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Analogy of the Fredholm Alternative for nonlinear operators

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Let us give the following simple motivation which arises in such a fundamental subject as the Sobolev imbedding theorems. It is well known that for $\Omega \subset \mathbb{R}^N$, a domain, the *continuous* imbedding

$$W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \tag{1}$$

holds provided $p \geq 1, q \geq 1$ and $N \geq 1$ satisfy certain relations and that under some additional restrictions this imbedding is *compact*

$$W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \tag{2}$$

(see e.g. Adams [A] or Kufner, John and Fučík [KJF]). Denoting by $\|\cdot\|_q$ and $\|\cdot\|_{1,p}$ the norm in $L^q(\Omega)$ and in $W_0^{1,p}(\Omega)$, respectively, the imbedding (1) expressed in terms of norms reads as follows: *there exists $C > 0$ independent of $u \in W_0^{1,p}(\Omega)$ such that*

$$\|u\|_q \leq C\|u\|_{1,p} \tag{3}$$

holds for any $u \in W_0^{1,p}(\Omega)$. Due to the Friedrichs inequality (see [A], [KJF]) the last assertion can be restated also as

$$\|u\|_q \leq C\|\nabla u\|_p \tag{4}$$

for any $u \in W_0^{1,p}(\Omega)$, where $C > 0$ does not depend on u . To make the notation clear we note that

$$\|\nabla u\|_p = \left(\int_\Omega |\nabla u(x)|^p \ dx\right)^{1/p}.$$

The following natural question arises when studying more carefully (4).

Question 1 *Does the best constant $C > 0$ exist in (4)?*

Let us remark that "the best constant" in (4) means the least number $C > 0$ for which the inequality (4) holds. In fact, such a constant can be characterized as

$$C = \sup \frac{\|u\|_q}{\|\nabla u\|_p}, \tag{5}$$

or equivalently as

$$\frac{1}{C} = \inf \frac{\|\nabla u\|_p}{\|u\|_q}, \tag{6}$$

where sup and inf are taken over all $u \in W_0^{1,p}(\Omega) \setminus \{0\}$. Due to the homogeneity of the fractions in (5), (6), the best constant $C_{p,q} > 0$ in (4) can be expressed as

$$C_{p,q}^{-1} = \inf\{\|\nabla u\|_p;\ \ u \in W_0^{1,p}(\Omega), \|u\|_q = 1\}. \tag{7}$$

Now, the answer to question 1 generates the following

Question 2 *Does $u \in W_0^{1,p}(\Omega)$ exist in which the infinum in (7) is achieved?*

If the imbedding (2) is compact then the standard minimizing argument provides the positive answer to question 2. So, let us assume (2) and denote by $u_{p,q} \in W_0^{1,p}(\Omega)$ the minimizer for (7). Then straightforward application of the Lagrange multiplier method yields that there exists real number $\lambda > 0$ such that

$$\int_\Omega |\nabla u_{p,q}(x)|^{p-2}\nabla u_{p,q}(x)\nabla\varphi(x)dx \ \ - \ \ \lambda\int_\Omega |u_{p,q}(x)|^{p-2}u_{p,q}(x)\varphi(x)dx = 0 \tag{8}$$

holds for any $\varphi \in W_0^{1,p}(\Omega)$. Substituting $\varphi = u_{p,q}$ in (8) one easily sees that

$$\lambda = C_{p,q}^{-p}. \tag{9}$$

Moreover, using the standard notation $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ for the p-Laplacian, the integral identity (8) means that $u_{p,q}$ is a nontrivial weak solution (i.e. eigenfunction) of the nonhomogeneous eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda|u|^{q-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{10}$$

and λ given by (9) is associated eigenvalue. Then the following question arises in a natural way.

Question 3 *What is the meaning of the spectrum of nonhomogeneous eigenvalue problem (10) and what are its fundamental properties?*

If we call nontrivial solutions of (10) the eigenfunctions and corresponding values of the spectral parameter λ associated eigenvalues of (10) then $u = u_{p,q}$ and $\lambda = C_{p,q}^{-p}$ are the principal eigenfunction and associated principal eigenvalue. Now we have to distinguish between two cases $p = q$ and $p \neq q$.

