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Perturbations of maximal monotone operators applied to the nonlinear Schrödinger and complex Ginzburg-Landau equations

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1. Introduction

Let Ω be a bounded or unbounded domain in $\mathbb{R}^N$ with compact C^2-boundary $\partial\Omega$. In $L^2(\Omega)$ we consider the nonlinear Schrödinger equation

$$(1.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} - i\Delta u + |u|^{p-1}u = 0, & (x,t) \in \Omega \times \mathbb{R}_+, \\ u = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}_+, & \\ u(x,0) = u_0(x), \quad x \in \Omega, & \end{cases}$$

where $i = \sqrt{-1}$, the exponent $p \geq 1$ is a constant and u is a complex-valued unknown function (cf. Lions [**11**]). The global existence of unique strong solutions to (1.1) was first proved by Pecher and von Wahl [**16**] under the following condition: $1 \leq p < \infty$ $(N = 1, 2)$ and

$$(1.2)\qquad 1 \leq p \leq \frac{N+2}{N-2} \quad (3 \leq N \leq 8).$$

They also conjecture that if $N \geq 3$ then $(N+2)/(N-2)$ is the largest possible exponent for the global existence of strong solutions (see [**16**, Remark I.3]). Applying her characterization theorem for maximal monotonicity, Shigeta [**17**] removed the restriction of $N \leq 8$ in condition (1.2).

The first purpose of this paper is to prove the global existence for all exponents $p \geq 1$ contrary to the conjecture. The previous arguments ([**16, 17**]) depending on the Gagliardo-Nirenberg inequality do not work in the case where $p > (N+2)/(N-2)$. So we have established a new inequality (see (1.3) below) similar to the sectorial estimate

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of $-\Delta$ in L^p (cf. [15]). Our approach here is much simpler than theirs and described as follows. We use a new type perturbation theorem for maximal monotone operators in a "complex" Hilbert space. In $L^2(\Omega)$ we introduce two operators as follows:

$$Su := -\Delta u \quad \text{for} \quad u \in D(S) := H^2(\Omega) \cap H_0^1(\Omega),$$
$$Bu := |u|^{p-1}u \quad \text{for} \quad u \in D(B) := L^2(\Omega) \cap L^{2p}(\Omega),$$

where $H^2(\Omega)$ and $H_0^1(\Omega)$ are the usual Sobolev spaces of L^2-type. Let $\varepsilon > 0$. Denoting by B_ε the Yosida approximation of B, we can show that for every $u \in D(S)$ and $p \geq 1$,

$$\big|\operatorname{Im}(Su, B_\varepsilon u)_{L^2}\big| \leq \frac{p-1}{2\sqrt{p}}\operatorname{Re}(Su, B_\varepsilon u)_{L^2}. \tag{1.3}$$

This inequality enables us to assert that $iS + B$ is maximal monotone in $L^2(\Omega)$.

The second purpose is to discuss another applicability of the inequality (1.3). Actually, we can improve the result of Unai and Okazawa [21] concerning the global existence for the complex Ginzburg-Landau equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - (\lambda + i\alpha)\Delta u + (\kappa + i\beta)|u|^{p-1}u - \gamma u = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+, \\ u = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}_+, \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{cases} \tag{1.4}$$

where $\lambda > 0$, $\kappa > 0$, $p \geq 1$ and α, β, $\gamma \in \mathbb{R}$ are constants. This equation has been widely studied by many authors using different methods (cf. Bu [3], Doering, Gibbon and Levermore [5], Ginibre and Velo [6, 7], Temam [19] and Yang [22]). Recently blow-up results for (1.4) with $\alpha = \gamma = 0$ and $\kappa < 0$ was given by Zaag [23]. Equation (1.4) is obviously reduced to a usual nonlinear Schrödinger equation when $\lambda = \kappa = \gamma = 0$ and to a nonlinear heat equation when $\alpha = \beta = \gamma = 0$.

The third purpose is to consider a parabolic regularization to (1.1). Namely, we turn our attention to the following equation

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - \Big(\dfrac{1}{n} + i\Big)\Delta u_n + |u_n|^{p-1}u_n = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+, \\ u_n = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}_+, \\ u_n(x,0) = u_0(x), \quad x \in \Omega, \end{cases} \tag{1.5}$$

where $n \in \mathbb{N}$. This is a special case of (1.4) and regarded as an approximate problem to (1.1).

Before stating our results we give some notations and definitions used in this paper. We shall use those spaces of complex-valued functions over Ω (or its closure $\overline{\Omega}$) such as $L^r(\Omega)$ $(r > 1)$, $C_0(\Omega)$, $C_0^\infty(\Omega)$, $C^1(\overline{\Omega})$, $C^{0,\alpha}(\overline{\Omega})$, $C^{1,\alpha}(\overline{\Omega})$ $(0 < \alpha \leq 1)$, etc. The norms of $L^r(\Omega)$ and $H^1(\Omega)$ are denoted by $\|\cdot\|_{L^r}$ and $\|\cdot\|_{H^1}$, respectively. Next we define two kinds of strong solution. One is bounded globally and the other may grow exponentially.

Definition 1. The global *strong solution* to (1.1) (or (1.5)) is defined as an $L^2(\Omega)$-valued function $u(t) := u(x,t)$ with the following properties:

(a) $u(t) \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^{2p}(\Omega)$ for all $t \geq 0$.

(b) $u(\cdot)$ is Lipschitz continuous on $[0,\infty)$: $u(\cdot) \in C^{0,1}([0,\infty); L^2(\Omega))$.

(c) The strong derivative $u'(t)$ exists for almost all $t \geq 0$ and is bounded in $L^2(\Omega)$: $u(\cdot) \in W^{1,\infty}(0,\infty; L^2(\Omega))$.

(d) $u(\cdot)$ satisfies (1.1) (or (1.5)) almost everywhere on $[0,\infty)$.

Definition 2. The global *strong solution* to (1.4) is defined as an $L^2(\Omega)$-valued function $u(t) := u(x,t)$ with the following properties:

(a) $u(t) \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^{2p}(\Omega)$ for all $t \geq 0$.

(b) $u(\cdot) \in C^{0,1}([0,T]; L^2(\Omega))$ $\forall\, T > 0$.

(c) $u(\cdot) \in W^{1,\infty}(0,T; L^2(\Omega))$ $\forall\, T > 0$.

(d) $u(\cdot)$ satisfies (1.4) a.e. on $[0,\infty)$.

We now state our main results in this paper.

