数理解析研究所講究録
1107 巻 1999 年 70-83

A CATEGORY OF CONTINUOUS MAPS

D.BUHAGIAR

1. Introduction

The study of General Topology is usually concerned with the category $\mathcal{TOP}$ of topological spaces as objects, and continuous maps as morphisms. The concepts of *space* and *map* are equally important and one can even look at a space as a map from this space onto a singleton space and in this manner identify these two concepts. With this in mind, a branch of General Topology which has become known as General Topology of Continuous Maps, or Fibrewise General Topology, was initiated. This field of research is concerned most of all in extending the main notions and results concerning topological spaces to those of continuous maps. In this way one can see some well-known results in a new and clearer light and one can also be led to further developments which otherwise would not have suggested themselves. The fibrewise viewpoint is standard in the theory of fibre bundles, however, it has been recognized relatively recently that the same viewpoint is also as important in other areas such as General Topology.

For an arbitrary topological space Y one considers the category $\mathcal{TOP}_Y$, the objects of which are continuous maps into the space Y, and for the objects $f : X \to Y$ and $g : Z \to Y$, a morphism from f into g is a continuous map $\lambda : X \to Z$ with the property $f = g \circ \lambda$. This situation is a generalization of the category $\mathcal{TOP}$, since the category $\mathcal{TOP}$ is isomorphic to the particular case of $\mathcal{TOP}_Y$ in which the space Y is a singleton space.

The carried out research showed a strong analogy in the behaviour of spaces and maps and it was possible to extend the main notions and results of spaces to that of maps. Since the considered case is of a wider generality (compared to that of spaces), the results obtained for maps are technically more complicated. Moreover, there are moments which are specific to maps. For example, there is no analogue

---

Date: April 23, 1999.

1991 *Mathematics Subject Classification.* Primary 54C05, 54C99; Secondary 54C10, 54B30, 54B35.

Key words and phrases. Fibrewise Topology, Categorical Topology, Continuous Map, Partial Product.

to Urysohn's Lemma for maps and so normality and functional normality do not coincide and as a consequence, there exist two theories of compactifications, one for Hausdorff compactifications and one for Tychonoff compactifications.

Some results in the General Topology of Continuous Maps were obtained quite some time ago. For example, in 1947, I.A.Vainstein [23] proposed the name of compact maps to perfect maps, G.T.Whyburn in 1953 [25, 26], as did G.L.Cain, N.Krolevets, V.M.Ulyanov [22] and others, considered compactifications of maps. In the meantime, until quite recently, there wasn't a connected unified theory for maps. One of the main reasons might have been the lack of separation axioms for maps, especially that of Tychonoffness (and complete regularity) and also that of (functional) normality and collectionwise normality.

Completely regular and Tychonoff maps, as well as (functionally) normal maps, were defined by B.A.Pasynkov in 1984 [18]. These definitions made it possible to generalize and obtain an analogue to the theorem on the embedding of Tychonoff spaces of weight τ into I^τ and to the existence of a compactification for a Tychonoff space having the same weight (see Theorem 1.4). It was also possible to construct a maximal Tychonoff compactification for a Tychonoff map (i.e. construct an analogue to the Stone-Čech compactification). Collectionwise normal maps were defined by the author [7] and enabled the definition of *metrizable type maps*, giving a satisfactory fibrewise version of the theory of metrizable spaces.

In most cases there is some choice in defining properties on maps and one usually prefers the simplest and the one that gives the most complete generalization of the corresponding results in the category $\mathcal{TOP}$. It would be beneficial to have a more systematic way of extending definitions and results from the category $\mathcal{TOP}$ to the category $\mathcal{TOP}_Y$ and some hope is provided by the link between Fibrewise Topology and Topos Theory [11, 12, 14, 15]. Unfortunately, as was noted in [10], this approach has several drawbacks. In defining *compact maps* [19, Proposition 2.2 (V.P.Norin)], *paracompact maps* [5], *metacompact maps*, *subparacompact maps*, *submetacompact maps* [6] and *metrizable type maps* [7], one can see a systematic method in defining notions in the category $\mathcal{TOP}_Y$ (or more general in the category $\mathcal{MAP}$) corresponding to definitions which involve coverings or bases of topological spaces. This construction gave satisfactory definitions which can be seen from the results obtained for such maps [5, 6, 7, 19]. One can also add that the definitions of paracompact maps, metacompact maps, subparacompact maps and submetacompact maps strengthened the result that paracompactness, metacompactness, subparacompactness and submetacompactness are all inverse invariant of perfect maps. Namely, it was proved that the inverse image of a paracompact T_2 (resp. subparacompact, metacompact, submetacompact) space by a paracompact T_2 (resp. subparacompact, metacompact, submetacompact) map is paracompact T_2 (resp. subparacompact, metacompact, submetacompact) [5, 6].

One of the most important operations on objects in $\mathcal{TOP}$ is the Tychonoff product which gives rise to many interesting results and examples. In particular, results concerning universal spaces. Recall that a space X is said to be *universal* for all spaces having a topological property $\mathcal{P}$ if the space X has property $\mathcal{P}$ and every space having property $\mathcal{P}$ is homeomorphically embeddable in X. Universal spaces are very useful since they reduce the study of a class of spaces having some topological property $\mathcal{P}$ to the study of subspaces of a fixed space. We are interested in obtaining analogues in the category $\mathcal{MAP}$ to the following three results obtained respectively by A.Tychonoff [21], P.S.Alexandroff [1] and N.Vedenissoff [24].

