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Some results on commutative semigroups and semigroup rings

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Let G be a torsion-free abelian (additive) group, and let S be a subsemigroup of G which contains 0. Then S is called a grading monoid ([No]). We will call a grading monoid simply a g-monoid.

For example, the direct sum $\mathbf{Z}_0 \oplus \cdots \oplus \mathbf{Z}_0$ of n-copies of the non-negative integers $\mathbf{Z}_0$ is a g-monoid.

Many terms in commutative ring theory may be defined analogously for S.

For example, a non-empty subset I of S is called an ideal of S if $S + I \subset I$.

Let I be an ideal of S with $I \subsetneqq S$. If $s_1 + s_2 \in I$ (for $s_1, s_2 \in S$) implies $s_1 \in I$ or $s_2 \in I$, then I is called a prime ideal of S.

Let Γ be a totally ordered abelian (additive) group. A mapping v of a torsion-free abelian group G onto Γ is called a valuation on G if $v(x+y) = v(x) + v(y)$ for all $x, y \in G$. The subsemigroup $\{x \in G \mid v(x) \geq 0\}$ of G is called the valuation semigroup of G associated to v.

The maximum number n so that there exists a chain $P_1 \subsetneqq P_2 \subsetneqq \cdots \subsetneqq P_n$ of prime ideals of S is called the dimension of S.

If every ideal I of S is finitely generated, that is, $I = \cup_i (S + s_i)$ for a finite number of elements $s_1, \cdots, s_n$ of S, then S is called a Noetherian semigroup.

Many propositions for commutative rings are known to hold for S.

For example, if S is a Noetherian semigroup, then every finitely generated extension g-monoid $S[x_1, \cdots, x_n] = S + \sum_i \mathbf{Z}_0 x_i$ is also Noetherin [M3, Proposition 3], and the integral closure of S is a Krull semigroup [M4].

Ideal theory of S is interesting itself and important for semigroup rings.

Let R be a commutative ring, and let S be a g-monoid. There arises the semigroup ring $R[S]$ of S over R: $R[S] = R[X;S] = \{\sum_{finite} a_s X^s \mid a_s \in R, s \in S\}$.

If S is the direct sum $\mathbf{Z}_0 \oplus \cdots \oplus \mathbf{Z}_0$ of n-copies of $\mathbf{Z}_0$, then $R[S]$ is isomorphic to the polynomial ring $R[X_1, \cdots, X_n]$ of n-variables over R.

Assume that the semigroup ring $D[S]$ over a domain D is a Krull domain. Then D.F. Anderson [A] and Chouinard [C] showed that $C(D[S]) \cong C(D) \oplus C(S)$, where C denotes ideal class group. Thus they were able to construct Krull domains that have various ideal class groups.

For another example, assume that D is integrally closed and S is integrally closed. Then we have $(I_1 \cap \cdots \cap I_n)^v = I_1^v \cap \cdots \cap I_n^v$ for every finite number of finitely generated ideals $I_1, \cdots, I_n$ of $D[S]$ if and only if $(I_1 \cap \cdots \cap I_n)^v = I_1^v \cap \cdots \cap I_n^v$ for every finite number of finitely generated ideals $I_1, \cdots, I_n$ of D and $(I_1 \cap \cdots \cap I_n)^v = I_1^v \cap \cdots \cap I_n^v$ for every finite number of finitely generated ideals $I_1, \cdots, I_n$ of S ([M1]), where v is the v-operation.

1

Let D be a Noetherian integral domain with the integral closure $\overline{D}$, and K the quotient field of D.

The Krull-Akizuki theorem states that, if $\dim(D) = 1$, then any ring between D and K is Noetherian and its dimension is at most 1.

The Mori-Nagata theorem states that $\overline{D}$ is a Krull ring for any Noetherian domain D.

Moreover, Nagata proved that, if D is of dimension 2, then $\overline{D}$ is Noetherian (cf. [Na]).