Theorem 1.1. *Let $p \geq 1$. Then for any $u_0 \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^{2p}(\Omega)$ there exists a unique global strong solution $u(t) := u(x,t)$ to (1.1) in $L^2(\Omega)$ such that*

$$u(\cdot) \in L^\infty(0,\infty; H^2(\Omega) \cap L^{2p}(\Omega)), \tag{1.6}$$

$$u(\cdot) \in C^{0,1/2}([0,\infty); H_0^1(\Omega)) \cap C^{0,1/(p+1)}([0,\infty); L^{p+1}(\Omega)), \tag{1.7}$$

$$\|u(t)\|_{H^1} \leq \|u_0\|_{H^1}, \tag{1.8}$$

$$\|u(t) - v(t)\|_{L^2} \leq \|u_0 - v_0\|_{L^2}, \tag{1.9}$$

$$\|\nabla u(t) - \nabla v(t)\|_{L^2}^2 \leq c_1 \|u_0 - v_0\|_{L^2}, \tag{1.10}$$

$$\|u(t) - v(t)\|_{L^{p+1}}^{p+1} \leq 2^{p-1} c_1 \|u_0 - v_0\|_{L^2}, \tag{1.11}$$

where $v(t)$ is a solution to (1.1) with initial value $v_0 \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^{2p}(\Omega)$ and

$$c_1 := \sqrt{p}\,(\|\Delta u_0\|_{L^2} + \|u_0\|_{L^{2p}}^p + \|\Delta v_0\|_{L^2} + \|v_0\|_{L^{2p}}^p).$$

The next theorem improves the main result in [**21**]. In fact, we have eliminated the condition $\lambda\kappa+\alpha\beta>0$ assumed there.

Theorem 1.2. *Let $\lambda>0$, $\kappa>0$ and $p\geq 1$. If $|\beta|\leq \dfrac{2\sqrt{p}}{p-1}\kappa$, then for any $u_0\in H^2(\Omega)\cap H_0^1(\Omega)\cap L^{2p}(\Omega)$ there exists a unique global strong solution $u(t):=u(x,t)$ to* (1.4) *in $L^2(\Omega)$ such that*

$$u(\cdot)\in L^\infty(0,T;H^2(\Omega)\cap L^{2p}(\Omega)), \tag{1.12}$$

$$u(\cdot)\in C^{0,1/2}([0,T];H_0^1(\Omega))\cap C^{0,1/(p+1)}([0,T];L^{p+1}(\Omega)), \tag{1.13}$$

$$\|u(t)\|_{H^1}\leq e^{\gamma t}\|u_0\|_{H^1}, \tag{1.14}$$

$$\|u(t)-v(t)\|_{L^2}\leq e^{\gamma t}\|u_0-v_0\|_{L^2}, \tag{1.15}$$

$$\|\nabla u(t)-\nabla v(t)\|_{L^2}^2\leq c_2 e^{2\gamma_+ t}\|u_0-v_0\|_{L^2}, \tag{1.16}$$

$$\|u(t)-v(t)\|_{L^{p+1}}^{p+1}\leq 2^{p-1}c_3 e^{2\gamma_+ t}\|u_0-v_0\|_{L^2}, \tag{1.17}$$

where $v(t)$ is a solution to (1.4) *with initial value $v_0\in H^2(\Omega)\cap H_0^1(\Omega)\cap L^{2p}(\Omega)$. Setting $\gamma_+:=\max\{0,\gamma\}$, c_2 and c_3 are given by*

$$\begin{aligned}
c_2 &:= \lambda^{-1}\big[\|(\lambda+i\alpha)\Delta u_0-(\kappa+i\beta)|u_0|^{p-1}u_0+\gamma u_0\|_{L^2}+\gamma_+\|u_0\|_{L^2}\big]\\
&\quad+\lambda^{-1}\big[\|(\lambda+i\alpha)\Delta v_0-(\kappa+i\beta)|v_0|^{p-1}v_0+\gamma v_0\|_{L^2}+\gamma_+\|v_0\|_{L^2}\big],\\
c_3 &:= \kappa^{-1}\big(\lambda+\sqrt{\lambda^2+\alpha^2}\big)c_2.
\end{aligned}$$

Theorem 1.3. *Let $u(t):=u(x,t)$ and $u_n(t):=u_n(x,t)$ be unique global strong solutions to* (1.1) *and* (1.5)*, respectively. Then for all $t\geq 0$,*

$$\|u(t)-u_n(t)\|_{L^2}\leq (t/2n)^{\frac{1}{2}}\|\nabla u_0\|_{L^2},\quad n\in\mathbb{N}, \tag{1.18}$$

$$\|\nabla u(t)-\nabla u_n(t)\|_{L^2}^2\leq (t/2n)^{\frac{1}{2}}c_4(u_0),\quad n\in\mathbb{N}, \tag{1.19}$$

$$\|u(t)-u_n(t)\|_{L^{p+1}}^{p+1}\leq 2^{p-1}(t/2n)^{\frac{1}{2}}c_4(u_0),\quad n\in\mathbb{N}, \tag{1.20}$$

where $c_4(u_0):=\sqrt{p}\,\|\nabla u_0\|_{L^2}(3\|\Delta u_0\|_{L^2}+2\|u_0\|_{L^{2p}}^p)$.

Remark 1. 1) Our method can be applied also to (1.1) (or (1.4)) with generalized nonlinear term $f(|u|^2)u$. Here we assume that $f\in C([0,\infty);\mathbb{R})\cap C^1((0,\infty);\mathbb{R})$ with $f'\geq 0$ and $sf'(s)\leq cf(s)$ for some constant $c>0$. The details will be published elsewhere.

2) In the case where $N\leq 3$ the solution to (1.1) (or (1.4)) is of class C^1; this can be shown by regarding (1.1) (or (1.4)) as a semilinear evolution equation (cf. [**16, 20, 22**]).

This paper is organized as follows. In Section 2 we prove a new type perturbation theorem for maximal monotone operators in a Hilbert space, assuming that an abstract version of the key inequality is satisfied. Section 3 is devoted to the key inequality. Once the key inequality is established, Theorems 1.1 – 1.3 are immediate consequences of the abstract results in Section 2.

2. Perturbation theorems

First we give definitions of nonlinear maximal monotone operators and semigroups of type ω in a complex Hilbert space X with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$.

An operator A with domain $D(A)$ and range $R(A)$ in X is said to be *monotone* if $\operatorname{Re}(Au_1 - Au_2, u_1 - u_2) \geq 0$ for every $u_1,\ u_2 \in D(A)$. If, in addition, $R(A+\zeta) = X$ for some (and hence for every) $\zeta \in \mathbb{C}$ with $\operatorname{Re}\zeta > 0$, we say that A is *maximal monotone* in X.

A semigroup of type ω on $\overline{D(A)}$ (the closure of $D(A)$ in X) is defined as a one-parameter family $\{U(t); t \geq 0\}$ with the following properties:

(i) $U(0) = 1,\ \ U(t+s) = U(t)U(s),\ \ t,s \geq 0.$

(ii) $U(t)v \to v\ (t \downarrow 0),\ \ v \in \overline{D(A)}.$

(iii) $\|U(t)v_1 - U(t)v_2\| \leq e^{\omega t}\|v_1 - v_2\|,\ \ v_1, v_2 \in \overline{D(A)},\ \ t \geq 0.$

In particular, a semigroup of type 0 is a contraction semigroup.

The next lemma may be already known (at least when $\omega = 0$), but we can give it a simple proof.