Theorem 1.1. *The Tychonoff cube $I^{\mathfrak{m}}$ is universal for all Tychonoff spaces of weight $\mathfrak{m} \geqslant \aleph_0$.*

Theorem 1.2. *The Alexandroff cube $F^{\mathfrak{m}}$ is universal for all T_0-spaces of weight $\mathfrak{m} \geqslant \aleph_0$.*

Theorem 1.3. *The Cantor cube $D^{\mathfrak{m}}$ is universal for all zero-dimensional spaces of weight $\mathfrak{m} \geqslant \aleph_0$.*

As is the case in $\mathcal{TOP}$, one of the most important operations on objects in the category $\mathcal{TOP}_Y$ is the fibrewise product of maps defined by B.A.Pasynkov [16, 17, 18]. As was mentioned above, the definitions of completely regular and Tychonoff maps made it possible to generalize and obtain an analogue to Theorem 1.1 in the category $\mathcal{TOP}_Y$ [18].

Theorem 1.4. *A Tychonoff map $f : X \to Y$ has weight $\mathfrak{W}(f) \leqslant \mathfrak{m}$ ($\mathfrak{m} \geqslant \aleph_0$) if and only if, the map f is homeomorphically embeddable into the projection p of a partial topological product $P = P(Y, \{Z_\alpha\}, \{O_\alpha\} : \alpha \in \mathcal{A})$, where $Z_\alpha = I$ for every $\alpha \in \mathcal{A}$ and $|\mathcal{A}| \leqslant \mathfrak{m}$.*

The following result was also given as a corollary to Theorem 1.4 in [18].

Corollary 1.5. *A continuous map is Tychonoff if and only if it is homeomorphically embeddable into the projection of a partial topological product, all the fibres of which are segments.*

For more details and undefined terms on the General Topology of Continuous Maps one can consult [5, 2, 3, 4, 6, 7, 9, 10, 13, 18, 19].

2. The category $\mathcal{MAP}$

A category of maps $\mathcal{MAP}$ in which one does not restrain oneself with a fixed base space Y was introduced by the author in [2]. The objects of $\mathcal{MAP}$ are continuous maps from any topological space into any topological space. For two objects $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$, a morphism from f_1 into f_2 is a pair of continuous maps $\{\lambda_T, \lambda_B\}$, where $\lambda_T : X_1 \to X_2$ and $\lambda_B : Y_1 \to Y_2$, such that the diagram

$$\begin{tikzcd} X_1 \arrow[r, "\lambda_T"] \arrow[d, "f_1"'] & X_2 \arrow[d, "f_2"] \\ Y_1 \arrow[r, "\lambda_B"'] & Y_2 \end{tikzcd}$$

is commutative. It is not difficult to see that this definition of a morphism in $\mathcal{MAP}$ satisfies the necessary axioms that morphisms should satisfy in any category (see, for example, [20]).

Let $\mathcal{P}_T$ and $\mathcal{P}_B$ be two topological/set theoretic properties of maps (for example: closed, open, 1-1, onto, etc.). If λ_T has property $\mathcal{P}_T$ and λ_B has property $\mathcal{P}_B$ then we say that $\{\lambda_T, \lambda_B\}$ is a $\{\mathcal{P}_T, \mathcal{P}_B\}$-*morphism*. If $\mathcal{P}_T$ is the *continuous* property, then we say that $\{\lambda_T, \lambda_B\}$ is a $\{*, \mathcal{P}_B\}$-*morphism*, similarly for $\mathcal{P}_B$. Therefore, a $\{*, *\}$-morphism is just a morphism. Also, if $\mathcal{P}_T = \mathcal{P}_B = \mathcal{P}$ then a $\{\mathcal{P}_T, \mathcal{P}_B\}$-morphism is called a $\mathcal{P}$-*morphism.*

As noted in the introduction, separation axioms for maps have already been defined in the category $\mathcal{TOP}_Y$ and since these axioms involve only one map, they have also been defined for the category $\mathcal{MAP}$.

We now give the definition of a submap as an analogue of subspace. Since we do not restrict ourselves to a fixed base space Y our definition slightly differs from that given in the category $\mathcal{TOP}_Y$ [18]. This definition was introduced in [2].

Definition 2.1. The map $g : A \to B$ is said to be a (closed, open, everywhere dense, etc.) submap of the map $f : X \to Y$, if g is the restriction of the map f on the (closed, open, everywhere dense, etc.) subset A of the space X and $g(A) = f(A) \subset B \subset Y$.

Remember that in $\mathcal{MAP}$ (as in $\mathcal{TOP}_Y$), by a *compact map* we mean a *perfect map*, namely, a closed map with compact fibres. It is evident that a closed submap of a compact map is compact.

Finally, we give the definitions of base and weight for a continuous map, both given by B.A.Pasynkov [16, 18].

Definition 2.2. Let $f : X \to Y$ be a map of topological spaces. A set $U \subset X$ is said to be *f-functionally open*, if there exists an open subset O of Y such that $U \subset f^{-1}O$ and U is functionally open in $f^{-1}O$.

