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Left Regular Bands and Semilattices in Finite Transformations[1]

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Introduction

Let X be a finite set and let $T(X)$ denote the full transformation semigroup on X, i.e., the semigroup of all maps from X into itself (under composition of maps). Let $G(X)$ be the symmetric group on X which is the biggest subgroup in $T(X)$. The set of all subsemigroups of $T(X)$ is denoted by $ST(X)$.

To investigate finite transformation semigroups is important for not only semigroup theory but automata theory. All semigroups treated here are finite.

Let $\mathbf{V}$ be a variety of semigroups (a class of semigroups closed under the formation of subsemigroups, homomorphic images and direct products). There arise the following questions:

(Q1) Determine all semigroups in $\mathbf{V}\cap ST(X)$, especially all maximal semigroups in it.

(Q2) Let $S,T\in\mathbf{V}\cap ST(X)$. Is there $\gamma\in G(X)$ such that $S=\gamma^{-1}T\gamma$ if $S\cong T$?

In consequencs of (Q2),

(Q2') Is T maximal if S is maximal and $S\cong T$?

A semigroup B is called a *band* if every element in B is an idempotent. A commutative band is called a *semilattice*. A band B is said to be *left regular* if $\alpha\beta\alpha=\alpha\beta$ for every $\alpha,\beta\in B$. The classes of left regular bands and semilattices are varieties, which are denoted by **LR** and **SL**, respectively.

The purpose of this paper is to solve the above quetions for **LR** and **SL**.

The quetion (Q1) for **SL** has been solved by M. Kunze and S. Crvenković (1989) (see [3], [4]). We here solve it by induction on $|X|$, that is, we give an algorithm to determine $\mathbf{SL}\cap ST(X_{k+1})$ from $\mathbf{SL}\cap ST(X_k)$, where $|X|$ denotes the cardinal number of X and $k=|X_k|$. Then (Q1) for **SL**, can be solved, since $\mathbf{SL}\cap ST(X_1)=T(X_1)$.

1. Left regular bands

[1] This is an abstract and the details will be published elsewhere.

For $\alpha \in T(X)$, let $im(\alpha) = \{x \in X | y\alpha = x \text{ for some } y \in X\}$ and $fix(\alpha) = \{x \in X | x\alpha = x\}$. The identity map ($fix(\alpha) = X$) and the constant map to x ($im(\alpha) = \{x\}$) on X are denoted by id_X and $c(x)$, respectively. The set of constant maps in $T(X)$ is denoted by $C(X)$.

A semigroup S is called a *left zero semigroup* if $\alpha\beta = \alpha$ for every $\alpha, \beta \in S$. Hereafter every semigroup is a subsemigroup of $T(X)$. The following facts are known:

Fact 1. (1) *$\alpha \in T(X)$ is an idempotent if and only if $fix(\alpha) = im(\alpha)$.*

(2) *S is a left zero semigroup if and only if $fix(\alpha) = im(\alpha)$ and $fix(\alpha) = fix(\beta)$ for every $\alpha, \beta \in S$.*

(3) *Let B be a band. Then $B \in$ **LR** if and only if $fix(\alpha\beta) = fix(\alpha) \cap fix(\beta)$ for every $\alpha, \beta \in B$.*

(4) *Let $B \in$ **LR**. Then $\alpha\beta = \alpha$ if and only if $fix(\alpha) \subseteq fix(\beta)$ for every $\alpha, \beta \in B$.*

Let $(X, \leq)$ be the partially ordered set X under an order relation $\leq$. The set of minimal elements in $(X, \leq)$ is denoted by $Min(\leq)$. A subset I of X is called an *o-ideal* if (1) $Min(\leq) \subseteq I$ and (2) $x \in I$ and $y \leq x$ imply $y \in I$. For $x \in X$, let $lb(x) = \{y \in X | y \leq x\}$ and $I(x) = lb(x) \cup Min(\leq)$. Then $I(x)$ is an o-ideal which is called the *principal ideal* generated by x. If $(X, \leq)$ has the least element, then $I(x) = lb(x)$. The set of o-ideals and principal ideals in $(X, \leq)$ denoted by $I(X, \leq)$ and $PI(X, \leq)$ or simply $I(\leq)$ and $PI(\leq)$, respectively. We state some properties of o-ideals in $(X, \leq)$.