Lemma 2.1. *Let A be a nonlinear operator in X and $\omega \in \mathbb{R}$. Assume that $A+\omega$ is maximal monotone in X. Let $\{U(t); t \geq 0\}$ be the semigroup of type ω on $\overline{D(A)}$ generated by $-A$. Then for every $u \in D(A)$ and $t \geq 0$,*

$$\|AU(t)u\| \leq e^{\omega t}\|Au\|.$$

Proof. Let $0 < \varepsilon < |\omega|^{-1}$ and $u \in D(A)$. Then we see from the maximal monotonicity of $A+\omega$ that $(1+\varepsilon A)^{-1}$ is Lipschitz continuous on X with Lipschitz constant $(1-\varepsilon\omega)^{-1}$. Hence we obtain

$$\begin{aligned}\|A(1+\varepsilon A)^{-1}u\| &= \varepsilon^{-1}\|(1+\varepsilon A)^{-1}(1+\varepsilon A)u - (1+\varepsilon A)^{-1}u\| \\ &\leq (1-\varepsilon\omega)^{-1}\|Au\|.\end{aligned}$$

This implies that for every $t \geq 0$ and $n \in \mathbb{N}$ with $n \geq \omega t$,

(2.1) $$\|A[1+(t/n)A]^{-n}u\| \leq [1-(\omega t/n)]^{-n}\|Au\|.$$

Let $\{U(t); t \geq 0\}$ be the semigroup of type ω on $\overline{D(A)}$ generated by $-A$. Then it is well-known that for every $v \in \overline{D(A)}$,

(2.2) $$[1+(t/n)A]^{-n}v \to U(t)v \quad \text{in} \quad X \quad (n \to \infty),$$

where the convergence is uniform with respect to t on every finite subinterval of $[0, \infty)$ (see Crandall and Liggett [4] or Miyadera [13]). In view of (2.1) and (2.2) we see from the demi-closedness of A (see Kato [9, Lemma 2.5]) that $U(t)u \in D(A)$ and

$$A[1+(t/n)A]^{-n}u \to AU(t)u \quad \text{weakly in } X.$$

So we obtain

$$\|AU(t)u\| \leq \liminf_{n\to\infty} \|A[1+(t/n)A]^{-n}u\| \leq e^{\omega t}\|Au\|.$$

This completes the proof of the desired inequality. □

For the next lemma see e.g. [13, pp. 145-148].

Lemma 2.2. *Let* A, ω *and* $\{U(t); t \geq 0\}$ *be the same as in* Lemma 2.1. *If* $u_0 \in D(A)$, *then* $u(t) := U(t)u_0$ *is a unique strong solution to the initial value problem*

(IVP) $$u'(t) + Au(t) = 0, \quad u(0) = u_0,$$

in the following sense:

(a) $u(t) \in D(A)$ *for all* $t \geq 0$.
(b) $\|u(t) - u(s)\| \leq e^{\omega_+(t+s)}\|Au_0\| \cdot |t-s|, \ t, s \geq 0, \ \omega_+ := \max\{0, \omega\}$.
(c) $u'(t)$ *exists a.e. on* $[0, \infty)$, *with* $\|u'(t)\| \leq e^{\omega t}\|Au_0\|$ *(a.e.)*.
(d) $u(\cdot)$ *satisfies* (IVP) *a.e. on* $[0, \infty)$.

Here we note that the Lipschitz constant in (b) is determined by the estimate of $u'(t)$ in (c).

Now we state our first perturbation theorem for maximal monotone operators which will be applied to (1.1).

Theorem 2.3. *Let S be a nonnegative selfadjoint operator in X and B a nonlinear maximal monotone operator in X, with $D(S) \cap D(B) \neq \phi$. Assume that there exists a constant θ_1 with $0 < \theta_1 < \pi/2$ such that for every $u \in D(S)$ and $\varepsilon > 0$,*

$$|\mathrm{Im}(Su, B_\varepsilon u)| \leq (\tan\theta_1)\mathrm{Re}(Su, B_\varepsilon u), \tag{2.3}$$

where B_ε is the Yosida approximation of B: $B_\varepsilon := \varepsilon^{-1}[1 - (1 + \varepsilon B)^{-1}]$. Then

(a) *$iS + B$ is maximal monotone in X.*

(b) *For every $v \in D(S^{1/2})$ and $\zeta \in \mathbb{C}$ with $\mathrm{Re}\,\zeta > 0$,*

$$\|S^{1/2}(iS + B + \zeta)^{-1}v\| \leq (\mathrm{Re}\,\zeta)^{-1}\|S^{1/2}v\|, \tag{2.4}$$

where $S^{1/2}$ is the square root of S.

(c) *For every $u_0 \in D(S) \cap D(B)$ and $t \geq 0$,*

$$\|SU(t)u_0\|^2 + \|BU(t)u_0\|^2 \leq \frac{1 + \sin\theta_1}{1 - \sin\theta_1}\|(iS + B)u_0\|^2, \tag{2.5}$$

where $\{U(t)\}$ is the contraction semigroup on $\overline{D(S) \cap D(B)}$ generated by $-(iS + B)$.

Proof. Let $u \in D(S)$ and $\varepsilon > 0$. First we shall show that

$$\|Su\|^2 + \|B_\varepsilon u\|^2 \leq \frac{1 + \sin\theta_1}{1 - \sin\theta_1}\|(iS + B_\varepsilon)u\|^2. \tag{2.6}$$

In fact, we see from (2.3) that

$$\begin{aligned}
\|Su\|^2 + \|B_\varepsilon u\|^2 &= \|(iS + B_\varepsilon)u\|^2 - 2\mathrm{Re}(iSu, B_\varepsilon u) \\
&\leq \|(iS + B_\varepsilon)u\|^2 + 2|\mathrm{Im}(Su, B_\varepsilon u)| \\
&\leq \|(iS + B_\varepsilon)u\|^2 + 2(\tan\theta_1)\mathrm{Re}(Su, B_\varepsilon u) \\
&= \|(iS + B_\varepsilon)u\|^2 + 2(\tan\theta_1)\mathrm{Re}(Su, (iS + B_\varepsilon)u) \\
&\leq \|(iS + B_\varepsilon)u\|^2 + 2(\tan\theta_1)\|Su\| \cdot \|(iS + B_\varepsilon)u\|.
\end{aligned}$$

This implies that

$$\|Su\| \leq (\tan\theta_1 + \sqrt{1 + \tan^2\theta_1})\|(iS + B_\varepsilon)u\| = \frac{\cos\theta_1}{1 - \sin\theta_1}\|(iS + B_\varepsilon)u\|.$$

Therefore we obtain (2.6).

Now we shall prove that $iS + B$ is maximal monotone in X based on the argument in Kato [10, Section 10]. Since B_ε is monotone and Lipschitz continuous on X, it follows that $iS + B_\varepsilon$ is also maximal monotone in X. For $f \in X$ and $\varepsilon > 0$ let $u_\varepsilon \in D(S)$ be a unique solution of the equation

$$(iS + B_\varepsilon)u_\varepsilon + u_\varepsilon = f. \tag{2.7}$$

Since we have assumed that $D(S) \cap D(B) \neq \phi$, it is easy to show that $\{\|u_\varepsilon\|\}$ is bounded as $\varepsilon \downarrow 0$. Hence we see from (2.6) and (2.7) that $\{\|B_\varepsilon u_\varepsilon\|\}$ is bounded. Therefore it follows from Brezis, Crandall and Pazy [**2**, Theorem 2.1] (see [**18**, Proposition IV.2.1]) that $iS + B$ is maximal monotone in X.