If $p = q$ (i.e. the problem (10) is homogeneous but nonlinear if $p \neq 2$) and $p > 1$ most properties of the principal eigenvalue and associated eigenfunction are the same regardless $p = 2$ or not (Anane [AN], Lindqvist [L]). The properties like the positivity, isolatedness and simplicity of the principle eigenvalue as well as the positivity of the principle eigenfunction are preserved. Also the second eigenvalue can be characterized variationally and associated eigenfunction splits Ω into two nodal subdomains, see Anane and Tsouli [AT]. Also a sequence of variational eigenvalues $\{\lambda_n\}$ of (10) satisfying a standard minimax characterization can be constructed but if $N > 1$ it is not known if this represents a complete list of the eigenvalues. For $N = 1$ completeness follows from the uniqueness theorem for associated initial value problem and was proved e.g. by Drábek [D1], Ôtani [O] and DelPino, Elgueta and Manásevich [DEM].

The case $p \neq q$ (i.e. the problem (10) is nonhomogeneous) is more complicated. First of all it follows from a simple renormalization argument that if $\lambda_0 > 0$ is an eigenvalue of (10) then any $\lambda > 0$ is also an eigenvalue of (10) and the corresponding eigenfunctions are real multiples of those associated with λ_0. Hence speaking about the eigenvalue of (10) we have always to add what is the normalizing condition for the corresponding eigenfunction. So we can restrict our attention for instance to the eigenfunctions sitting on the unit sphere $\|u\|_q = 1$. It was proved in García and Peral [GP1] that for $1 < p < N, 1 < q < p^*$, where $p^* := \frac{Np}{N-p}$, the problem (10) has a sequence of variational eigenvalues. However, completeness of the set of eigenvalues as well as its basic properities (even of the principal eigenvalue) are not clear at all. It was proved in Huang [H] that the principal eigenvalue of (10) is simple if $p < q$. The proof follows more or less the same lines as that for $p = q$ and does not work for $p > q$. In fact, an example of ring-shaped domain Ω is given in García and Peral [GP2], for which the principal eigenvalue of (10) is not simple if q is close enough to p^*. On the other hand the simplicity persists if besides the normalizing condition we look for the eigenfunctions with minimal energy

$$E(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{1}{q}\int_\Omega |u|^q dx$$

(see Drábek [D2]). For $N = 1$ we can benefit again from the global existence and uniqueness theorem for associated inital value problem and to get very transparent picture of the whole spectrum including an analytic expressions for the eigenvalues and associated eigenfunctions (see Drábek and Manásevich [DM]). This picture also suggests the idea how some bifurcation diagrams should look like also in PDE case.

Let us go back to the case $p = q, \quad p > 1$. In this case $C_p := C_{pq}$ satisfies

$$C_p^{-1} = \inf \ \{\|\nabla u\|_p; u \in W_0^{1,p}(\Omega), \|u\|_p = 1\} \tag{11}$$

and there exists unique positive in Ω function $u_1 \in W_0^{1,p}(\Omega), \|u_1\|_p = 1$, such that

$$C_p^{-1} = \|\nabla u_1\|_p.$$

We derive easily that $\lambda_1 = C_p^{-p}$ and u_p are the principle eigenvalue and associated eigenfunction of the homogeneous (for $p = 2$ linear) eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda |u|^p u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{12}$$

It is well known from the linear Fredholm alternative that the boundary value problem

$$\begin{cases} -\Delta u - \lambda_1 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{13}$$

has a weak solution *if and only if* $f \in W^{-1,p'}(\Omega)$ satisfies

$$\int_\Omega f u_1 dx = 0. \tag{14}$$

Moreover the solution set is an unbounded one dimensional linear set in $W_0^{1,p}(\Omega)$. Several questions arise if we consider a similar situation for general $p > 1$. Namely, consider the boundary value problem

$$(15) \qquad \begin{cases} -\Delta_p u - \lambda_1 |u|^{p-2}u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Question 4 *How does the condition (14) affect the solvability of (15)?*

