数理解析研究所講究録
1105 巻 1999 年 81-90

Upper and lower bounds for the Hausdorff dimension of the attractor for reaction-diffusion equations in $\mathbb{R}^n$

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Abstract

We consider a reaction-diffusion equation (RDE) of the form

$$\begin{cases} \partial_t u = \nu\Delta u - f(u) - \lambda_0 u - g \\ u|_{t=0} = u_0(x), \quad x \in \mathbb{R}^n \end{cases}$$

with $u = u(t,x) \in \mathbb{R}$ scalar and $x \in \mathbb{R}^n$. Here $g = g(x), u_0(x), \lambda_0 > 0$ and the nonlinearity $f \in C^1(\mathbb{R} \times \mathbb{R}^n)$ are supposed to be given. Under appropriate conditions on f, g and u_0 we prove both existence of a global attractor for equation (1) in the whole space $\mathbb{R}^n$ and lower and upper bounds (in terms of physical parameter ν) for the Hausdorff dimension of the attractor. Note that the case of $\mathbb{R}^n$ has specific difficulties. First, the semigroup generated by the above equation is not compact. The second difficulty is connected with the fact that the Laplace operator has continuous spectrum. We overcome these difficulties by systematic use of weighted Sobolev spaces.

Introduction and Results

One of the most important objects used to describe large-time dynamics of infinite-dimensional dynamical systems are global attractors and their dimensions. The global attractor of an evolution equation is the maximal, "compact", finite dimensional (in the sense of Hausdorff), invariant set in the phase space which attracts bounded sets. Upper and lower bounds for the Hausdorff dimension of the attractor (in terms of some physical parameters) imply that even infinite-dimensional dynamical systems possess an asymptotic behavior determined by a finite number of degrees of freedom. For a precise definition of the attractor and concepts of dimension see [2]. According to a conjecture of Landau and Ruelle-Takens, the non-trivial dynamics on the attractors of the Navier-Stokes system determines turbulent behavior of fluids. Hence both existence of attractors and upper and lower bounds (in terms of physical parameters) on their dimensions, are of great interest. Note that at present the existence of global attractors as well as estimates on their Hausdorff dimensions were obtained for many equations of mathematical physics in bounded domains, see [2]. In this paper, we consider a reaction-diffusion equation (RDE) of the form

$$(1)\qquad \begin{cases} \partial_t u = \nu\Delta u - f(u) - \lambda_0 u - g \\ u|_{t=0} = u_0(x), \quad x \in \mathbb{R}^n \end{cases}$$

where $u = u(t,x) \in \mathbb{R}$ is scalar and $x \in \mathbb{R}^n$. Here $g = g(x)$, $u_0(x)$, $\lambda_0 > 0$ and the nonlinearity $f \in C^1(\mathbb{R} \times \mathbb{R}^n)$ are supposed to be given. We specify conditions on f, u_0, and g later. Our main goal is both, to prove existence of global attractors for RDE (1), and to obtain lower and upper bounds (in terms of the Reynolds number ν^{-1}) for the Hausdorff dimensions of the attractors. For a proof of existence of the attractor in this case we follow [3]. There the following restrictions are imposed on the nonlinearity $f = f(u)$.

Condition 1.

1. $f \in C^1$ and $f'(u) \geq -C$

2. $f(u) \cdot u \geq 0$ for all $u \in \mathbb{R}$

3. $|f(u)| \leq |u|^{1+\alpha}(1 + |u|^{p_2})$,

where $0 \leq \alpha,\ 0 \leq p_2$ and $p_2 + \alpha \leq p_0 = \min\{\frac{4}{n}, \frac{2}{n-2}\}$, if $n \geq 3$. For $n \leq 2$ we can take $p_0 = \frac{4}{n}$.

Let $H_{l,\gamma}(\mathbb{R}^n)$ be the weighted Sobolev spaces with norms

$$\|u\|_{l,\gamma}^2 = \sum_{|\alpha| \leq l} \|\partial^\alpha u\|_{0,\gamma}^2,$$

where

$$\|u\|_{0,\gamma}^2 = \int_{\mathbb{R}^n} (1 + |\epsilon x|^2)^\gamma \, |u(x)|^2 dx$$

and $\epsilon > 0$ is a small enough, but fixed number.