Next we prove (2.4). Since $B_\varepsilon u \to Bu$ $(\varepsilon \downarrow 0)$ in X for every $u \in D(B)$, we see from (2.3) that for every $u \in D(S) \cap D(B)$,

$$|\mathrm{Im}(Su, Bu)| \leq (\tan\theta_1)\mathrm{Re}(Su, Bu). \tag{2.8}$$

Let $v \in D(S^{1/2})$ and $\zeta \in \mathbb{C}$ with $\mathrm{Re}\,\zeta > 0$. According to the maximality of $iS + B$ there exists a unique solution $u_\zeta \in D(S) \cap D(B)$ of the equation $(iS + B)u_\zeta + \zeta u_\zeta = v$. It then follows that

$$\mathrm{Re}(Su_\zeta, Bu_\zeta) + (\mathrm{Re}\,\zeta)\|S^{1/2}u_\zeta\|^2 = \mathrm{Re}(S^{1/2}u_\zeta, S^{1/2}v).$$

In view of (2.8) we see that $\|S^{1/2}u_\zeta\| \leq (\mathrm{Re}\,\zeta)^{-1}\|S^{1/2}v\|$. This is nothing but (2.4).

Finally we prove (2.5). Letting $\varepsilon \downarrow 0$ in (2.6) with $u = U(t)u_0$ $(u_0 \in D(S) \cap D(B)$, $t \geq 0)$, we have

$$\|SU(t)u_0\|^2 + \|BU(t)u_0\|^2 \leq \frac{1 + \sin\theta_1}{1 - \sin\theta_1}\|(iS + B)U(t)u_0\|^2.$$

Applying Lemma 2.1 to the right-hand side, we obtain (2.5). □

Remark 2. For the maximal monotonicity of $S + B$ in term of $\mathrm{Re}(Su, B_\varepsilon u)$ see e.g. [**14**, Lemma 6.2].

Corollary 2.4. *In* Theorem 2.3 *assume further that $D(S) \cap D(B)$ is dense in X. Then $U(t)$ leaves $D(S^{1/2})$ invariant and for every $v \in D(S^{1/2})$ and $t \geq 0$,*

$$\|S^{1/2}U(t)v\| \leq \|S^{1/2}v\|. \tag{2.9}$$

In particular, if $0 \in D(B)$ and $B0 = 0$, then for every $v \in X$ and $t \geq 0$,

$$\|U(t)v\| \leq \|v\|. \tag{2.10}$$

Proof. We see from (2.4) that for every $v \in D(S^{1/2})$, $t \geq 0$ and $n \in \mathbb{N}$,

$$\|S^{1/2}[1 + (t/n)(iS + B)]^{-n}v\| \leq \|S^{1/2}v\|.$$

Since $D(S^{1/2}) \subset \overline{D(S) \cap D(B)}$, (2.9) follows from (2.2) with $A = iS + B$ and the weak-closedness of $S^{1/2}$. $\square$

The next is our second perturbation theorem for maximal monotone operators which will be applied to (1.4).

Theorem 2.5. *In* Theorem 2.3 *assume further that there exists a constant* θ_2 *with* $0 < \theta_2 < \pi/2$ *such that for every* $u_1, u_2 \in D(B)$,

$$|\mathrm{Im}(Bu_1 - Bu_2, u_1 - u_2)| \leq (\tan\theta_2)\mathrm{Re}(Bu_1 - Bu_2, u_1 - u_2). \tag{2.11}$$

For $\lambda > 0$, $\kappa > 0$ *and* α, β, $\gamma \in \mathbb{R}$ *let*

$$A := (\lambda + i\alpha)S + (\kappa + i\beta)B - \gamma, \quad D(A) := D(S) \cap D(B).$$

Then

(a) $A + \gamma$ *is maximal monotone in* X, *provided that* $|\beta| \leq (\tan\theta_0)^{-1}\kappa$, *where* $\theta_0 := \max\{\theta_1, \theta_2\}$.

(b) *For every* $v \in D(S^{1/2})$ *and* $\zeta \in \mathbb{C}$ *with* $\mathrm{Re}\,\zeta > \gamma$,

$$\|S^{1/2}(A + \zeta)^{-1}v\| \leq (\mathrm{Re}\,\zeta - \gamma)^{-1}\|S^{1/2}v\|. \tag{2.12}$$

(c) *For every* $u_0 \in D(A)$ *and* $t \geq 0$,

$$\|SU(t)u_0\| \leq \lambda^{-1}(e^{\gamma t}\|Au_0\| + \gamma_+\|U(t)u_0\|), \tag{2.13}$$

where $\{U(t)\}$ *is the semigroup of type* γ *on* $\overline{D(A)}$ *generated by* $-A$.

Proof. Let $\lambda > 0$, $\kappa > 0$ and α, $\beta \in \mathbb{R}$. Suppose that

$$|\beta| \leq (\tan\theta_0)^{-1}\kappa \leq (\tan\theta_j)^{-1}\kappa \quad (j = 1, 2).$$

Then it follows from (2.3) that

$$\mathrm{Re}(Su, (\kappa + i\beta)B_\varepsilon u) \geq \left[(\tan\theta_1)^{-1}\kappa - |\beta|\right]|\mathrm{Im}(Su, B_\varepsilon u)| \geq 0. \tag{2.14}$$

This implies that for every $u \in D(S)$,

$$\lambda\|Su\| \leq \|(\lambda + i\alpha)Su + (\kappa + i\beta)B_\varepsilon u\|. \tag{2.15}$$

In fact, we see from (2.14) that

$$\lambda\|Su\|^2 \le \mathrm{Re}(Su, (\lambda + i\alpha)Su + (\kappa + i\beta)B_\varepsilon u). \tag{2.16}$$

On the other hand, it follows from (2.11) that $(\kappa + i\beta)B$ is also monotone in X:

$$\begin{aligned}&\mathrm{Re}((\kappa + i\beta)(Bu_1 - Bu_2), u_1 - u_2)\\ &\ge \left[(\tan\theta_2)^{-1}\kappa - |\beta|\right] |\mathrm{Im}(Bu_1 - Bu_2, u_1 - u_2)| \ge 0.\end{aligned}$$

This implies further that $(\kappa + i\beta)B_\varepsilon$ is monotone in X:

$$\mathrm{Re}((\kappa + i\beta)(B_\varepsilon v_1 - B_\varepsilon v_2), v_1 - v_2) \ge \varepsilon\kappa\|B_\varepsilon v_1 - B_\varepsilon v_2\|^2 \ge 0.$$

Hence we see that $(\lambda + i\alpha)S + (\kappa + i\beta)B_\varepsilon$ is also maximal monotone in X. Therefore for every $f \in X$ and $\varepsilon > 0$ there exists a unique solution $u_\varepsilon \in D(S)$ of the equation

$$(\lambda + i\alpha)Su_\varepsilon + (\kappa + i\beta)B_\varepsilon u_\varepsilon + u_\varepsilon = f.$$

Since (2.15) plays the role of (2.6) in Theorem 2.3, we can conclude that

$$A + \gamma = (\lambda + i\alpha)S + (\kappa + i\beta)B$$

is maximal monotone in X.