Definition 2.3. Let $f : X \to Y$ be a map of topological spaces. A collection $\mathfrak{B}_f$ of open (resp. f-functionally open, functionally open) subsets of X is called a *base* (resp. *f-functionally open base, functionally open base*), *for the map f* if for every point $x \in X$ and every neighborhood U_x of x in X there exists a neighborhood O_y of the point $y = f(x)$ in Y and an element $V \in \mathfrak{B}_f$ such that $x \in f^{-1}O_y \cap V \subset U_x$.

Definition 2.4. The minimal cardinal number of the form $|\mathfrak{B}_f|$, where $\mathfrak{B}_f$ is a base (resp. f-functionally open base, functionally open base) for the map f (if such bases exist), is called the *weight* (resp. *f-functional weight, functional weight*) *of the continuous map f* and is denoted by $\mathfrak{w}(f)$ (resp. $\mathfrak{W}(f), \mathfrak{W}'(f)$).

A proof for the following propostion can be found in [19].

Proposition 2.1. *The map $f : X \to Y$ is completely regular if and only if there exists an f-functionally open base of f.*

The above proposition shows in particular that for a Tychonoff map f, the weight $\mathfrak{W}(f)$ is defined.

3. Elementary partial topological products

The notion of *elementary partial topological product* was introduced by B.A.Pasynkov in 1964 [16, 17]. By taking fan products of elementary partial topological products, which are called partial topological products, he proved Theorem 1.4, the analogue of Theorem 1.1 in the category $\mathcal{TOP}_Y$. In this section we give the definition of elementary partial topological products, as given by B.A.Pasynkov, and in the following sections we go on to define partial topological products for both the Tychonoff product of maps and fan product relative to an inverse system [3], the two types of products in the category $\mathcal{MAP}$ introduced in [2]. In the following sections we use these definitions to obtain analogues of Theorems 1.1, 1.2 and 1.3 (and so also Theorem 1.4) in the category $\mathcal{MAP}$. The proofs of the results in the following sections are found in [3].

Definition 3.1. Let Y and Z be topological spaces and let O be an open subset of Y. Consider the disjoint union D of the sets $Y \setminus O$ and

$O \times Z$ and define a map $p : D \to Y$ by letting $p(y) = y$ if $y \in Y \setminus O$ and $p(y, z) = y$ if $(y, z) \in O \times Z$. Let Ω_Y and $\Omega_{O\times Z}$ be the topologies of Y and $O \times Z$ respectively. The elementary partial topological product ($\equiv$ EPTP) with base space Y, fibre Z and open set O is the set D endowed with the topology generated by the base $p^{-1}\Omega_Y \cup \Omega_{O\times Z}$ and is denoted by $P(Y, Z, O)$. The continuous map $p : P(Y, Z, O) \to Y$ is called the *projection of the EPTP* $P(Y, Z, O)$. The projection q of the product $O \times Z \subset P(Y, Z, O)$ onto the factor Z is called the *side projection of the EPTP* $P(Y, Z, O)$.

Thus, the EPTP $P(Y, Z, O)$ induces on $O \times Z$ the topology of the topological product $O \times Z$, and on $Y \setminus O$, the subspace topology as a subspace of Y. Also, the projection p is continuous, open and its restriction on $Y \setminus O$ is a homeomorphic embedding. The following result can be found in [19].

Proposition 3.1. *The projection* $p : P \to Y$ *of the EPTP* $P = P(Y, Z, O)$ *satisfies the inequality* $\mathfrak{w}(p) \leqslant \mathfrak{w}(Z) + 1$. *If the fibre* Z *is a* T_i*-space, then the projection* p *is a* T_i*-map, for* $i \leqslant 3$. *If the fibre* Z *is completely regular, then the projection* p *is completely regular and* $\mathfrak{W}(p) = \mathfrak{w}(Z) + 1$. *If moreover, the set* $O \subset Y$ *is functionally open, then the weight* $\mathfrak{W}'(p)$ *is defined and* $\mathfrak{W}'(p) = \mathfrak{W}(p)$.

4. Tychonoff products

Tychonoff products of maps is taken to be the Tychonoff product of objects in the category $\mathcal{MAP}$ [2, 3]. Recently, Tychonoff products of maps were used to obtain an analogue in the category $\mathcal{MAP}$, of the Tamano Theorem on an external characterization for paracompact spaces [4]. We recall the definition.

Definition 4.1. Let $\{f_\alpha : \alpha \in \mathcal{A}\}$ be a collection of continuous maps, where $f_\alpha : X_\alpha \to Y_\alpha$. The Tychonoff product of the maps $\{f_\alpha : \alpha \in \mathcal{A}\}$, which is denoted by $\prod\{f_\alpha : \alpha \in \mathcal{A}\}$, is the continuous map which assigns to the point $x = \{x_\alpha\} \in \prod\{X_\alpha : \alpha \in \mathcal{A}\}$ the point $\{f_\alpha(x_\alpha)\} \in \prod\{Y_\alpha : \alpha \in \mathcal{A}\}$.