Theorem 1 ([3]) *Let $\gamma > 0$, $g \in H_{0,\gamma}$, $u_0 \in H_{1,\gamma}$ and let the nonlinearity $f = f(u)$ satisfy Condition 1. Then there exists a unique solution $u(t,x)$ of RDE (1), which belongs to*

$$L_2([0,T], H_{2,\gamma}) \cap L_\infty([0,T], H_{1,\gamma}).$$

Moreover the mappings $S_t : u_0(x) \mapsto u(t,x)$ form a semigroup which possesses a global attractor $\mathcal{A} \subset H_{0,\gamma}$.

Let us consider the following special cases of Theorem 1.

Proposition 1 *The global attractor $\mathcal{A}$ consists of only one point in the following cases*

1. $g = 0$ and f satisfies Condition 1.

2. $g \in H_{0,\gamma}(\mathbb{R}^n)$ and $f'(u) \geq -\lambda_0$.

Proof: First consider the case 1), that is,

$$\left\{\begin{array}{l} \partial_t u = \nu\Delta u - f(u) - \lambda_0 u \\ \\ u|_{t=0} = u_0(x). \end{array}\right. \tag{2}$$

The condition $f(u) \cdot u \geq 0$ implies that $u \equiv 0$ is an equilibrium. On the other hand, multiplying both sides of (2) by $\varphi \cdot u$, where $\varphi(x) = (1 + |\epsilon x|^2)^\gamma$ and integrating with respect to x, we obtain that any solution of (2) tends to $u \equiv 0$ as $t \to \infty$ in $H_{0,\gamma}$. Hence $\mathcal{A} = \{0\}$. Let us next consider case 2). We start with the special case of 2), where $f \equiv 0$ and $g \in H_{0,\gamma}$. Then (1) takes the form

$$\left\{\begin{array}{l} \partial_t u = \nu\Delta u - \lambda_0 u - g(x) \\ \\ u|_{t=0} = u_0(x). \end{array}\right.$$

Let $u_*(x)$ be the unique solution of

$$\nu\Delta u_* - \lambda_0 u_* = g \tag{3}$$

As $g \in H_{0,\gamma}$, we have $u_* \in H_{2,\gamma}(\mathbb{R}^n)$. On the other hand, equation (3) can be rewritten as

$$\frac{\partial u_*}{\partial t} = \nu\Delta u_* - \lambda_0 u_* - g$$

and as a result we obtain

$$\frac{\partial(u - u_*)}{\partial t} = \nu\Delta(u - u_*) - \lambda_0(u - u_*),$$

which in turn implies $\mathcal{A} = \{u_*\} = \{(\nu\Delta - \lambda_0 I)^{-1} g\}$. We next proof that in the general case 2) the attractor also consists of only one single point.

Let S be the set of all equilibria of (1). We prove that S consists of a single point. Indeed, let u_1 and u_2 be two distinct equilibria, that is,

$$\nu\Delta u_1 - f(u_1) - \lambda_0 u_1 - g(x) = 0 \tag{4}$$

$$\nu\Delta u_2 - f(u_2) - \lambda_0 u_2 - g(x) = 0. \tag{5}$$

Subtracting (5) from (4) we obtain

$$\nu\Delta(u_1 - u_2) - (f(u_1) - f(u_2)) - \lambda_0(u_1 - u_2) = 0. \tag{6}$$

Multiplying both sides of (6) by $(u_1 - u_2)$ and integrating with respect to x, we obtain, using $u_i \in H_{0,\gamma}, \gamma > 0$, that

$$\nu \int |\nabla(u_1 - u_2)|^2 dx + \int (\Psi(u_1) - \Psi(u_2))(u_1 - u_2)\, dx \geq 0 \tag{7}$$

where $\Psi(u) := f(u) - \lambda_0 u$. Note that $\Psi'(u) \geq 0$. Therefore the integrand in (7) is positive and $u_1 \equiv u_2$. Then the assertion of Proposition 1, Case 2 is an easy consequence of the gradient structure of (1). $\bowtie$

Proposition 1 leads to the following natural question: How rich is the global attractor $\mathcal{A}$ for reaction-diffusion equation (1). Our main result gives a partial answer to this question. In order to formulate it, we introduce a class of nonlinearities $\mathcal{M}$: we say $f \in \mathcal{M}$, iff there exists $\xi \in \mathbb{R}$ such that $f'(\xi) < -\lambda_0$.