In [M2] we proved the Krull-Akizuki theorem for semigroups.

In [M4] we proved the Mori-Nagata theorem for semigroups.

Let T be an extension g-monoid of S. An element t of T is called integral over S if $nt \in S$ for some positive integer n. The set of integral elements of T is called the integral closure of S in T. The integral closure $\overline{S}$ in the quotient group $q(S) = \{s - s' \mid s, s' \in S\}$ is called the integral closure of S, and is denoted by $\overline{S}$. If $\overline{S} = S$, then S is called integrally closed.

In 1, we proved the following Theorem and answered to the following question in the negative.

Theorem. Let S be a 2-dimensional Noetherian semigroup. Then the integral closure $\overline{S}$ of S is a Noetherian semigroup.

Let P be a prime ideal of S. Then the maximum number n so that there exists a chain $P_1 \subsetneqq P_2 \subsetneqq \cdots \subsetneqq P_n = P$ of prime ideals of S is called the height of P, and is denoted by $ht(P)$.

Question. If P is a prime ideal of height r in a Noetherian semigroup S, then is P a prime ideal minimal among containing an r-generated ideal of S?

This is "yes" for rings.

Now, to answer to the Question, let $x_1 + x_2 = x_3 + x_4$ be a unique relation of letters x_1, x_2, x_3 and x_4. Set $S = \mathbf{Z}_0x_1 + \mathbf{Z}_0x_2 + \mathbf{Z}_0x_3 + \mathbf{Z}_0x_4$. Then S is a g-monoid. $M = (x_1, x_2, x_3, x_4) = \cup_i(S + x_i)$ is a unique maximal ideal of S. Then S is a Noetherian semigroup of dimension 3. M is not a prime ideal minimal among containng a 3-generated ideal of S.

2

Larsen-McCarthy's Multiplicative Theory of Ideals [LM] is one of the basic references of multiplicative ideal theory for commutative rings. In 2, we proved or disproved all the Theorems in [LM] for semigroups. We will state two Theorems.

Let M be a non-empty set. Assume that, for every $s \in S$ and $a \in M$, there is defined $s + a \in M$ such that, for every $s_1, s_2 \in S$ and $a \in M$, we have $(s_1 + s_2) + a = s_1 + (s_2 + a)$ and $0 + a = a$. Then M is called an S-module.

Theorem. Let S be a Noetherian semigroup, M a finitely generated S-module, L and N submodules of M, and I an ideal of S. Then there exists a positive integer r such that for every $n > r$, we have

$$(nI + L) \cap N = (n - r)I + ((rI + L) \cap N).$$

This is a semigroup version of the Artin-Rees Lemma for rings.

Let M be an S-module. If $s_1 + a = s_2 + a$ (for $s_1, s_2 \in S$ and $a \in M$) implies $s_1 = s_2$, then M is called cancellative.

Theorem implies that if M is a finitely generated cancellative module over a Noetherian semigroup S, then $\cap_{n=1}^{\infty}(nI + M) = \emptyset$ for every proper ideal I of S.

An element s of a g-monoid S is called unit if $-s \in S$. Let s be a non-unit of S. If $s = s_1 + s_2$ implies that s_1 or s_2 is a unit, then s is called irreducible. If every element of S is expressed as a sum of irreducible elements uniquely (up to units and permutation), then S is called factorial (or a UFS).

If there exists a family $\{V_\lambda \mid \lambda\}$ of $\mathbf{Z}$-valued valuation semigroups on $q(S)$ so that $S = \cap_\lambda V_\lambda$ and each element of S is a unit for almost all λ, then S is called a Krull semigroup.

An S-submodule I of $q(S)$ is called a fractional ideal of S, if $s + I \subset S$ for some $s \in S$. Let $F(S)$ be the set of fractional ideals of S. For every fractional ideal I of S, we set $div(I) = \{J \in F(S) \mid J^v = I^v\}$, and set $D(S) = \{div(I) \mid I \in F(S)\}$, and $C(S) = D(R)/\{div(x) \mid x \in q(S)\}$, where I^v is the intersection of principal fractional ideals of S containing I. If $I^v = I$, then I is called divisorial.