If $pr_T^\alpha : \prod\{X_\alpha : \alpha \in \mathcal{A}\} \to X_\alpha$ and $pr_B^\alpha : \prod\{Y_\alpha : \alpha \in \mathcal{A}\} \to Y_\alpha$ are the projections, then the diagram

$$\begin{array}{ccc}
\prod\{X_\alpha : \alpha \in \mathcal{A}\} & \xrightarrow{pr_T^\alpha} & X_\alpha \\
{\scriptstyle \prod\{f_\alpha:\alpha\in\mathcal{A}\}}\big\downarrow & & \big\downarrow{\scriptstyle f_\alpha} \\
\prod\{Y_\alpha : \alpha \in \mathcal{A}\} & \xrightarrow[pr_B^\alpha]{} & Y_\alpha
\end{array}$$

is commutative. Therefore, the pair $\{pr^{\alpha}_{T}, pr^{\alpha}_{B}\}$ is a $\{\text{onto}, \text{onto}\}$-morphism of $\prod\{f_\alpha : \alpha \in \mathcal{A}\}$ into f_α.

We now introduce and define Tychonoff partial topological products.

Definition 4.2. . Let $P_\alpha = P(Y_\alpha, Z_\alpha, O_\alpha)$ be an EPTP with base space Y_α, fibre Z_α and open set O_α for every α in some indexing set $\mathcal{A}$ and let $p_\alpha : P_\alpha \to Y_\alpha$ be the corresponding projection of the EPTP P_α. The Tychonoff product $\prod P_\alpha \equiv \prod\{P_\alpha : \alpha \in \mathcal{A}\}$ is called the *Tychonoff partial topological product ($\equiv$ TPTP) of the EPTPs $P_\alpha, \alpha \in \mathcal{A}$*. The Tychonoff product $\prod p_\alpha \equiv \prod\{p_\alpha : \alpha \in \mathcal{A}\}$ of the projections p_α is called the *projection of the TPTP $\prod P_\alpha$ onto its base*. The projection of the TPTP $\prod P_\alpha$ onto the EPTP P_α is denoted by pr_α.

Next, we formulate the main theorem of this section, an analogue of Theorem 1.1 in the category $\mathcal{MAP}$ with respect to Tychonoff products. By I we denote the *unit interval* $[0,1] \subset \mathbb{R}$.

Theorem 4.1. *For a Tychonoff map $f : X \to Y$ the following are equivalent:*

1. *The map f has weight $\mathfrak{W}(f) \leqslant \mathfrak{m}$ ($\mathfrak{m} \geqslant \aleph_0$);*
2. *There exists a homeomorphic embedding-morphism of the map f into the projection of a TPTP $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$, where the EPTP $P_\alpha = P(Y, I, O_\alpha)$ and $|\mathcal{A}| \leqslant \mathfrak{m}$;*
3. *There exists a homeomorphic embedding-morphism of the map f into the projection of a TPTP $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$, where the EPTP $P_\alpha = P(Y_\alpha, I, O_\alpha)$ and $|\mathcal{A}| \leqslant \mathfrak{m}$.*

We can write down the following corollaries to the above theorem. Since a $T_{2\frac{1}{2}}$ compact map is Tychonoff, we have:

Corollary 4.2. *For a $T_{2\frac{1}{2}}$ compact map $f : X \to Y$ into a Hausdorff space Y the following are equivalent:*

1. *The map f has weight $\mathfrak{W}(f) \leqslant \mathfrak{m}$ ($\mathfrak{m} \geqslant \aleph_0$);*
2. *There exists a $\{$closed homeomorphic embedding, homeomorphic embedding$\}$-morphism of the map f into the projection of a TPTP $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$, where the EPTP $P_\alpha = P(Y, I, O_\alpha)$ and $|\mathcal{A}| \leqslant \mathfrak{m}$;*
3. *There exists a $\{$closed homeomorphic embedding, homeomorphic embedding$\}$-morphism of the map f into the projection of a TPTP $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$, where the EPTP $P_\alpha = P(Y_\alpha, I, O_\alpha)$ and $|\mathcal{A}| \leqslant \mathfrak{m}$.*

Corollary 4.3. *For a continuous map $f : X \to Y$ the following are equivalent:*

1. *The map f is Tychonoff;*

2. *There exists a homeomorphic embedding-morphism of the map f into the projection of a TPTP $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$, where the EPTP $P_\alpha = P(Y, I, O_\alpha)$;*
3. *There exists a homeomorphic embedding-morphism of the map f into the projection of a TPTP $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$, where the EPTP $P_\alpha = P(Y_\alpha, I, O_\alpha)$.*

Corollary 4.4. *For a continuous map $f : X \to Y$ into a Hausdorff space Y the following are equivalent:*

1. *The map f is $T_{2\frac{1}{2}}$ and compact;*
2. *There exists a {closed homeomorphic embedding, homeomorphic embedding}-morphism of the map f into the projection of a TPTP $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$, where the EPTP $P_\alpha = P(Y, I, O_\alpha)$;*
3. *There exists a {closed homeomorphic embedding, homeomorphic embedding}-morphism of the map f into the projection of a TPTP $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$, where the EPTP $P_\alpha = P(Y_\alpha, I, O_\alpha)$.*

We end this section by a universal type theorem for T_0-maps in $\mathcal{MAP}$, an analogue to Theorem 1.2 in $\mathcal{TOP}$. By the space F we denote the two point set $\{0, 1\}$ with the topology consisting of the empty set, the set $\{0\}$ and the whole space.