Proposition 1 showed that

Corollary 1 *If f satisfies Condition 1, but $f \notin M$, then the attractor $\mathcal{A}$ consists of a singleton. In particular,* $\dim \mathcal{A} = 0$.

Next we formulate our main result.

Theorem 2 *Let* $u_0 \in H_{1,\gamma}(\mathrm{IR}^n), \gamma > 0$ *and* $f \in \mathcal{M}$ *satisfy Condition 1. Then there exists* $L_0 > 0$ *such that for all* $L \geq L_0$ *there is (an explicitly given)* $g_L(x) \in H_{0,\gamma}$, *and the global attractor* $\mathcal{A}_L$ *of RDE (1) with* $g = g_L(x)$ *admits the following double-sided estimates*

$$C_1 \nu^{-n/2} L^n \leq \dim \mathcal{A}_L \leq C_2 \nu^{-n/2} L^n.$$

Here the constants C_1 *and* C_2 *depend on* λ_0 *but not on* ν *and* L.

Proof: We start with $n = 1$. Let L be any given positive number and $z(x) \in C_0^\infty(\mathrm{IR})$ such that $z(x) = \xi$ if $-L \leq x \leq L$ and $z(x) = 0$ if $|x| \geq L + 1$. Here $\xi \in \mathrm{IR}$ is chosen such that $f'(\xi) < -\lambda_0$. Define

$$g_L(x) := \nu \Delta z(x) - f(z(x)) - \lambda_0 z(x),$$

Note that $g_L(x) \in H_{0,\gamma}(\mathrm{IR}^n), \gamma > 0$. Consider now

$$\text{(8)} \qquad \begin{cases} \partial_t u = \nu \Delta u - f(u) - \lambda_0 u - g_L(x) \\ u|_{t=0} = u_0(x) \end{cases}$$

Obviously $u_*(x) := z(x)$ is an equilibrium for (8). Let $M_+(z)$ be the unstable manifold at this equilibrium (see [2]). Note that the unstable manifold exists, and is finite dimensional, since the essential spectrum of the linearization, determined by $\nu\Delta - f'(z(x)) - \lambda_0$, is strictly left of the imaginary axis. Since $M_+(z) \subset \mathcal{A}$, a lower bound for $\dim M_+(z)$ yields a lower bound for the dimension of $\mathcal{A}$ as well. Therefore we have to estimate $\dim M_+(z)$. To this end, we have to study the linearized equation at $z(x)$, that is, the linearized operator,

$$A'(z(x))w = \nu\Delta w - f'(z(x))w - \lambda_0 w.$$

Since $z \in C_0^\infty(\mathrm{IR})$ it is not difficult to see that

1. $\langle A'(z(x))w, h\rangle \ = \ \langle w, A'(z(x))h\rangle$ for all $w, h \in H_{2,\gamma}$

2. $\langle A'(z(x))w, w\rangle \ \leq \ \beta\langle w, w\rangle$, for some $\beta > 0$ and for all $w \in H_{2,\gamma}$.

