) Since $\{\rho_n\}$ converges to $\rho_\infty$, $\rho_\infty$ has no accidental elliptic, thus $\rho_\infty(KP)$ is an accidental parabolic.

Since $Ih(\rho_\infty(KP)) = Ih(\rho_\infty(P))$ and $Ih(\rho_\infty(KP)^{-1}) = Ih(\rho_\infty(P^K))$, the limit set $\Lambda(\langle\rho_\infty(K^2), \rho_\infty(KP)\rangle)$ is equal to $l_a \cup \{\infty\}$, where $l_a$ is the line which contains $c(\rho_\infty, P)$ and is parallel to the real axis. It is known that the limit set of $\operatorname{Im}\rho$ is contaied in one side of $\Lambda(\langle\rho_\infty(K^2), \rho_\infty(KP)\rangle)$. Thus we can see that $s(P)$ is an end pivot and $\nu_{\infty,\epsilon} = s(P)$. □

### 4.2.4 Proof of Proposition 4.1(2)

(i) By 4.2.2, $\nu_{\infty,\epsilon} = s(P) \in \mathbb{Q} \cup \{\infty\}$ if and only if the $\epsilon$-end of $\Sigma_\infty$ is type (i) with $\nu_{\infty,\epsilon} = s(P)$, or the $\epsilon$-end of $\Sigma_\infty$ is type (ii) with infinite pivot

$s(P)$. The case of type (i) end is easy. In the case of type (ii) end, the argument in [1] is applicable and we can prove the proposition.

(ii) By 4.2.2, $\nu_{\infty,\epsilon} \in \partial\mathbb{H}^2 - \mathbb{Q} \cup \{\infty\}$ if and only if the $\epsilon$-end of $\Sigma_\infty$ is type (iii). Let $P_1, P_2, \ldots$ be elliptic generators such that $s(P_1), s(P_2), \ldots$ are the mutually distinct pivots of the $\epsilon$-end of $\Sigma_\infty$. Then each $s(P_j)$ is also a pivot of $\Sigma(\rho_n)$ for sufficiently large $n$. In this case the argument in [1] is applicable and we can see that there exists a universal constant $R > 0$ such that $r(\rho_n, P_j) > R$, thus $r(\rho_\infty, P_j) \geq R$. By a direct calculation, we can see for each $i \neq j$ that

$$r(\rho_\infty, P_iP_j) = r(\rho_\infty, P_i)r(\rho_\infty, P_j)/|c(\rho_\infty, P_i) - c(\rho_\infty, P_j)|.$$

On the other hand, by Jorgensen's inequality, we have the inequality

$$r(\rho_\infty, P_iP_j) \leq 1.$$

Thus, we have $|c(\rho_\infty, P_i) - c(\rho_\infty, P_j)| \geq R^2$. Hence the closed geodesic $\gamma_i$ which represents $\rho_\infty(P_i)$ exits the $\epsilon$-end as $i \to \infty$. Therefore we have the equality

$$\nu_\epsilon^M(\rho_\infty) = \lim_{i\to\infty} s(P_i) = \nu_{\infty,\epsilon}.$$

(See Figure 11.) This completes the proof.

# References


[1] T. Jorgensen, *On pairs of once-punctured tori*, unfinished manuscript.

[2] Y. N. Minsky, *The classification of punctured-torus groups*, preprint.

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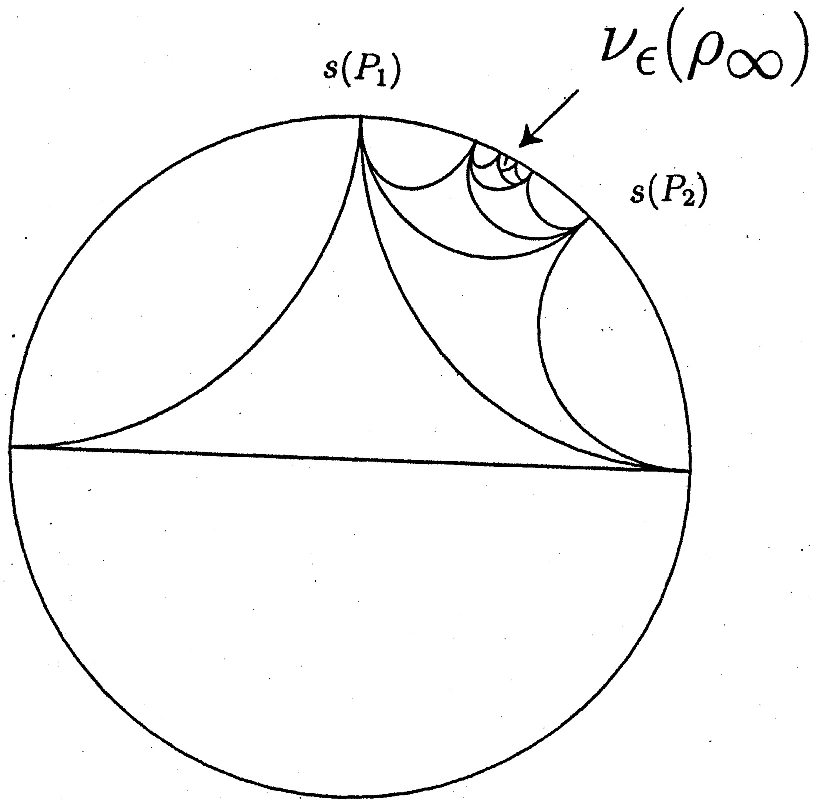


Figure 11