Fact 2 (1) *$I(\leq)$ forms a lattice under $\cup$ and $\cap$, and $I(x) = \bigcap\{I \in I(\leq) | I \ni x\}$.*

(2) *Let $|X| = n$ and $|Min(\leq)| = m$. For any $I \in I(\leq)$, there exists a maximal chain including I of the length $n - m + 1$:*

$Min(\leq) = I_m \subset I_{m+1} \subset \ldots \subset I = I_k \subset \ldots \subset I_n = X$, where $|I_k| = k$, and where $J \subset I$ means $J \subseteq I$ and $J \neq I$.

(3) *Let $I \in I(\leq)$ with $I \neq Min(\leq)$. Then I is principal ideal if and only if there exists a unique $J \in I(\leq)$ such that $J \subset I$ and $|J| = |I| - 1$.*

Proposition 1.1. *Let $J(\leq)$ be a $\cap$-closed subset of $I(\leq)$. For $I \in J(\leq)$, let $LZ(I) = \{\alpha \in T(X) | fix(\alpha) = I$ and $x\alpha \in I \cap lb(x)$ if $x \notin I$ for every $x \in X\}$, and let $LR(J(\leq)) = \bigcup\{LZ(I) | I \in J(\leq)\}$. Then :*

(1) *$LR(\leq)$ is a left regular band and each $LZ(I)$ is a left zero semigroup. In this case, $|LZ(I)| = \prod_{x \notin I} |I \cap lb(x)|$.*

(2) *$LR(\leq)$ is maximal if and only if $(X, \leq)$ has the least element and $J(\leq) = I(\leq)$.*

Let $B \in \mathbf{LR}$ which contains id_X. Define a relation $\leq_B$ on X by $x \leq_B y$ if and only if $y\alpha = x$ for some $\alpha \in B$. Then $\leq_B$ is an order relation on X. Let $\alpha \in B$ and let $x \in fix(\alpha)$ and $y \in X$ with $y \leq_B x$. Then $x\beta = y$ for some $\beta \in B$, so that $y\alpha = x\beta\alpha = x\alpha\beta\alpha = x\alpha\beta = y$. Thus $y \in fix(\alpha)$. Since $x\alpha \leq_B x$ for all $x \in X$, we have that $Min(\leq_B) \subseteq fix(\alpha)$. We conclude that $fix(\alpha)$ is an o-ideal in $(X, \leq_B)$ for every $\alpha \in B$. Let $J(\leq_B) = \{fix(\alpha) | \alpha \in B\}$. By (3) of Fact 1, $J(\leq_B)$ is $\cap$-closed, so that we can construct $LR(J(\leq_B))$ as in Proposition 1.1. Then clearly $B \subseteq LR(J(\leq_B))$. It is clear that $(X, \leq_B)$ has the least element n if and only if $c(n) \in B$.

From the above facts and Proposition 1.1, we obtain:

Theorem 1.2. *Let $B \in \mathbf{LR}$ and $\leq_B$ defined above. Then B is maximal if and only if $c(n) \in B$ for some $n \in X$ and $B = LR(I(\leq_B))$.*

Let A and B be algebras and let ϕ be a homomorphism from A onto B. Then ϕ is said to be *split* if there exists a homomorphism ψ from B to A such that $\psi\phi = id_B$. In this case, $x\psi$ for $x \in B$ is called the *skeleton* of $x\phi^{-1}$.