Here by $\langle\cdot,\cdot\rangle$ we denote the scalar product in $H_{0,\gamma}(\mathrm{IR}^n)$. Let R_k be a k-dimensional subspace of $H_{1,\gamma}(\mathrm{IR}^n)$, defined as

$$R_k := \mathrm{span}[w_1(x), \cdots, w_k(x)]$$

where $w_j(x)$ are eigenvalues of $-\Delta$, with Dirichlet boundary conditions, that is

$$\begin{cases} -\Delta w_j = \lambda_j w_j \\ w_j(-L) = w_j(L) = 0 \end{cases} \tag{9}$$

with $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$, and k is chosen such that $\lambda_k < -f'(\xi) - \lambda_0$. An explicit computation shows $\lambda_j = \left(\frac{\pi j}{L}\right)^2$ and $w_j(x) = \sin\frac{\pi j}{L}x, |x| \leq L$. We continue $w_j(x)$ on the whole real line by $w_j(x) \equiv 0$ in $|x| \geq L$. Thus, in order to fulfill $\lambda_k < -f'(\xi) - \lambda_0$, we can choose

$$k = [L(-f'(\xi) - \lambda_0)^{1/2} \cdot \pi^{-1}], \tag{10}$$

where $[x]$ denotes the integer part of x. We next show that

$$\langle A'(z(x))w, w\rangle|_{w\in R_k} > 0.$$

Since $(A'(z(x))w_j, w_k) = 0$ if $j \neq k$, it is sufficient to check (10) at $w = w_j, j = 1, \cdots k$. Indeed,

$$\begin{aligned}
&\langle A'(z(x))w_j, w_j\rangle = -\int_{-L}^{L}\left((w_j'(x))^2 + (f'(\xi)+\lambda_0)w_j^2(x)\right)dx = \\
&= -\int_{-L}^{L}\left(\frac{\pi^2 j^2}{L^2}w_j^2 + (f'(\xi)+\lambda_0)w_j^2\right)dx = \\
&= -\left(\frac{\pi^2 j^2}{L^2} + f'(\xi) + \lambda_0\right)\int_{-L}^{L} w_j^2(x)dx > 0
\end{aligned}$$

for all $j = 1, \cdots, k$ according to (10). Due to Courant's Minimax Principle, we have

$$\dim M_+(z) \geq k = [L(-f'(\xi) - \lambda_0)^{1/2} \cdot \pi^{-1}].$$

Hence in case of $n = 1$ we obtain

$$\dim \mathcal{A} \geq LC_1(\lambda_0)$$

for sufficiently large L. Replacing $-\Delta$ by $-\nu\Delta$ in (9) yields

$$\dim \mathcal{A} \geq L\nu^{-1/2}C_1(\lambda_0). \tag{11}$$

It remains to show that

$$\dim \mathcal{A} \leq C_2(\lambda_0)L \cdot \nu^{-1/2} \tag{12}$$

for sufficiently large L. A proof of (12) is based on the estimate (see [3])

$$\dim \mathcal{A} \leq C(\lambda_0)\nu^{-1/2}\|g\|^2_{L_2(\mathbb{R})}$$

where $g(x)$ is the forcing function in (1). If we set $g = g_L(x)$, then $\|g\|^2_{L_2(\mathbb{R})} = O(L)$ and

$$\dim \mathcal{A} \leq C_2(\lambda_0)\nu^{-1/2} \cdot L \tag{13}$$

for sufficently large L. This proves Theorem 2 in the case $n = 1$.

In the case $n > 1$, the proof of Theorem 2 is based on the estimate of eigenvalues of $-\Delta$ in a ball. Consider, instead of (9), the Dirchlet problem in a ball $B_L = \{x \in \mathbb{R}^n | \|x\| \leq L\}$

$$\begin{cases} -\Delta w_j = \lambda_j w_j \\ w_j|_{\partial B_L} = 0 \end{cases}$$

It is well-known that $\lambda_j \sim j^{\frac{2}{n}}$ for $j \to \infty$, so that the estimates (12) and (13) take the form

$$C_1(\lambda_0)\nu^{-n/2}L^n \leq \dim \mathcal{A} \leq C_2(\lambda_0)\nu^{-n/2}L^n$$

for sufficiently large L.

Corollary 2 *The dimension of the attractor can be made arbitrarily large.*

Acknowledgement. We would like to express our gratitude to A. Babin and B. Fiedler for inspiring discussions and helpful remarks. We are also grateful to J. F. Toland and S. Zelick for useful comments during their visit at the Free University of Berlin.

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