To prove (2.12) let $v \in D(S^{1/2})$ and $\zeta \in \mathbb{C}$ with $\mathrm{Re}\,\zeta > \gamma$. According to the maximality of $A + \gamma$ there exists a unique solution $u_\zeta \in D(A)$ of the equation $Au_\zeta + \zeta u_\zeta = v$, i.e.,

$$(\lambda + i\alpha)Su_\zeta + (\kappa + i\beta)Bu_\zeta - \gamma u_\zeta + \zeta u_\zeta = v.$$

Making the inner product of this equation with Su_ζ, we have

$$\mathrm{Re}(Su_\zeta, (\kappa + i\beta)Bu_\zeta) + (\mathrm{Re}\,\zeta - \gamma)\|S^{1/2}u_\zeta\|^2 \le \mathrm{Re}(S^{1/2}u_\zeta, S^{1/2}v).$$

Letting ε tend to zero in (2.14) with $u \in D(S) \cap D(B)$, we see that $\mathrm{Re}(Su, (\kappa + i\beta)Bu) \ge 0$. Therefore we obtain (2.12).

Finally we prove (2.13). Letting $\varepsilon \downarrow 0$ in (2.16) with $u \in D(A)$, we have

$$\begin{aligned}\lambda\|Su\|^2 &\le \mathrm{Re}(Su, (A + \gamma)u)\\ &\le \|Su\|(\|Au\| + \gamma_+\|u\|).\end{aligned}$$

This implies that for every $u_0 \in D(A)$ and $t \ge 0$,

$$\lambda\|SU(t)u_0\| \le \|AU(t)u_0\| + \gamma_+\|U(t)u_0\|.$$

Applying Lemma 2.1 to the first term on the right-hand side, we obtain (2.13). □

As a consequence of (2.12), we have

Corollary 2.6. *In* Theorem 2.5 *assume further that $D(A)$ is dense in X. Then $U(t)$ leaves $D(S^{1/2})$ invariant and for every $v \in D(S^{1/2})$ and $t \geq 0$,*

$$\|S^{1/2}U(t)v\| \leq e^{\gamma t}\|S^{1/2}v\|. \tag{2.17}$$

In particular, if $0 \in D(B)$ and $B0 = 0$, then for every $v \in X$ and $t \geq 0$,

$$\|U(t)v\| \leq e^{\gamma t}\|v\|. \tag{2.18}$$

Concerning approximation to the resolvent and semigroup we have

Theorem 2.7. *Let A and S be the same as in* Theorem 2.3. *Then for every $n \in \mathbb{N}$, $v \in D(S^{1/2})$ and $\zeta \in \mathbb{C}$ with* $\operatorname{Re}\zeta > 0$,

$$\left\|(iS + B + \zeta)^{-1}v - \left[(n^{-1} + i)S + B + \zeta\right]^{-1}v\right\| \leq c_n\|S^{1/2}v\|, \tag{2.19}$$

where $c_n := (\operatorname{Re}\zeta)^{-3/2}(2\sqrt{n})^{-1}$. Let $\{U_n(t); t \geq 0\}$ be the contraction semigroup on $\overline{D(S) \cap D(B)}$ generated by $-\left[(n^{-1} + i)S + B\right]$. Then for every $n \in \mathbb{N}$, $u_0 \in D(S) \cap D(B)$ and $t \geq 0$,

$$\|SU_n(t)u_0\|^2 + \|BU_n(t)u_0\|^2 \leq \frac{1 + \sin\theta_1}{1 - \sin\theta_1}\left\|\left[(n^{-1} + i)S + B\right]u_0\right\|^2. \tag{2.20}$$

Assume further that $D(S) \cap D(B)$ is dense in $D(S^{1/2})$ [i.e., $D(S) \cap D(B)$ is a core for $S^{1/2}$]. Then for every $n \in \mathbb{N}$, $v \in D(S^{1/2})$ and $t \geq 0$,

$$\|U(t)v - U_n(t)v\| \leq (t/2n)^{1/2}\|S^{1/2}v\|, \tag{2.21}$$

where $\{U(t)\}$ is the contraction semigroup on X generated by $-(iS + B)$.

Proof. First Theorem 2.5 applies to conclude that $(n^{-1} + i)S + B$ is maximal monotone in X. Now let $v \in D(S^{1/2})$ and $\zeta \in \mathbb{C}$ with $\operatorname{Re}\zeta > 0$. Then there exist unique solutions $u_n, u \in D(S) \cap D(B)$ of the respective equations

$$\left[(n^{-1} + i)S + B\right]u_n + \zeta u_n = v, \quad (iS + B)u + \zeta u = v.$$

Hence (2.19) follows from the monotonicity of $iS + B$ and (2.4):

$$\begin{aligned}
(\operatorname{Re}\zeta)\|u - u_n\|^2 &\leq \operatorname{Re}(n^{-1}Su_n, u - u_n) \\
&\leq n^{-1}\|S^{1/2}u_n\| \cdot \|S^{1/2}u\| - n^{-1}\|S^{1/2}u_n\|^2 \\
&\leq (1/4n)\|S^{1/2}u\|^2 \\
&\leq (1/4n) \cdot (\operatorname{Re}\zeta)^{-2}\|S^{1/2}v\|^2.
\end{aligned}$$

Next let $u \in D(S)$. Noting that

$$\|(iS + B_\varepsilon)u\| \le \big\|[(n^{-1} + i)S + B_\varepsilon]u\big\|,$$

we see from (2.6) that for every $u \in D(S) \cap D(B)$,

$$\|Su\|^2 + \|B_\varepsilon u\|^2 \le \frac{1+\sin\theta_1}{1-\sin\theta_1}\big\|[(n^{-1} + i)S + B_\varepsilon]u\big\|^2.$$

Thus the proof of (2.20) is parallel to that of (2.5).

Finally we shall prove (2.21). Since $D(S)\cap D(B)$ is dense in $D(S^{1/2})$, it suffices to prove (2.21) for the elements in $D(S) \cap D(B)$. Let $u_0 \in D(S) \cap D(B)$. Then $u_n(t) := U_n(t)u_0$ and $u(t) := U(t)u_0$ are unique strong solutions to the respective initial value problems:

$$u_n'(t) + [(n^{-1} + i)S + B]u_n(t) = 0, \quad \text{a.a. } t \ge 0, \quad u_n(0) = u_0,$$
$$u'(t) + (iS + B)u(t) = 0, \quad \text{a.a. } t \ge 0, \quad u(0) = u_0.$$

So we see from the monotonicity of $iS + B$ that for a.a. $s \ge 0$,

$$\begin{aligned}(d/ds)\|u(s) - u_n(s)\|^2 &= 2\mathrm{Re}\,(u'(s) - u_n'(s), u(s) - u_n(s))\\ &\le 2\mathrm{Re}(n^{-1}Su_n(s), u(s) - u_n(s))\\ &\le (1/2n)\|S^{1/2}u(t)\|^2.\end{aligned}$$