Here the striking difference between the case $p = 2$ and $p \neq 2$ appears. The condition (14) is *not necessary* for the solvability of (15). A counterexample in ODE case ($N = 1$) is constructed in Binding, Drábek and Huang [BDH] and DelPino, Drábek and Manásevich [DDM]. In the latter paper it is shown even more than that:
if $\Omega = (0,1)$ *there is a function* $f_0 \in C^2[0,1]$ *and* $\rho > 0$ *such that for any* $f \in L^\infty(0,1)$, $\|f - f_0\|_\infty < \rho$ *we have* $\int_0^1 f u_1 dx \neq 0$ *and the boundary value problem*

$$(16) \qquad \begin{cases} -(|u'|^{p-2}u')' - \lambda_1 |u|^{p-2}u = f & \text{in (0,1)}, \\ u(0) = u(1) = 0 \end{cases}$$

has at least two solutions.
In particular, this result and the homogeneity of the left hand side of (16) imply that the range of the operator A: $W_0^{1,p}(0,1) \to W^{-1,p'}(0,1)$,

$$A\colon u \mapsto -(|u'|^{p-2}u')' - \lambda_1 |u|^{p-2}u$$

contains a cone with nonempty interior if $p \neq 2$. Similar result but if λ_1 in (16) is substituted by a certain higher eigenvalue is proved in Drábek and Takáč [DT]. On the other hand it is well known that the range of A for $p = 2$ is a linear subspace of $W^{-1,p'}(0,1)$ of codimension 1 (and hence it has an empty interior).
It should be pointed out here that the range of A *is not the whole* $W^{-1,p'}(0,1)$ for $p \neq 2$. For example, taking $f \equiv 1$ one can show that (16) has no solution (see DelPino and Manásevich [DMA]).

Comming back to the meaning of the condition (14) another interesting phenomenon occurs. Namely, this condition appears *to be sufficient* in a certain sense. More precisely, it is proved in [DDM] that *given* $f \in C^1[0,1]$ *satisfying* $\int_0^1 f u_1 dx = 0$ *the boundary value problem* (16) *has at least one solution.*
The following question then appears in a natural way.

Question 5 *What is the solution set of (16) in that case (i.e. if* $f \in C^1[0,1]$ *satisfies (14))?*

Also here the case $p \neq 2$ is very different. It is proved in [DDM] that the set of all solutions of (16) is bounded in C^1 norm.

The picture of nonlinear Fredholm alternative can be completed by considering solvability of

$$(17) \qquad \begin{cases} -(|u'|^{p-2}u')' - \lambda |u|^{p-2}u = f & \text{in (0,1)}, \\ u(0) = u(1) = 0 \end{cases}$$

when λ is *not an eigenvalue.* It is well known that for $p = 2$ the boundary value problem (17) has *unique solution* for any $f \in W^{-1,p'}(0,1)$. It follows from the Leray-Schauder degree theory that for $p \neq 2$ the problem (17) has *at least one solution* for any $f \in W^{-1,p'}(1,0)$. Uniqueness, however, holds only for $\lambda \leq 0$ due to the monotonicity of the operator $u \mapsto -(|u'|^{p-2}u)' - \lambda|u|^{p-2}u$. If $\lambda > 0$ and $p \neq 2$ one can find f such that the problem (17) has *at least two* distinct solutions as shown in Fleckinger, Hernández, Takáč and deThélin [FHTT] and Drábek and Takáč [DT].

Let us consider now the *energy functional* $E_{f,\lambda}\colon W^{1,p}(\Omega) \to \mathbb{R}$ associated with the boundary value problem

$$\begin{cases} -\Delta_p u - \lambda|u|^{p-2}u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{18}$$

Obviously,

$$E_{f,\lambda}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{p}\int_\Omega |u|^p dx - \int_\Omega fu dx,$$

and the critical points of $E_{f,\lambda}$ are in one-to-one correspondence with the weak solutions of (18). The functional $E_{f,\lambda}$ has a global minimum (and in fact it is coercive) if $\lambda < \lambda_1$ due to the variational characterization (11) while $E_{f,\lambda}$ has a saddle point geometry if $\lambda > \lambda_1, \lambda$ not an eigenvalue. For $p = 2, E_{f,\lambda}$ has always unique critical point in above mentioned cases, for $p \neq 2, E_{f,\lambda}$ has unique critical point only if $\lambda \leq 0$. A counterexample showing that there exists f for which $E_{f,\lambda}$ has at least two distinct critical points is given in [DEM] (for $\lambda \in (0, \lambda_1)$ and $p > 2$), [FHTT] (for $\lambda \in (0, \lambda_1)$ and $p \in (1,2)$) and [DT] (for $\lambda > 0$ and $p > 1, p \neq 2$).