Proposition 1.3. *Let $B \in \mathbf{RL}$ and let $J(\leq_B) = \{fix(\alpha) | \alpha \in B\}$. Suppose that $B = LR(\leq_B)$. Then the map $\phi : B \to J(\leq_B), \alpha \mapsto \alpha\phi$ defined by $\alpha\phi = fix(\alpha) \in J(\leq_B)$ for $\alpha \in B$ is a splitting homomorphism from $(B, \cdot)$ onto $J(\leq_B), \cap)$.*

Theorem 1.4 *Let $B, C \in \mathbf{LR}$ with $B \cong C$. If B is maximal, then so is C and there exists $\gamma \in G(X)$ such that $C = \gamma^{-1}B\gamma$.*

In Theorem 1.4, B is said to be *strongly maximal*, that is, there are no $C, D \in \mathbf{LR}$ such that $B \cong C \subset D$. Therefore enery maximal left regular band is strongly maximal.

2. Semilattices

We first state briefly the results of Kunze and Crvenković.

Let $(X, \leq)$ be a partially ordered set. An o-ideal F in $(X, \leq)$ is called an *F-ideal* if $F \cap lb(x)$ has the greatest element g_F for every $x \in X$. The set of F-ideals in $(X, \leq)$ is denoted by $F(X, \leq)$ or simply $F(\leq)$. Then $F(\leq)$ is $\cap$-closed. For $F \in F(\leq)$, define $\gamma_F \in T(X)$ by $x\gamma_F = g_F$ for every $x \in X$, and let $SL(F(\leq)) = \{\gamma_F | F \in F(\leq)\}$. Then $SL(F(\leq))$ is a semilattice and $(S, \cdot) \cong (F(\leq), \cap)$.

On the other hand, let S be a semilattice. Define $\leq_S$ on X by $x \leq_S y$ if and only if $y\alpha = x$ for some $\alpha \in S \cup \{id_X\}$. Then (1) $(X, \leq_S)$ is a partially ordered set, (2) $fix(\alpha)$ is an F-ideal for every $\alpha \in S$ and $F(\leq_S) = \{fix(\alpha) | \alpha \in S\}$, (3) $S \subseteq SL(F(\leq_S))$ and (4) If S is a maximal semilattice, then $(X, \leq_S)$ has the least element..

They determined all maximal semilattices by the types of ordered set $(X, \leq)$.

Since $F(\leq) \subseteq I(\leq)$ and it is $\cap$-closed, we can construct $LR(F(\leq))$. Then $SL(F(\leq))$ is the skeletons of the homomorphism $\phi : (LR(F(\leq), \cdot) \to (F(\leq), \cap), \alpha \mapsto fix(\alpha)$, since $(SL(F(\leq)), \cdot) \cong (F(\leq), \cap)$, so that it is isomorphic to the skeletons $\{\alpha_I | I \in F(\leq)\}$ defined in Proposition 1.3, which is denoted by $Sk_1(\phi)$.

An ordered set $(X, \leq)$ is said to be *simplest* if it has the least element n and every $x \in X \backslash \{n\}$ covers n, i.e., there is no y such that $n < y < x$, which is denoted by $(X, \leq_{sim})$. Then all subsets of $(X, \leq_{sim})$ containing n are o-ideals and F-ideals, and $LR(I(\leq_{sim})) = SL(I(\leq_{sim}))$. If $(X, \leq)$ has the least element n, then $Sk_1(\phi)$ is a subsemilattice of $SL(F(\leq_{sim}))$. Since $(X, \leq_S)$ has the least element if S is a maximal semilattice, we obtain:

Proposition 2.1. *Every maximal semilattice S can be embedded in the semilattice $SL(F(\leq_{sim}))$ determined by the simplest ordered set, that is,* $S \cong Sk_1(\phi) \subseteq SL(F(\leq_{sim}))$.

Proposition 2.1 shows that $\mathbf{SL} \cap T(X)$ has unique strongly maximal elemet $SL(I(\leq_{sim}))$

Let X_n be a finite set with $|X_n| = n$ and let $S \in \mathbf{SL} \cap ST(X_n)$. Let n be any fixed element in $Min(\leq_S)$. Then $S \cup \{c(n)\} \cup \{id_{X_n}\}$ is also a semilattice in $T(X_n)$. Hereafter we assume that every semilattice contains $c(n)$ and id_{X_n}. In this case, $c(n)$ and id_{X_n} are the zero and the identity of S, respectively. Therefore $(X_n, \leq_S)$ has the least element n. Suppose that n covered with $m \in X_n$. Then the principal ideals $I(m) = \{m, n\}$ and $I(n) = \{n\}$ are an F-ideals in $(X_n, \leq_S)$. Let $\gamma 111_{I(m)}$ and $\gamma_{I(n)}$ be as above, and let
$X_{(m)} = \{x \in X_n | x\gamma_{I(m)} = m\}$ and $X_{(n)} = \{x \in X_n | x\gamma_{I(m)} = n\}$.