Theorem 4.5. *For a T_0-map $f : X \to Y$ the following are equivalent:*

1. *The map f has weight $\mathfrak{w}(f) \leqslant \mathfrak{m}$ ($\mathfrak{m} \geqslant \aleph_0$);*
2. *There exists a homeomorphic embedding-morphism of the map f into the projection of a TPTP $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$, where the EPTP $P_\alpha = P(Y, F, Y)$ and $|\mathcal{A}| \leqslant \mathfrak{m}$;*
3. *There exists a homeomorphic embedding-morphism of the map f into the projection of a TPTP $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$, where the EPTP $P_\alpha = P(Y_\alpha, F, O_\alpha)$ and $|\mathcal{A}| \leqslant \mathfrak{m}$.*

Corollary 4.6. *For a continuous map $f : X \to Y$ the following are equivalent:*

1. *The map f is T_0;*
2. *There exists a homeomorphic embedding-morphism of the map f into the projection of a TPTP $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$, where the EPTP $P_\alpha = P(Y, F, Y)$;*
3. *There exists a homeomorphic embedding-morphism of the map f into the projection of a TPTP $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$, where the EPTP $P_\alpha = P(Y_\alpha, F, O_\alpha)$.*

5. Fan products

We recall the definition of fan product with respect to a collection of maps and an inverse system, introduced in [2].

Suppose we are given a collection of maps $f_\sigma : X_\sigma \to Y_\sigma$ for every $\sigma \in \Sigma$, where the indexing set Σ is directed by the relation $\leqslant$. We further suppose that we are given an inverse system $\{Y_\sigma, \lambda_\rho^\sigma, \Sigma\}$. We denote by P, the subspace of the Tychonoff product $\prod\{X_\sigma : \sigma \in \Sigma\}$ given by

$$\left\{\{x_\sigma\} : \lambda_\rho^\sigma(f_\sigma x_\sigma) = f_\rho x_\rho \text{ for every } \sigma, \rho \in \Sigma \text{ satisfying } \rho \leqslant \sigma\right\}.$$

We call this space, the *fan product of the spaces X_σ with respect to the maps f_σ and the inverse system* $\{Y_\sigma, \lambda_\rho^\sigma, \Sigma\}$. The space P is denoted by $\prod\left\{X_\sigma, f_\sigma, \{Y_\sigma, \lambda_\rho^\sigma, \Sigma\}\right\}$.

For every $\sigma \in \Sigma$, the restriction of the projection $pr_\sigma : \prod\{X_\sigma : \sigma \in \Sigma\} \to X_\sigma$ on the subspace P will be denoted by π_σ and is called the *projection of the fan product P to X_σ*. From the definition of fan product we have $\lambda_\rho^\sigma \circ f_\sigma \circ \pi_\sigma = f_\rho \circ \pi_\rho$ for every $\sigma, \rho \in \Sigma$ satisfying $\rho \leqslant \sigma$. In this way one can define a map $p : P \to \varprojlim\{Y_\sigma, \lambda_\rho^\sigma, \Sigma\}$, called the *projection of the fan product P to the limit of the inverse system* $\{Y_\sigma, \lambda_\rho^\sigma, \Sigma\}$, by

$$p = \prod\{f_\sigma \circ \pi_\sigma : \sigma \in \Sigma\}.$$

It is evident that the projections p and $\pi_\sigma, \sigma \in \Sigma$, are continuous maps. The projection p is called the *fibrewise product of the maps f_σ with respect to the inverse system* $\{Y_\sigma, \lambda_\rho^\sigma, \Sigma\}$ and is denoted by $\prod\left\{f_\sigma, \{Y_\sigma, \lambda_\rho^\sigma, \Sigma\}\right\}$. It is not difficult to see that for every point $y = \{y_\sigma\} \in \varprojlim\{Y_\sigma, \lambda_\rho^\sigma, \Sigma\}$, the preimage $p^{-1}y$ is homeomorphic to the Tychonoff product of the fibres $f_\sigma^{-1}y_\sigma$, that is $\prod\{f_\sigma^{-1}y_\sigma : \sigma \in \Sigma\}$.

Fan partial topological products were introduced in [3].

Definition 5.1. . Let $P_\sigma = P(Y_\sigma, Z_\sigma, O_\sigma)$ be an EPTP with base space Y_σ, fibre Z_σ and open set O_σ for every σ in some directed set Σ and let $p_\sigma : P_\sigma \to Y_\sigma$ be the corresponding projection of the EPTP P_σ. Also, let there be given an inverse system $\{Y_\sigma, \lambda_\rho^\sigma, \Sigma\}$. The fan product $P = \prod\left\{P_\sigma, p_\sigma, \{Y_\sigma, \lambda_\rho^\sigma, \Sigma\}\right\}$ is called the *Fan partial topological product ($\equiv$ FPTP) of the EPTPs $P_\sigma, \sigma \in \Sigma$, with respect to the inverse system* $\{Y_\sigma, \lambda_\rho^\sigma, \Sigma\}$. The fibrewise product $p = \prod\left\{p_\sigma, \{Y_\sigma, \lambda_\rho^\sigma, \Sigma\}\right\}$ of the projections p_σ with respect to the inverse system $\{Y_\sigma, \lambda_\rho^\sigma, \Sigma\}$ is called the *projection of the FPTP P onto its base.* The projection of the FPTP P onto the EPTP P_σ is denoted by π_σ.