Let us consider $\lambda = \lambda_1$ and study the energy functional $E_{f,\lambda}$.

Observation 1 *If $f \in W^{-1,p'}(\Omega), \int_\Omega fu_1 dx \neq 0$ then E_{f,λ_1} is unbounded from below (in the direction of u_1).*

Observation 2 *If $p = 2$ and $f \in W^{-1,2}(\Omega), \int_\Omega fu_1 dx = 0$ then E_{f,λ_1} is bounded from below.*

Again the following question arizes in a natural way.

Question 6 *Let $p \in (1,2) \cup (2,\infty), f \in W^{-1,p'}(\Omega), \int_\Omega fu_1 dx = 0$. Is E_{f,λ_1} bounded from below?*

The answer is known in ODE case ($N = 1, \Omega = (0,1)$) and it is quite interesting. The following assertions are proved in [DDM].

Let $p \in (1,2), f \in C^1[0,1], \int_0^1 fu_1 dx = 0$. Then E_{f,λ_1} is unbounded from below.

Let $p \in (2,\infty), f \in C^1[0,1], \int_0^1 fu_1 dx = 0$. Then E_{f,λ_1} is bounded from below.

Observation 3 *Let $p = 2, f \in W^{-1,2}(\Omega), \int_\Omega fu_1 dx \neq 0$. Then E_{f,λ_1} has no critical point.*

Question 7 *Let $p \in (1,2) \cup (2,\infty)$ and $\int_\Omega fu_1 dx \neq 0$. Does E_{f,λ_1} have any critical point?*

In above mentioned papers [BDH] and [DDM] examples are given showing that the answer is positive for certain $f \in W^{-1,p'}(0,1)$. On the other hand there are $f's$ for which E_{f,λ_1} has no critical point (cf. [DM]).

Observation 4 *Let $p = 2, f \in W^{-1,2}(\Omega), \int\limits_{\Omega} f u_1 dx = 0$. Then E_{f,λ_1} has an unbounded continuum (linear set of dimension one) of critical points.*

Question 8 *Let $p \in (1,2) \cup (2,\infty), f \in W^{-1,p'}(\Omega), \int\limits_{\Omega} f u_1 dx = 0$. What is the structure of the set of all critical points of E_{f,λ_1}?*

The answer is known in ODE case ($N = 1, \Omega = (0,1)$) and it is proved in [DDM] that for $p \in (1,2) \cup (2,\infty), f \in C^1[0,1], \int\limits_0^1 f u_1 dx = 0$ the set of all critical points of E_{f,λ_1} is nonempty and bounded in C^1 norm.

Let us conclude the introduction by mentioning the relation between above mentioned results and the sensitivity of *optimal Poincaré inequality* under a *linear perturbations.* It follows from (7) and the simplicity of the first eigenvalue of

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda |u|^{p-2}u & \text{in } (0,1), \\ u(0) = u(1) = 0 \end{cases}$$

that the (Poincaré) inequality

$$C_p^p \int_0^1 |u'|^p dx - \int_0^1 |u|^p dx \geq 0 \tag{19}$$

minimizes (and equals zero) just in the one dimensional linear subspace of $W_0^{1,p}(0,1)$ spanned by u_1. Let us add perturbation term $-\int\limits_0^1 f u dx$ to the left hand side of (19) and consider $f \in W^{-1,p'}(0,1), \int\limits_0^1 f u_1 dx = 0$.