Since $x\alpha \leq x$ for every $x \in X_n$ and every $\alpha \in S$, we have that either $m\alpha = n\alpha = n$ or $m\alpha = m, n\alpha = n$ for every $\alpha \in S$.

Let $S_{com(m,n)} = \{\alpha \in S | m\alpha = n\alpha = n\}$ and $S_{sep(m,n)} = \{\alpha \in S | m\alpha = m, n\alpha = n\}$.

Then they are subsemilattices of S. In particular, $S_{com(m,n)}$ is an ideal of S.

Lemma 2.1. *$S_{com(m,n)} = \{\alpha \in S | fix(\alpha) \subseteq X_{(n)}\}$ and $S_{sep(m,n)} = \{\alpha \in S | X_{(n)}\alpha \subseteq X_{(n)}$ and $X_{(m)}\alpha \subseteq X_{(m)}\}$.*

Lemma 2.2. *Let S and $X_{(m)}, X_{(n)}$ be as above and let $U \in \mathbf{SL} \cap T(X_n)$ with $S \subseteq U$. Then n is the least element covered with m in $(X_n, \leq_U)$ and*
$U_{com(m,n)} = \{\alpha \in U | fix(\alpha) \subseteq X_{(n)}\}$,
$U_{sep(m,n)} = \{\alpha \in U | X_{(m)}\alpha \subseteq X_{(m)}$ and $X_{(n)}\alpha \subseteq X_{(n)}\}$.

Define $\phi \in T(X_n)$ by $x\phi = x$ if $x \neq n$ and $n\phi = m$. Then it is easy to see that $(\alpha\beta)\phi_{=}(\alpha\phi)(\beta\phi)$, so that ϕ is a homomorphism of S to $S\phi$. Since **SL** is a variety, $S\phi$ is a semilattice. For $\alpha \in S$, let $\alpha\phi|_{X_{n-1}}$ be the restriction of $\alpha\phi$ to $X_{n-1} = X_n \backslash \{n\}$. Then $S\phi \cong \{\alpha\phi|_{X_{n-1}} | \alpha \in S\}$. Therefore we regard $S\phi$ as a semilattice in $T(X_{n-1})$. In this case, $S\phi$ is called the *ϕ-contraction* of S, and S is called a *ϕ-extension* of $S\phi$.

Let $T = S\phi, M = X_{(m)}, N = (X_{(n)} \backslash \{n\}) \cup \{m\}$ and let $T_N = \{\alpha \in T | fix(\alpha) \subseteq N\}$, $T_M = \{\alpha \in T | M\alpha \subseteq M$ and $N\alpha \subseteq N\}$. Then it is easy to see that (1) $T \in \mathbf{SL} \cap T(X_{n-1})$ and m is the least element in $(X_{n-1}, \leq_T)$, (2) $(S_{com(m,n)})\phi = T_N$ and $(S_{sep(m,n)})\phi = T_M$.

Lemma 2.3. *The maps : $S_{sep(m,n)} \to T_M, \alpha \mapsto \alpha\phi$ and $S_{com(m,n)} \to T_N$, $\beta \mapsto \beta\phi$ are isomorphisms.*

We now construct a semilattice in $T(X_n)$ from any semilattice in $T(X_{n-1})$. Let $T \in \mathbf{SL} \cap ST(X_{n-1})$. Suppose that $(X_{n-1}, \leq_T)$ has the least element m.