We now formulate the main theorem of this section, an analogue of Theorem 1.1 in the category $\mathcal{MAP}$ with respect to fan products. Recall that in the above context, if Y_0 is a topological space and $Y_\sigma = Y_0$ for every $\sigma \in \Sigma$, and we further have the binding maps $\lambda_\rho^\sigma = \mathrm{id}_{Y_0}$ for every $\sigma, \rho \in \Sigma$ satisfying $\rho \leqslant \sigma$, then the inverse system $S(Y_0, \Sigma) = \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$ is called the *constant inverse system* of the space Y_0 on the set Σ and we have that the limit $\varprojlim\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$ is homeomorphic to Y_0.

Theorem 5.1. *For a Tychonoff map $f : X \to Y$ the following are equivalent:*

1. *The map f has weight $\mathfrak{W}(f) \leqslant \mathfrak{m}$ ($\mathfrak{m} \geqslant \aleph_0$);*
2. *There exists a homeomorphic embedding-morphism of the map f into the projection of a FPTP $P = \prod\{P_\sigma, p_\sigma, S(Y, \Sigma)\}$, where the EPTP $P_\sigma = P(Y, I, O_\sigma)$ and $|\Sigma| \leqslant \mathfrak{m}$;*
3. *There exists a homeomorphic embedding-morphism of the map f into the projection of a FPTP $P = \prod\{P_\sigma, p_\sigma, \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}\}$, where the EPTP $P_\sigma = P(Y_\sigma, I, O_\sigma)$ and $|\Sigma| \leqslant \mathfrak{m}$.*

We have the following corollaries to the above theorem. Since a $T_{2\frac{1}{2}}$ compact map is Tychonoff, we have:

Corollary 5.2. *For a $T_{2\frac{1}{2}}$ compact map $f : X \to Y$ the following are equivalent:*

1. *The map f has weight $\mathfrak{W}(f) \leqslant \mathfrak{m}$ ($\mathfrak{m} \geqslant \aleph_0$);*
2. *There exists a {closed homeomorphic embedding, homeomorphic embedding}-morphism of the map f into the projection of a FPTP $P = \prod\{P_\sigma, p_\sigma, S(Y, \Sigma)\}$, where the EPTP $P_\sigma = P(Y, I, O_\sigma)$ and $|\Sigma| \leqslant \mathfrak{m}$;*
3. *There exists a {closed homeomorphic embedding, homeomorphic embedding}-morphism of the map f into the projection of a FPTP $P = \prod\{P_\sigma, p_\sigma, \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}\}$, where the EPTP $P_\sigma = P(Y_\sigma, I, O_\sigma)$ and $|\Sigma| \leqslant \mathfrak{m}$.*

Corollary 5.3. *For a continuous map $f : X \to Y$ the following are equivalent:*

1. *The map f is Tychonoff;*
2. *There exists a homeomorphic embedding-morphism of the map f into the projection of a FPTP $P = \prod\{P_\sigma, p_\sigma, S(Y, \Sigma)\}$, where the EPTP $P_\sigma = P(Y, I, O_\sigma)$;*
3. *There exists a homeomorphic embedding-morphism of the map f into the projection of a FPTP $P = \prod\{P_\sigma, p_\sigma, \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}\}$, where the EPTP $P_\sigma = P(Y_\sigma, I, O_\sigma)$.*

Corollary 5.4. *For a continuous map $f : X \to Y$ the following are equivalent:*

1. *The map f is $T_{2\frac{1}{2}}$ and compact;*

2. *There exists a* {*closed homeomorphic embedding, homeomorphic embedding*}*-morphism of the map* f *into the projection of a FPTP* $P = \prod\{P_\sigma, p_\sigma, \boldsymbol{S}(Y_0, \Sigma)\}$, *where the EPTP* $P_\sigma = P(Y, I, O_\sigma)$;
3. *There exists a* {*closed homeomorphic embedding, homeomorphic embedding*}*-morphism of the map* f *into the projection of a FPTP* $P = \prod\{P_\sigma, p_\sigma, \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}\}$, *where the EPTP* $P_\sigma = P(Y_\sigma, I, O_\sigma)$.

Remark 5.1. One can note that contrary to Corollaries 4.2 and 4.4, in Corollaries 5.2 and 5.4 the Hausdorffness of the space Y is not necessary to ensure closedness of the *top* homeomorphic embedding.

Finally, we end this section by a universal type theorem for T_0-maps in $\mathcal{MAP}$ for fan poducts corresponding to Theorem 4.5. This is an analogue of Theorem 1.2 in the category $\mathcal{MAP}$ with respect to fan products.

Theorem 5.5. *For a* T_0*-map* $f : X \to Y$ *the following are equivalent:*

1. *The map* f *has weight* $\mathfrak{w}(f) \leqslant \mathfrak{m}$ $(\mathfrak{m} \geqslant \aleph_0)$;
2. *There exists a homeomorphic embedding-morphism of the map* f *into the projection of a FPTP* $P = \prod\{P_\sigma, p_\sigma, \boldsymbol{S}(Y, \Sigma)\}$, *where the EPTP* $P_\sigma = P(Y, F, Y)$ *and* $|\Sigma| \leqslant \mathfrak{m}$;
3. *There exists a homeomorphic embedding-morphism of the map* f *into the projection of a FPTP* $P = \prod\{P_\sigma, p_\sigma, \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}\}$, *where the EPTP* $P_\sigma = P(Y_\sigma, F, O_\sigma)$ *and* $|\Sigma| \leqslant \mathfrak{m}$.