Theorem. If S is a g-monoid, then the following conditions are equivalent:

(1) S is a factorial semigroup.

(2) S is a Krull semigroup and $C(S) = 0$.

(3) S is a Krull semigroup and every prime divisorial ideal of S is principal.

3

Kaplansky's Commutative Rings [Kap] is one of the basic rerences of commutative ring theory. We know that all the Theorems in Chapters 1 and 2 of [Kap] hold for S [TM].

In 3, we showed that all the Theorems in Chapter 3 of [Kap] hold for g-monoids. We will state some Theorems.

Let A be an S-module and $s \in S$. If $s + a_1 = s + a_2$ (for $a_1, a_2 \in A$) implies $a_1 = a_2$, then s is called a non-zerodivisor on A. If s is not a non-zerodivisor, then s is called a zerodivisor on A. The set of zerodivisors on A is denoted by $Z(A)$. Let B be a submodule of an S-module A, and $s \in R$. If $s + a \in B$ (for $a \in A$) implies $a \in B$, then s is called a non-zerodivisor on A modulo B (or a non-zerodivisor on A/B). If s is not a non-zerodivisor on A/B, then s is called a zerodivisor. The set of zerodivisors on A/B is denoted by $Z(A/B)$.

The ordered sequence of elements $x_1, \cdots, x_n$ of S is called a regular sequence on A, if $(x_1, \cdots, x_n) + A \subsetneqq A$ and if $x_1 \notin Z(A)$, $x_2 \notin Z(A/((x_1) + A))$, $\cdots$, $x_n \notin Z(A/((x_1, \cdots, x_{n-1}) + A))$.

Let A be an S-module. If $Z(A) = \emptyset$, then A is called torsion-free.

Let A be an S-module, and I an ideal of S. Let $x_1, \cdots, x_n$ be a regular sequence in I on A. If $x_1, \cdots, x_n, x$ is not a regular sequence on A for each $x \in I$, then $x_1, \cdots, x_n$ is called a maximal regular sequence in I on A.

Let A be an S-module, and I an ideal of S. Then the maximum of lengths of all regular sequences in I on A is called the grade of I on A, and is denoted by $G(I, A)$.

Let A be an S-module. If any two maximal regular sequences in I on A have the same length for every ideal I with $I + A \subsetneqq A$, then A is said to satisfy property (*). If A satisfies property (*), we say also that (S, A) satisfies property (*).

Theorem. Let S be a Noetherian semigroup, and A a finitely generated torsion-free cancellative S-module with property (*). Let $I = (x_1, \cdots, x_n)$ be a proper ideal of S. Then $G(I, A) = n$ if and only if $x_1, \cdots, x_n$ is a regular sequence on A.

Let S be a Noetherian semigroup with maximal ideal M. If $G(M, S) = dim(S)$, then R is called a Macaulay semigroup.

Theorem. Let S be a Macaulay semigroup such that (S, S) satisfies property (*). Then we have $G(I, S) = ht(I)$ for every ideal I of S.

Let S be a Noetherian semigroup with maximal ideal M. The cardinality of a minimal generators of M is called the V-dimension of S, and is denoted by $V(S)$.

A Noetherian semigroup S is called a regular semigroup if $V(S) = dim(S)$.

Theorem. Let S be a Noetherian semigroup with maximal ideal M. Assume that M is generated by a regular sequence $a_1, \cdots, a_k$ on S. Then $k = dim(S) = V(S)$, and S is a regular semigroup.

Theorem. Any regular semigroup is a Macaulay semigroup.

Theorem. The polynomial semigroup $S[X]$ is a Macaulay semigroup if and only if S is a Macaulay semigroup.