Therefore, (2.21) follows from (2.9). □

2. Proofs of Theorems 1.1–1.3

For the abstract setting of initial value problems (1.1), (1.4) and (1.5) we introduce two operators in the complex Hilbert space $X := L^2(\Omega)$ with inner product $(\cdot,\cdot) = (\cdot,\cdot)_{L^2}$ and norm $\|\cdot\| = \|\cdot\|_{L^2}$. Namely, we define the operators $S,\ B$ as stated in Section 1:

$$Su := -\Delta u \quad \text{for} \quad u \in D(S) := H^2(\Omega) \cap H_0^1(\Omega),$$
$$Bu := |u|^{p-1}u \quad \text{for} \quad u \in D(B) := L^2(\Omega) \cap L^{2p}(\Omega).$$

Then (1.1), (1.4) and (1.5) are regarded as the respective initial value problems for abstract evolution equations:

$$u'(t) + (iS + B)u(t) = 0, \quad u(0) = u_0, \tag{3.1}$$

$$u'(t) + [(\lambda + i\alpha)S + (\kappa + i\beta)B - \gamma]u(t) = 0, \quad u(0) = u_0, \tag{3.2}$$

$$u_n'(t) + [(n^{-1} + i)S + B]u_n(t) = 0, \quad u_n(0) = u_0. \tag{3.3}$$

To apply the results in Section 2, we shall show that the operators S and B satisfy the inequalities (2.3) and (2.11) with the same constant and that $D(S) \cap D(B)$ is a core for $S^{1/2}$.

It is well-known that S is a nonnegative selfadjoint operator in X. On the other hand, we have

Lemma 3.1. *B is a sectorial operator in X, i.e., for every $u_1, u_2 \in D(B)$,*

$$|\mathrm{Im}(Bu_1 - Bu_2, u_1 - u_2)| \leq \frac{p-1}{2\sqrt{p}} \mathrm{Re}(Bu_1 - Bu_2, u_1 - u_2). \tag{3.4}$$

Hence, if $|\theta| \leq \tan^{-1}[2\sqrt{p}/(p-1)]$, then $e^{i\theta}B$ is maximal monotone in X.

Proof. The constant factor in (3.4) has been determined by Liskevich and Perelmuter [**12**]. Apart from the constant factor it is not so difficult to prove (3.4).

It remains to show that B is maximal in X. Let $f \in X$ and $\varepsilon > 0$. Then for almost all $x \in \Omega$ the equation

$$z + \varepsilon |z|^{p-1} z = f(x) \tag{3.5}$$

in $\mathbb{C}$ has a unique solution $z = u_\varepsilon(x)$ such that $|u_\varepsilon(x)| \leq |f(x)|$ and

$$|u_\varepsilon(x) - \tilde{u}_\varepsilon(x)| \leq |f(x) - \tilde{f}(x)|, \tag{3.6}$$

where $\tilde{u}_\varepsilon(x)$ is a unique solution of (3.5) with f replaced with $\tilde{f}$. Using approximation by simple functions, we see from (3.6) that u_ε is measurable on Ω. (The measurability of $u_\varepsilon(x)$ was not mentioned in [**21**].) Therefore $u_\varepsilon \in D(B)$ and we obtain $R(1+\varepsilon B) = X$. □

Lemma 3.2. *$H_0^1(\Omega) \cap C^1(\overline{\Omega})$ is invariant under $(1 + \varepsilon B)^{-1}$ for every $\varepsilon > 0$. More precisely, put $u_\varepsilon(x) := (1 + \varepsilon B)^{-1} f(x)$ for $f \in H_0^1(\Omega) \cap C^1(\overline{\Omega})$ and $\varepsilon > 0$. Then $u_\varepsilon \in H_0^1(\Omega) \cap C^1(\overline{\Omega})$ and*

$$\begin{aligned} \nabla u_\varepsilon &= \frac{1}{\mathrm{Jac}} \{(1 + \varepsilon p |u_\varepsilon|^{p-1}) \nabla f - \varepsilon (p-1) |u_\varepsilon|^{p-3} u_\varepsilon \mathrm{Re}(\overline{u_\varepsilon} \nabla f)\} \\ &= (1 + \varepsilon |u_\varepsilon|^{p-1})^{-1} \nabla f - \frac{1}{\mathrm{Jac}} \varepsilon (p-1) |u_\varepsilon|^{p-3} u_\varepsilon \mathrm{Re}(\overline{u_\varepsilon} \nabla f), \end{aligned} \tag{3.7}$$

where $\mathrm{Jac} = (1 + \varepsilon |u_\varepsilon|^{p-1})(1 + \varepsilon p |u_\varepsilon|^{p-1})$.

Proof. For $\xi = {}^t(\xi_1, \xi_2) \in \mathbb{R}^2$ we set

$$b(\xi) := {}^t(|\xi|^{p-1}\xi_1, |\xi|^{p-1}\xi_2). \tag{3.8}$$

Then we see that b is monotone in $\mathbb{R}^2$:

$$(b(\xi)-b(\eta))\cdot(\xi-\eta)=\mathrm{Re}(|z|^{p-1}z-|w|^{p-1}w)(\overline{z-w})\geq 0,$$

where $z:=\xi_1+i\xi_2$ and $w:=\eta_1+i\eta_2$. This leads us to define

$$\Phi(\xi)={}^t(\Phi_1(\xi),\Phi_2(\xi)):=\xi+\varepsilon b(\xi),\quad \varepsilon>0. \tag{3.9}$$

It then follows from the monotonicity of b that $\Phi:\mathbb{R}^2\to\mathbb{R}^2$ is a bijection. Moreover we can show that Φ is a C^1-bijection. In fact, it is easy to see that Φ is of class C^1 and for every $\xi\in\mathbb{R}^2$,

$$\begin{aligned}\mathrm{Jac}(\xi)&:=\begin{vmatrix}\partial\Phi_1/\partial\xi_1 & \partial\Phi_1/\partial\xi_2\\ \partial\Phi_2/\partial\xi_1 & \partial\Phi_2/\partial\xi_2\end{vmatrix}\quad\text{(Jacobian determinant)}\\ &=(1+\varepsilon|\xi|^{p-1})(1+\varepsilon p|\xi|^{p-1})\geq 1.\end{aligned}$$

Therefore we can conclude by the inverse function theorem that Φ^{-1} is also of class C^1.