Corollary 5.6. *For a continuous map* $f : X \to Y$ *the following are equivalent:*

1. *The map* f *is* T_0;
2. *There exists a homeomorphic embedding-morphism of the map* f *into the projection of a FPTP* $P = \prod\{P_\sigma, p_\sigma, \boldsymbol{S}(Y, \Sigma)\}$, *where the EPTP* $P_\sigma = P(Y, F, Y)$;
3. *There exists a homeomorphic embedding-morphism of the map* f *into the projection of a FPTP* $P = \prod\{P_\sigma, p_\sigma, \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}\}$, *where the EPTP* $P_\sigma = P(Y_\sigma, F, O_\sigma)$.

6. Zero-dimensional maps

Zero-dimensional maps were defined by the author in [3]. We note that this definition of zero-dimensional maps differs from that given in [8]. Using this definition, it was shown in [3] that many properties of zero-dimensional spaces can be generalized from the category $\mathcal{TOP}$ to the category $\mathcal{MAP}$. Below we will mainly concern ourselves with a universal type theorem for zero-dimensional maps.

Definition 6.1. Let there be given a continuous map $f : X \to Y$. A set $U \subset X$ is said to be *f-closed-open (f-clopen)*, if there exists an open subset O of Y such that $U \subset f^{-1}O$ and U is clopen in $f^{-1}O$.

Definition 6.2. Let there be given a continuous map $f : X \to Y$, where $X \neq \emptyset$. The map f is called *zero-dimensional* if it is a T_1-map and has a base $\mathfrak{B}_f$ consisting of f-clopen sets, where a map $f : X \to Y$ is said to be a T_1-map if for every two distinct points $x, x' \in X$ lying in the same fibre, each of the points x, x' has a neighborhood in X which does not contain the other point.

Note that if the set U is f-clopen then it is also open in X but is not necessarily closed in X. It is not difficult to see that every zero-dimensional map is Tychonoff.

Theorem 6.1. *If $f : X \to Y$ is a zero-dimensional map, then so is any submap $g : A \to B$, where $A \neq \emptyset$.*

We have the following results concerning Tychonoff products and fibrewise products of zero-dimensional maps.

Proposition 6.2. *The Tychonoff product $f = \prod\{f_\alpha : \alpha \in \mathcal{A}\} : X = \prod\{X_\alpha : \alpha \in \mathcal{A}\} \to Y = \prod\{Y_\alpha : \alpha \in \mathcal{A}\}$, where $\mathcal{A} \neq \emptyset$ and $X_\alpha \neq \emptyset$ for every $\alpha \in \mathcal{A}$, is zero-dimensional if and only if all the maps f_α are zero-dimensional.*

Proposition 6.3. *Let $p : P = \prod\{X_\sigma, f_\sigma, \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}\} \to \varprojlim\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$ be the fibrewise product of the maps f_σ with respect to the inverse system $\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$, where $\varprojlim\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\} \neq \emptyset$. If all the maps f_σ are zero-dimensional then the map p is also zero-dimensional.*

The following is a universal type theorem for zero-dimensional maps. This is an analogue of Theorem 1.3 in the category $\mathcal{MAP}$. By the space D we understand the two point set $\{0, 1\}$ with the discrete topology.

Theorem 6.4. *For a zero-dimensional map $f : X \to Y$ the following are equivalent:*

1. *The map f has weight $\mathfrak{W}(f) \leqslant \mathfrak{m}$ ($\mathfrak{m} \geqslant \aleph_0$);*
2. *There exists a homeomorphic embedding-morphism of the map f into the projection of a TPTP $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$, where the EPTP $P_\alpha = P(Y, D, O_\alpha)$ and $|\mathcal{A}| \leqslant \mathfrak{m}$;*
3. *There exists a homeomorphic embedding-morphism of the map f into the projection of a TPTP $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$, where the EPTP $P_\alpha = P(Y_\alpha, D, O_\alpha)$ and $|\mathcal{A}| \leqslant \mathfrak{m}$;*

4. *There exists a homeomorphic embedding-morphism of the map f into the projection of a FPTP $P = \prod\{P_\sigma, p_\sigma, \boldsymbol{S}(Y,\Sigma)\}$, where the EPTP $P_\sigma = P(Y, D, O_\sigma)$ and $|\Sigma| \leqslant \mathfrak{m}$;*
5. *There exists a homeomorphic embedding-morphism of the map f into the projection of a FPTP $P = \prod\left\{P_\sigma, p_\sigma, \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}\right\}$, where the EPTP $P_\sigma = P(Y_\sigma, D, O_\sigma)$ and $|\Sigma| \leqslant \mathfrak{m}$.*

We can write down the following corollary to the above theorem.