4

Let D be an integral domain with quotient field K. Let $F(D)$ be the set of non-zero fractional ideals of D. A mapping $I \longmapsto I^*$ of $F(D)$ to $F(D)$ is called a star-operation on D if for all $a \in K - \{0\}$ and $I, J \in F(D)$;

(1) $(a)^* = (a)$ and $(aI)^* = aI^*$;
(2) $I \subset I^*$;
(3) If $I \subset J$, then $I^* \subset J^*$; and
(4) $(I^*)^* = I^*$.

Let $\Sigma(D)$ be the set of star-operations on D.

Let $F'(D)$ be the set of non-zero D-submodules of K. A mapping $I \longmapsto I^*$ of $F'(D)$ to $F'(D)$ is called a semistar-operation on D if for all $a \in K - \{0\}$ and $I, J \in F'(D)$;

(1) $(aI)^* = aI^*$;
(2) $I \subset I^*$;
(3) If $I \subset J$, then $I^* \subset J^*$; and
(4) $(I^*)^* = I^*$.

Let $\Sigma'(D)$ be the set of semistar-operations on D.

A valuation ring (or a valuation semigroup) V is said to be discrete if its value group is discrete.

In 4, we proved the following Theorems.

Theorem. Let D be a domain with dimension n. Then D is a discrete valuation ring if and only if $\mid \Sigma'(D) \mid = n+1$.

Let S be a g-monoid with quotient group G. A mapping $I \longmapsto I^*$ of $F(S)$ to $F(S)$ is called a star-operation on S if for all $a \in G$, and $I, J \in F(S)$; (1) $(a)^* = (a)$; (2) $(a+I)^* = a+I^*$; (3) $I \subset I^*$; (4) If $I \subset J$, then $I^* \subset J^*$; (5) $(I^*)^* = I^*$.

For example, let I^v be the intersection of principal fractional ideals containing I, then v is a star-operation on S which is called the v-operation on S. Let $\Sigma(S)$ be the set of star-operations on S.

Let $F'(S)$ be the set of submodules of G. A mapping $I \longmapsto I^*$ of $F'(S)$ to $F'(S)$ is called a semistar-operation on S if, for all $a \in G$ and $I, J \in F'(S)$; (1) $(a+I)^* = a+I^*$; (2) $I \subset I^*$; (3) If $I \subset J$, then $I^* \subset J^*$; (4) $(I^*)^* = I^*$.

Let $\Sigma'(S)$ be the set of semistar-operations on S.

Theorem. Let S be a g-monoid with dimension n. Then S is a discrete valuation semigroup if and only if $\mid \Sigma'(S) \mid = n+1$.

Theorem. Let V be a valuation semigroup of dimension n, v its valuation and Γ its value group. Let $M = P_n \supsetneqq P_{n-1} \supsetneqq \cdots \supsetneqq P_1$ be the prime ideals of V, and let $\{0\} \subsetneqq H_{n-1} \subsetneqq \cdots \subsetneqq H_1 \subsetneqq \Gamma$ be the convex subgroups of Γ. Let m be a positive integer such that $n+1 \leq m \leq 2n+1$. Then the followings are equivalent:

(1) $\mid \Sigma'(V) \mid = m$.

(2) The maximal ideal of the g-monoid $V_{P_i} = \{s-t \mid s \in V, t \in V-P_i\}$ is principal for exactly $2n+1-m$ of i.

(3) The ordered abelian group Γ/H_i has a minimal positive element for exactly $2n+1-m$ of i.

Theorem. Let V be a valuation ring of dimension n, v its valuation and Γ its value group. Let $M = P_n \supsetneqq P_{n-1} \supsetneqq \cdots \supsetneqq P_1 \supsetneqq (0)$ be the prime ideals of V, and let $\{0\} \subsetneqq H_{n-1} \subsetneqq \cdots \subsetneqq H_1 \subsetneqq \Gamma$ be the convex subgroups of Γ. Let m be a positive integer such that $n+1 \leq m \leq 2n+1$. Then the followings are equivalent:

(1) $\mid \Sigma'(V) \mid = m$.