Now let $u_\varepsilon(x)=v_\varepsilon(x)+iw_\varepsilon(x):=(1+\varepsilon B)^{-1}f(x)$ for $f=g+ih\in H_0^1(\Omega)\cap C^1(\overline{\Omega})$. Then we have

$$u_\varepsilon(x)+\varepsilon|u_\varepsilon(x)|^{p-1}u_\varepsilon(x)=f(x). \tag{3.10}$$

To show that $u_\varepsilon\in H_0^1(\Omega)\cap C^1(\overline{\Omega})$ put

$$U_\varepsilon(x):={}^t(v_\varepsilon(x),w_\varepsilon(x)),\quad F(x):={}^t(g(x),h(x)).$$

Then we see from (3.8) and (3.9) that (3.10) is equivalent to

$$\Phi(U_\varepsilon(x))=F(x).$$

Since $F:\overline{\Omega}\to\mathbb{R}^2$ is of class C^1, it follows from the chain rule that $U_\varepsilon=\Phi^{-1}\circ F:\overline{\Omega}\to\mathbb{R}^2$ is also of class C^1. In fact, we have

$$\begin{aligned}\nabla v_\varepsilon&=\frac{1}{\mathrm{Jac}}\Big[\{1+\varepsilon|u_\varepsilon|^{p-1}+\varepsilon(p-1)|u_\varepsilon|^{p-3}w_\varepsilon^2\}\nabla g-\varepsilon(p-1)|u_\varepsilon|^{p-3}v_\varepsilon w_\varepsilon\nabla h\Big]\\ &=\frac{1}{\mathrm{Jac}}\{(1+\varepsilon p|u_\varepsilon|^{p-1})\nabla g-\varepsilon(p-1)|u_\varepsilon|^{p-3}v_\varepsilon\mathrm{Re}(\overline{u_\varepsilon}\nabla f)\},\\ \nabla w_\varepsilon&=\frac{1}{\mathrm{Jac}}\{(1+\varepsilon p|u_\varepsilon|^{p-1})\nabla h-\varepsilon(p-1)|u_\varepsilon|^{p-3}w_\varepsilon\mathrm{Re}(\overline{u_\varepsilon}\nabla f)\},\end{aligned}$$

where $\mathrm{Jac}=\mathrm{Jac}(U_\varepsilon(x))=(1+\varepsilon|u_\varepsilon|^{p-1})(1+\varepsilon p|u_\varepsilon|^{p-1})$. Therefore $u_\varepsilon\in C^1(\overline{\Omega})$ and $\nabla u_\varepsilon=\nabla v_\varepsilon+i\nabla w_\varepsilon$ is given by (3.7). On the other hand, we see from (3.7) and (3.10) that $|u_\varepsilon(x)|\leq|f(x)|$ and $|\nabla u_\varepsilon(x)|\leq 2|\nabla f(x)|$. This proves that $u_\varepsilon\in H_0^1(\Omega)$. □

The above-mentioned expressions for ∇v_ε and ∇w_ε are well-ordered comparing with those given in [21]. The consequent simplicity of ∇u_ε in (3.7) leads us to the key inequality the proof of which is now given by

Lemma 3.3. *Let B_ε be the Yosida approximation of B. Then for every $u \in D(S)$,*

$$|\mathrm{Im}(Su, B_\varepsilon u)| \le \frac{p-1}{2\sqrt{p}}\mathrm{Re}(Su, B_\varepsilon u). \tag{3.11}$$

Consequently, for every $u \in D(S) \cap D(B)$,

$$|\mathrm{Im}(Su, Bu)| \le \frac{p-1}{2\sqrt{p}}\mathrm{Re}(Su, Bu).$$

Proof. Put $D_0 := H^2(\Omega) \cap H_0^1(\Omega) \cap C^1(\overline{\Omega})$. Then it follows from the regularity theorem and Morrey's theorem that

$$\begin{aligned} C_0(\Omega) &\subset (1+S)(H^2(\Omega) \cap H_0^1(\Omega) \cap C^{1,\alpha}(\overline{\Omega})) \quad (0<\alpha<1) \\ &\subset (1+S)D_0 \end{aligned}$$

(see e.g. Brezis [1, p. 198]). This implies that $(1+S)D_0$ is dense in X and hence D_0 is a core for S (see Kato [8, Problem III.5.19]). Therefore it suffices to prove (3.11) for the elements in D_0. Let $f \in D_0$. Setting $u_\varepsilon := (1+\varepsilon B)^{-1}f$, we see from Lemma 3.2 that $u_\varepsilon \in H_0^1(\Omega) \cap C^1(\overline{\Omega})$ and

$$\frac{1}{\varepsilon}(\nabla f - \nabla u_\varepsilon) = \frac{|u_\varepsilon|^{p-1}}{1+\varepsilon|u_\varepsilon|^{p-1}}\nabla f + \frac{1}{\mathrm{Jac}}(p-1)|u_\varepsilon|^{p-3}u_\varepsilon \mathrm{Re}(\overline{u_\varepsilon}\nabla f).$$

Since $B_\varepsilon f = \varepsilon^{-1}(f-u_\varepsilon)$, we have

$$\begin{aligned} (Sf, B_\varepsilon f) &= \varepsilon^{-1}(\nabla f, \nabla f - \nabla u_\varepsilon) \\ &= I_1(f) + (p-1)I_2(f), \end{aligned}$$

where

$$I_1(f) := \int_\Omega \frac{|u_\varepsilon|^{p-1}}{1+\varepsilon|u_\varepsilon|^{p-1}}|\nabla f|^2\,dx$$

and

$$I_2(f) := \int_\Omega \frac{1}{\mathrm{Jac}}|u_\varepsilon|^{p-3}(\overline{u_\varepsilon}\nabla f)\cdot \mathrm{Re}(\overline{u_\varepsilon}\nabla f)\,dx.$$

Hence we obtain

$$\mathrm{Re}(Sf, B_\varepsilon f) = I_1(f) + (p-1)\mathrm{Re}I_2(f), \tag{3.12}$$

$$\mathrm{Im}(Sf, B_\varepsilon f) = (p-1)\mathrm{Im}I_2(f). \tag{3.13}$$

Noting that

$$I_1(f) \geq \int_\Omega \frac{1}{\mathrm{Jac}} |u_\varepsilon|^{p-1} |\nabla f|^2 \, dx,$$

$$\mathrm{Re} I_2(f) = \int_\Omega \frac{1}{\mathrm{Jac}} |u_\varepsilon|^{p-3} |\mathrm{Re}(\overline{u_\varepsilon}\, \nabla f)|^2 \, dx,$$

we see by the Cauchy-Schwarz inequality that

$$\begin{aligned} |I_2(f)|^2 &\leq \int_\Omega \frac{1}{\mathrm{Jac}} |u_\varepsilon|^{p-1} |\nabla f|^2 \, dx \int_\Omega \frac{1}{\mathrm{Jac}} |u_\varepsilon|^{p-3} |\mathrm{Re}(\overline{u_\varepsilon}\, \nabla f)|^2 \, dx \\ &\leq I_1(f) \mathrm{Re} I_2(f). \end{aligned} \tag{3.14}$$

Now we can estimate $\mathrm{Im}(Sf, B_\varepsilon f)$ in the same way as in [15]. If $p = 1$, then by (3.13) $\mathrm{Im}(Sf, B_\varepsilon f) = 0$. Therefore we may assume that $p > 1$. It follows from (3.12)–(3.14) that

$$\begin{aligned} (p-1)^{-2} |\mathrm{Im}(Sf, B_\varepsilon f)|^2 &= |I_2(f)|^2 - |\mathrm{Re} I_2(f)|^2 \\ &\leq I_1(f) \mathrm{Re} I_2(f) - |\mathrm{Re} I_2(f)|^2 \\ &= \mathrm{Re}(Sf, B_\varepsilon f) \mathrm{Re} I_2(f) - p |\mathrm{Re} I_2(f)|^2 \\ &\leq \frac{1}{4p} |\mathrm{Re}(Sf, B_\varepsilon f)|^2. \end{aligned}$$

Noting that $\mathrm{Re}(Sf, B_\varepsilon f) \geq 0$, we obtain (3.11). □

We see from Lemmas 3.1 and 3.3 that the inequalities (2.3) and (2.11) hold with

$$\tan\theta_1 = \tan\theta_2 = \frac{p-1}{2\sqrt{p}}.$$

Noting that $C_0^\infty(\Omega) \subset D(S) \cap D(B)$ is a core for $S^{1/2}$, we can conclude that S and B satisfy all the assumptions stated in Section 2.