Case $p = 2$. We have

$$C_2^2 \int_0^1 |u'|^2 dx - \int_0^1 |u|^2 dx - \int_0^1 f u dx \geq C_f > -\infty \tag{20}$$

and the left hand side of (20) minimizes and equals C_f just on the linear set of all solutions of the boundary value problem

$$\begin{cases} -C_2^2 u'' - u = f & \text{in } (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

Case $p > 2$. For $f \in C^1[0,1]$ we have again

$$C_p^p \int_0^1 |u'|^p dx - \int_0^1 |u|^p dx - \int_0^1 f u dx \geq C_f > -\infty \tag{21}$$

but the left hand side of (21) minimizes and equals C_f on the bounded set of all solutions of the boundary value problem

$$\begin{cases} -C_p^p(|u'|^{p-2}u')' - |u|^{p-2}u = f & \text{in } (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

Case $1 < p < 2$. For $f \in C^1[0,1]$ we can always find a sequence $\{u_n\} \in W_0^{1,p}(0,1)$ such that

$$C_p^p \int_0^1 |u_n'|^p dx - \int_0^1 |u_n|^p dx - \int_0^1 f u_n dx \searrow -\infty$$

as $n \to \infty$.

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References

[A] R.A. Adams, *Sobolev Spaces,* Academic Press, Inc., New York 1975.

[AN] A. Anane, *Simplicité et isolation de la premiére valeur propre du p–Laplacien avec poids*, C.R. Acad. Sci. Paris, Sér. I. Math. 305 (1987), 725–728.

[AT] A. Anane and N. Tsouli, *On the second eigenvalue of the p-Laplacian*, in: Nonlinear Partial Differential Equations (From a Conference in Fes, Maroc, 1994), A. Benkirane and J.-P. Grossez Ed., Pitman Research Notes in Math. 343, Longman 1996.

[BDH] P. Binding, P. Drábek and Y.X. Huang, *On the Fredholm alternative for the p-Laplacian,* Proc. Amer. Math. Soc. 125 (12) (1997), 3555–3559.

[DDM] M. DelPino, P. Drábek and Manásevich, *The Fredholm alternative at the first eigenvalue for the one dimensional p-Laplacian,* to appear in J. Differential Equations.

[DEM] M.A. DelPino, M. Elgneta and R. Manásevich, *A homotopic deformation along p of a Leray–Schauder degree result and existence for* $(|u'|^{p-2}u')' + f(t,u) = 0, u(0) = u(T) = 0, p > 1$, J. Differential Equations 80 (1) (1989), 1–13.

[DMA] M. DelPino and R. Manásevich, *Multiple solutions for the p-Laplacian under global nonresonance,* Proc. Amer. Math. Soc. 112 (1) (1991), 131–138.

[D1] P. Drábek, *Ranges of a-homogeneous operators and their perturbations,* Časopis pro Pěstování Matematiky 105 (1980), 167–183.

[D2] P. Drábek, *On the uniqueness for a superlinear eigenvalue problem,* to appear in Appl. Math. Letters.

[DM] P. Drábek and R. Manásevich, *On the closed solution to some nonhomogeneous eigenvalue problems with p-Laplacian,* preprint.

[DT] P. Drábek, P. Takáč, *A counterexample to the Fredholm alternative for the p-Laplacian,* to appear in Proc. Amer. Math. Society.

[FHTT] J. Fleckinger, J. Hernández, P. Takáč and F. deThélin, *Uniqueness and positivity for solutions of equations with the p-Laplacian,* Proceedings of the Conference on Reaction - Diffusion Equations, Trieste, Italy, October 1985. Marcel Dekker, Inc., New York - Basel, 1997.

[GP1] J. García and I. Peral, *Existence and non-uniqueness for the p-Laplacian: Nonlinear eigenvalues,* Comm. Part. Diff. Equations 12 (1987), 1389–1430.

[GP2] J. García and I. Peral, *On limits of solutions of elliptic problems with nearly critical exponent,* Commun. Part. Differential Equations, 17 (11 & 12), (1992), 2113–2126.

[H] Y.X. Huang, *A note on the asymptotic behavior of positive solutions for some elliptic equation,* Nonlinear Analysis, T.M.A. 29 (5) (1997), 533–537.

[KJF] A. Kufner, O. John and S. Fučík, *Function Spaces,* Academia, Prague 1977.

[L] P. Lindqvist, *On the equation div* $(|\nabla u|^{p-2}\nabla u) + \lambda |u|^{p-2}u = 0$, Proc. Amer. Math. Soc. 109 (1990), 157–164.

[O] M.Ôtani, *A remark on certain nonlinear elliptic equations,* Proc. Fac. Tokai Univ., Vol. XIX (1984), 23–28.