(2) The maximal ideal of V_{P_i} is principal for exactly $2n+1-m$ of i.

(3) Γ/H_i has a minimal positive element for exactly $2n+1-m$ of i.

5

Let R be a commutative ring, and let K be its total quotient ring; $K = \{a/b \mid a \in R, b \text{ is a non-zerodivisor of } R\}$. Let S be a g-monoid, and let G be the quotient group of S.

An element $\alpha \in G$ is called almost integral over S if there exists an element s of S such that $s + n\alpha \in S$ for every positive integer n. The set of almost integral elements of G over S is called the complete integral closure (or the CIC) of S. If the complete integral closure of S coincides with S, then S is called completely integrally closed (or CIC).

R is said to be root closed if whenever $x^n \in R$ for some $x \in K$ and positive integer n, then $x \in R$.

The maximal number n so that there exists a set of n-elements in G which is independent over $\mathbf{Z}$ is called the torsion-free rank of G, and is denoted by t.f.r.(G).

In 5, we proved the following Theorems.

Theorem. $R[X;S]$ is integrally closed if and only if S is integrally closed, R is integrally closed, $K[X_1]$ is integrally closed and $q(K[X_1, \cdots, X_{n-1}])[X_n]$ is integrally closed for every n with $n \leq$ t.f.r.(G).

Theorem. $R[X;S]$ is CIC if and only if S is CIC, R is CIC and $R[X_1, \cdots, X_n]$ is CIC for every positive integer $n \leq$ t.f.r.(G).

Theorem. $R[X;S]$ is root closed if and only if S is integrally closed, R is root closed, $K[X_1]$ is root closed and $q(K[X_1, \cdots, X_{n-1}])[X_n]$ is root closed for every n with $n \leq$ t.f.r.(G).

If, for each element a of R, there exists an element b of R such that $a = a^2b$, then R is called a von Neumann regular ring.

Theorem. Assume that K is a von Neumann regular ring. Then $R[X;S]$ is integrally closed if and only if S is integrally closed and R is integrally closed.

Theorem. Assume that K is a von Neumann regular ring. Then $R[X;S]$ is CIC if and only if S is CIC and R is CIC.

Let R be a Noetherian reduced ring. Then $R[X;S]$ is CIC if and only if S is CIC and R is CIC.

Theorem. Assume that K is a von Neumann regular ring. Then $R[X;S]$ is root closed if and only if S is integrally closed and R is root closed.

6

We denote the unit group of S by H. Let R be a ring. Let $U(R)$ be the unit group of R. The group of units $f = \sum a_s X^s$ of $R[X;S]$ with $\sum a_s = 1$ is denoted by $\mathrm{V}(R[X;S])$.

The following is a semigroup version of Karpilovsky's Problem [Kar, chapter 7, problem 9]:

Problem. Find necessary and sufficient conditions for $R[X;S]$ under which,

(1) H has a torsion-free complement in $\mathrm{V}(R[X;S])$.
($V(R[X;S]) = \{X^h \mid h \in H\} \otimes W$, where W is torsion-free.)
(2) H has a free complement in $\mathrm{V}(R[X;S])$.
($V(R[X;S]) = \{X^h \mid h \in H\} \otimes W$, where W is free.)
(3) $\mathrm{U}(R[X;S])$ is free modulo torsion.
($U(R[X;S])/\{\text{torsion elements}\}$ is free.)

In 6, we proved the following,

Theorem (An answer to Problem for reduced rings). Let R be reduced. Then,

(1) H has a torsion-free complement in $\mathrm{V}(R[X;S])$.

(2) H has a free complement in $\mathrm{V}(R[X;S])$ if and only if H is free.

(3) $\mathrm{U}(R[X;S])$ is free modulo torsion if and only if $\mathrm{U}(R)$ is free modulo torsion and H is free.

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