Let M, N be any subsets of X_{n-1} such that $X_{n-1} = N \cup M$ and $M \cap N = \{m\}$ and let $T_N = \{\alpha \in T | fix(\alpha) \subseteq N\}$, $T_M = \{\alpha \in T | N\alpha \subseteq N, M\alpha \subseteq M\}$ and let $T_{M,N} = T_M \cup T_N$.

Then $T_{M,N}$ is a subsemilattice of T, but $T_N \cap T_M \neq \emptyset$. In particular, if $M = X_{n-1}$ and $N = \{m\}$, then $T_M = T$ and $T_N = \{c(m)\}$, and if $M = \{m\}$ and $N = X_{n-1}$, then $T_M = T_N = T$.

Let $T \in \mathbf{SL} \cap T(X_{n-1})$ and let M, N be as above. Then T is said to be *(M, N)-maximal* if $T = T_{M,N}$ and $T_M = U_M$ and $S_N = U_N$ for every $U \in \mathbf{SL} \cap ST(X_{n-1})$ with $T \subseteq U$.

Lemma 2.4 *Let $T, U \in \mathbf{SL} \cap T(X_{n-1})$ with $T \subseteq U$. Then T is (M, N)-maximal if and only if, for every $\alpha \in U \backslash T$, $fix(\alpha) \cap M \backslash \{m\} \neq \emptyset$, and $x\alpha \in$*

$N\backslash\{m\}$ *for some* $x \in M$ *or* $y\alpha \in M\backslash\{m\}$ *for some* $y \in N$.

For $\alpha \in T_N$, define $\alpha_{e1} \in T(X_n)$ by $x\alpha_{e1} = n$ if $x\alpha = m$, otherwise $x\alpha_{e1} = x\alpha$ for every $x \in X_{n-1}$ and $n\alpha_{e1} = n$.

For $\alpha \in T_M$, define $\alpha_{e2} \in T(X_n)$ by $x\alpha_{e2} = n$ if $x\alpha = m$ and $x \in N$, otherwise $x\alpha_{e2} = x\alpha$ for every $x \in X_{n-1}$ and $n\alpha_{e2} = n$.

Let $(T_N)_{e1} = \{\alpha_{e1} | \alpha \in T_N\}$, $(T_N)_{e2} = \{\alpha_{e2} | \alpha \in T_M\}$ and $(T_{M,N})_e = (S_N)_{e1} \cup (S_M)_{e2}$.

Theorem 2.2. *Let* $(T_{M,N})_e$ *be as above. Then:*

(1) $(T_{M,N})_e$ *is a semilattice in* $T(X_n)$ *and* n *is the least element covered with* m *in* $(X_n, \leq_{T_e})$.

(2) $((T_{M,N})_e)_{com(m,n)} = (T_N)_{e1}$ *and* $((T_{M,N})_e)_{sep(m,n)} = (T_M)_{e2}$.

(3) *Let* $S \in \mathbf{SL} \cap T(X_n)$ *and let* $X_{(m)}, X_{(n)}$ *be as above. If* $S\phi = T$, *then* $S \subseteq (T_{M,N})_e$, *where* $M = X_{(m)}$ *and* $N = (X_{(n)}\backslash\{n\}) \cup \{m\}$.

(4) *In* (3), S *is maximal in* $T(X_n)$ *if and only if* $S = (T_{M,N})_e$ *and* T *is* (M, N) *– maximal in* $T(M_{n-1}$.

In Theorem 2.2, T_e is not a ϕ-extension of T, but it is a ϕ-extension of $T_{M,N}$.

Suppose that all maximal semilattices in $T(X_{n-1})$ have been determined. Then by Lemma 2.4, all (M, N)-maximal semilattices in $T(X_{n-1})$ can be determuned. for any subsets M, N of X_{n-1} with $M \cup N = X_{n-1}$ and $N \cap M = \{m\}$. Thus by Theorem 2.2, all maximal semilattices in $T(X_n)$ can be constructed. Since $T(X_1)$ is trivially a maximal semilattice in $T(X_1)$, we conclude that all maximal semilattices in finite transformations can be obtained by induction on $n = |X_n|$.

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