We are now in a position to prove Theorems 1.1 – 1.3.

Proof of Theorem 1.1. As stated at the beginning of this section, (1.1) is written in the form of (3.1). We see from Theorem 2.3 and Lemma 3.3 that $iS + B$ in (3.1) is a maximal monotone operator with domain $D(S) \cap D(B)$ dense in X. Now let $\{U(t)\}$ be the contraction semigroup on X generated by $-(iS + B)$. Then for every $u_0 \in D(S) \cap D(B)$, $u(t) := U(t)u_0$ is a unique solution to (3.1) in the sense of Lemma 2.2 (with $A = iS + B$ and $\omega = 0$). This implies that (1.1) admits a unique global strong solution $u(x,t)$ in the sense of Definition 1.

It remains to prove (1.6)–(1.11). Since $B0 = 0$, we obtain (1.8) as a combination of (2.9) and (2.10). Noting that $(1+\sin\theta_1)/(1-\sin\theta_1) = p$, we see from (2.5) that for all $t \geq 0$,

$$\|\Delta u(t)\|_{L^2}^2 + \|u(t)\|_{L^{2p}}^{2p} \leq p\,(\|\Delta u_0\|_{L^2} + \|u_0\|_{L^{2p}}^p)^2. \tag{3.15}$$

(1.6) is a consequence of this inequality and (1.8). (1.9) is a property of the contraction semigroup $\{U(t)\}$. Now it follows from the Cauchy-Schwarz inequality that for every $u, v \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^{2p}(\Omega)$,

$$\|\nabla u - \nabla v\|_{L^2}^2 \leq \|u - v\|_{L^2}(\|\Delta u\|_{L^2} + \|\Delta v\|_{L^2}), \tag{3.16}$$

$$\|u - v\|_{L^{p+1}}^{p+1} \leq \|u - v\|_{L^2}(\|u\|_{L^{2p}} + \|v\|_{L^{2p}})^p. \tag{3.17}$$

Consequently, (1.10) and (1.11) follow from (1.9) and (3.15). To prove (1.7) let $t, s \in [0, \infty)$. Then we see from Lemma 2.2 (b) with $\omega = 0$ that

$$\|u(t) - u(s)\|_{L^2} \leq (\|\Delta u_0\|_{L^2} + \|u_0\|_{L^{2p}}^p)|t - s|.$$

Therefore (1.7) follows from (3.15), (3.16) and (3.17) (with u and v replaced with $u(t)$ and $u(s)$):

$$\|\nabla u(t) - \nabla u(s)\|_{L^2}^2 \leq 2\sqrt{p}\,(\|\Delta u_0\|_{L^2} + \|u_0\|_{L^{2p}}^p)^2|t - s|,$$
$$\|u(t) - u(s)\|_{L^{p+1}}^{p+1} \leq 2^p\sqrt{p}\,(\|\Delta u_0\|_{L^2} + \|u_0\|_{L^{2p}}^p)^2|t - s|.$$

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. First we note that (1.4) is written in the form of (3.2). We see from Theorem 2.5, Lemmas 3.1 and 3.3 that

$$A + \gamma = (\lambda + i\alpha)S + (\kappa + i\beta)B$$

is a maximal monotone operator with domain $D(A)$ dense in X. Now let $\{U(t)\}$ be the semigroup of type γ on X generated by $-A$. Then for every $u_0 \in D(A)$, $u(t) := U(t)u_0$ is a unique solution to (3.2) in the sense of Lemma 2.2 (with $\omega = \gamma$). This implies that (1.4) admits a unique global strong solution $u(x,t)$ in the sense of Definition 2.

It remains to prove (1.12)–(1.17). Combining (2.17) with (2.18), we obtain (1.14). (1.15) is a property of the semigroup $\{U(t)\}$ of type γ. Next we prove that $\Delta u(\cdot)$ and $Bu(\cdot)$ are bounded on $[0,T]$. First, (2.13) together with (2.18) yields that for all $t \geq 0$,

$$\|\Delta u(t)\|_{L^2} \leq \lambda^{-1}(\|Au_0\|_{L^2} + \gamma_+\|u_0\|_{L^2})e^{\gamma t}. \tag{3.18}$$

Second, noting that

$$\kappa\|Bu(t)\|_{L^2}^2 = \mathrm{Re}((A-(\lambda+i\alpha)S+\gamma)u(t), Bu(t))_{L^2},$$

we see that for all $t \geq 0$,

$$\kappa\|Bu(t)\|_{L^2} \leq \|Au(t)\|_{L^2} + \|(\lambda+i\alpha)Su(t)\|_{L^2} + \gamma_+\|u(t)\|_{L^2}.$$

Applying Lemma 2.1, (3.18) and (2.18) to the right-hand side, we obtain

$$\|u(t)\|_{L^{2p}}^p \leq \kappa^{-1}(1+\sqrt{1+(\alpha/\lambda)^2})(\|Au_0\|_{L^2} + \gamma_+\|u_0\|_{L^2})e^{\gamma t}. \tag{3.19}$$

Now (1.12) is a consequence of these estimates and (1.14). Furthermore, (3.18) and (3.19) guarantee that (1.16) and (1.17) follow from (3.16) and (3.17), respectively. Finally, let $t, s \in [0, T]$. Then we see from Lemma 2.2(b) that

$$\|u(t) - u(s)\|_{L^2} \leq e^{2\gamma_+ T}\|Au_0\|_{L^2}|t-s|.$$

Therefore we can prove (1.13) in the same way as in the proof of Theorem 1.1 (combine (3.16), (3.17) with (3.18), (3.19), respectively). □

Proof of Theorem 1.3. Let $\{U(t)\}$ be the same as in the proof of Theorem 1.1. Let $\{U_n(t)\}$ be the contraction semigroup on X generated by $-[(n^{-1}+i)S+B]$. Then (1.18) is nothing but (2.21). Setting $u_n(t) = U_n(t)u_0$, we see from (2.20) that

$$\|\Delta u_n(t)\|_{L^2}^2 + \|u_n(t)\|_{L^{2p}}^{2p} \leq p\,(2\|\Delta u_0\|_{L^2} + \|u_0\|_{L^{2p}}^p)^2,$$

This is the estimate corresponding to (3.15). By virtue of these estimates we can derive (1.19), (1.20) from (3.16), (3.17) and (1.18). □

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