Corollary 6.5. *For a continuous map $f : X \to Y$, where $X \neq \emptyset$, the following are equivalent:*

1. *The map f is zero-dimensional;*
2. *There exists a homeomorphic embedding-morphism of the map f into the projection of a TPTP $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$, where the EPTP $P_\alpha = P(Y, D, O_\alpha)$;*
3. *There exists a homeomorphic embedding-morphism of the map f into the projection of a TPTP $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$, where the EPTP $P_\alpha = P(Y_\alpha, D, O_\alpha)$;*
4. *There exists a homeomorphic embedding-morphism of the map f into the projection of a FPTP $P = \prod\{P_\sigma, p_\sigma, \boldsymbol{S}(Y,\Sigma)\}$, where the EPTP $P_\sigma = P(Y, D, O_\sigma)$;*
5. *There exists a homeomorphic embedding-morphism of the map f into the projection of a FPTP $P = \prod\left\{P_\sigma, p_\sigma, \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}\right\}$, where the EPTP $P_\sigma = P(Y_\sigma, D, O_\sigma)$.*

Finally, the following result concerning Tychonoff compactifications was also given in [3]. Recall that a compact map $bf : b_fX \to Y$ is said to be a compactification of $f : X \to Y$ if there exists a {dense homeomorphic embedding}-morphism $\{\lambda, \mathrm{id}_Y\} : f \to bf$ [25, 26]. In this situation we usually identify X with $\lambda(X)$ and so $b_fX = [X]_{b_fX}$ and $f = bf|_X$, where by $[X]_{b_fX}$ we mean the closure of X in b_fX. For details concerning compactifications of Tychonoff maps, in particular the construction of βf, one can consult [18, 19, 13].

Theorem 6.6. *Every zero-dimensional map $f : X \to Y$ of weight $\mathfrak{W}(f) = \mathfrak{m} \geqslant \aleph_0$ has a zero-dimensional compactification $bf : b_fX \to Y$ of weight $\mathfrak{W}(bf) = \mathfrak{m}$.*

References

[1] P.S. Alexandroff, *Zur Theorie der topologischen Räume*, C.R. (Doklady) Acad. Sci. URSS **11** (1936), 55–58.

[2] D. Buhagiar, *The category $\mathcal{MAP}$*, submitted for publication.

[3] ———, *Partial topological products in $\mathcal{MAP}$*, submitted for publication.

[4] ———, *The Tamano Theorem in $\mathcal{MAP}$*, to appear in Comment. Math. Univ. Carolinae.

[5] ———, *Paracompact maps*, Q & A in General Topology **15** (1997), no. 2, 203–223.
[6] D. Buhagiar and T. Miwa, *Covering properties on maps*, Q & A in General Topology **16** (1998), no. 1, 53–66.
[7] D. Buhagiar, T. Miwa, and B.A. Pasynkov, *On metrizable type (MT-) maps and spaces*, to appear in Topology Appl.
[8] R. Engelking, *General Topology*, revised ed., Heldermann, Berlin, 1989.
[9] I.M. James, *Spaces*, Bull. London Math. Soc. **18** (1986), 529–559.
[10] ———, *Fibrewise topology*, Cambridge Univ. Press, Cambridge, 1989.
[11] P.T. Johnstone, *The Gleason cover of a topos II*, J. Pure and Appl. Algebra **22** (1981), 229–247.
[12] ———, *Wallman compactification of locales*, Houston J. Math. **10** (1984), 201–206.
[13] H.P.A. Künzi and B.A. Pasynkov, *Tychonoff compactifications and R-completions of mappings and rings of continuous functions*, Categorical Topology (L'Aquila, 1994), Kluwer Acad. Publ., Dordrecht, 1996, pp. 175–201.
[14] D. Lever, *Continuous families: Categorical aspects*, Cahiers de topologie et géométrie différentielle **24** (1983), 393–432.
[15] ———, *Relative topology*, Categorical Topology, Proc. Conference Toledo, Ohio, 1983, Heldermann, Berlin, 1984.
[16] B.A. Pasynkov, *Partial topological products*, Doklady Akad. Nauk SSSR **154** (1964), 767–770.
[17] ———, *Partial topological products*, Trans. Moscow Math. Soc. **13** (1965), 153–272.
[18] ———, *On extension to mappings of certain notions and assertions concerning spaces*, Mappings and Functors, Izdat. MGU, Moscow, 1984, in Russian, pp. 72–102.
[19] ———, *Elements of the general topology of continuous maps*, On Compactness and Completeness Properties of Topological Spaces, "FAN" Acad. of Science of the Uzbek. Rep., Tashkent, 1994, in Russian, pp. 50–120.
[20] G. Preuss, *Theory of topological structures (an approach to categorical topology)*, Mathematics and its Applications, D.Reidel Publishing Company, Dordrecht, Holland, 1987.
[21] A. Tychonoff, *Über die topologische Erweiterung von Räumen*, Math. Ann. **102** (1930), 544–561.
[22] V.M. Ulyanov, *On compact extensions of countable character and absolutes*, Math. Sb. **98** (1975), no. 2, 223–254, in Russian.
[23] I.A. Vainstein, *On closed maps of metric spaces*, Doklady Akad. Nauk SSSR **57** (1947), 319–321, in Russian.
[24] N. Vedenissoff, *Remarks on the dimension of topological spaces*, Uch. Zapiskii Mosk. Univ. **30** (1939), 131–140, in Russian.
[25] G.T. Whyburn, *A unified space for mappings*, Trans. Amer. Math. Soc. **74** (1953), no. 2, 344–350.
[26] ———, *Compactification of mappings*, Math. Ann. **166** (1966), no. 1, 168–174.

David Buhagiar, Department of Mathematics, Okayama University, Okayama 700-8530, Japan
E-mail address: buhagiar@math.okayama-u.ac.jp