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Normally supercompact spaces and completely distributive poset *

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Abstract

A Hausdorff space X is called normally supercompact (NS for short) if it has a subbase $\mathcal{S}$ such that (1) every cover consisting of elements of $\mathcal{S}$ has a subcover consisting of at most two elements, and (2) for any pair A, B of elements of $\mathcal{S}$ if $A \cup B = X$ then there exist $C, D \in \mathcal{S}$ such that $C \cap D = \emptyset$ and $A \cup C = B \cup D = X$. A poset L is called a completely distributive poset (CDP for short) if (3) every nonempty subset has the inf, (4) every subset in which every pair has an upper bounded has the sup, and (5) the distributive law holds for any existing sups and existing infs. In this paper, we prove that the category of all NS spaces and the category of all CDP's are isomorphic. As a result we deduce that the order in a connected compact linearly ordered space is unique. Moreover, we also set a corresponding result for zero-dimensional NS spaces. In particular, we show that a space is zero-dimensional NS if and only if it has a subbase consisting of clopen sets satisfying (1) and (2).

Key words: Normally supercompact space, completely distributive poset, Lawson topology, zero-dimensional space, Cantor cube

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§1 Introduction

In this paper, all spaces are assumed to be Hausdorff topological spaces. In a space X, a family $\mathcal{S}$ of subsets is called a *closed subbase* if $\{X \backslash S : S \in \mathcal{S}\}$ is a subbase for X. A family $\mathcal{S}$ of subsets is called *linked* if every pair of elements of $\mathcal{S}$ has a nonempty intersection. A

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family of subsets is called *binary* if its every linked subfamily has a nonempty intersection. A space is called *supercompact* if it has a binary closed subbase [5]. By the Alexander subbase lemma every supercompact space is compact. All continuous images of linearly ordered compacta are supercompact [3]. On the other hand, there exist many compact spaces which are not supercompact. In fact, in [13] and [14] the authors proved that in every supercompact space every non-P-point[1] is the limit of a nontrivial sequence. A family $\mathcal{S}$ is called *normal* if for every pair of disjoint elements A, B of $\mathcal{S}$ there exist $C, D \in \mathcal{S}$ such that $C \cup D = X$ and $D \cap A = C \cap B = \emptyset$. A space X with a normal binary closed subbase is called *normally supercompact* (*NS* for short) [7]. NS spaces have very rich "geometric" structures. For example, if they are connected then they are locally connected and generalized arcwise connected. Moreover, a NS space is an absolute retract if and only if it is an absolute neighborhood retract, if and only if it is connected and metrizable. In [6] van Mill defined a partial order on a NS space. Many people have studied NS spaces using this partial order [7][9][11][12]. In [12] we used this order to set up a correspondence between NS spaces and partially ordered sets which are very like completely distributive lattices. In the present paper, we prove that this correspondence can be extended to an isomophic between a category of NS spaces and a category of CDPs. As a corollary, we deduce that in a connected linearly ordered space the family of all closed intervals is the unique normal binary closed subbase which is closed under arbitrary intersections. Moreover, we show that every zero-dimensional NS space has a normal binary closed subbase consisting of clopen sets.

§2 Preliminaries

In this section we present some basic concepts and results on order theory. Please see [4] for more information about this topic.

Let L be a partially ordered set (*poset* for short). For $A \subset L$, we denote the infimum of A in L, if it exists, by $infA$. If $A = \{a_1, a_2, \cdots, a_n\}$, $a_1 \wedge a_2 \wedge \cdots \wedge a_n$ instead of $infA$. Similarly, for supremum, by $supA$ and $a_1 \vee a_2 \vee \cdots \vee a_n$ respectively. The least element of L is denoted by $\perp$ if it exists. For $a, b \in L$, we say that a is *way-below to* b, in symbole $a \ll b$, if for every directed set $D \subset L$ with $supD \geq b$, there exists $d \in D$ such that $d \geq a$. If $a \ll a$, then a is called *compact*. The set of all compact elements of L is denoted by $C(L)$. An element $m \neq \perp$ is called a *co-prime element* if $m \leq a \vee b$ implies $m \leq a$ or $m \leq b$. The set of all co-prime elements of L is denoted by $M(L)$. For $A \subset L$, let $\downarrow A = \{x \in L : x \leq a$ for some $a \in A\}$. In particular, $\downarrow a = \downarrow \{a\}$. Let $\Downarrow a = \{x \in L : x \ll a\}$. Dually, we can define $\uparrow A, \uparrow a$ and $\Uparrow a$. A complete lattice is called a *continuous lattice* (*completely distributive lattice* or *CDL* for short, respectively) if the distributive law for arbitrary infs and arbitrary directed sups (arbitrary sups, respectively) holds. It is well-known that a complete lattice L is a continuous lattice (completely distributive lattice, respectively) if

[1]A point p in a space is called a *P-point* if $p \notin (\bigcup \mathcal{C})^- \setminus \bigcup \mathcal{C}$ for any countable family $\mathcal{C}$ of closed sets.

and only if $x = sup \Downarrow x$ ($x = sup(M(L) \cap \Downarrow x)$, respectively) for any $x \in L$. In [12], in order to characterize NS spaces we defined the concept of completely distributive poset. A subset A of a poset L is called *relatively directed* if for every pair $a, b \in A$ there exists $x \in L$ such that $a, b \leq x$. A poset is called a *completely distributive poset* (*CDP* for short) if
(CDP 1) every nonempty set has the inf,
(CDP 2) every relatively directed set has the sup, and
(CDP 3) the distributive law holds for arbitrary infs and arbitrary relatively directed sups.
A CDP is called *algebraic* if every element is the sup of compact elements. A subset U of a poset L is called *Scott-open* if $U = \uparrow U$ and $L \setminus U$ is closed under directed sups. The family of all Scott-open sets and all sets of the form $L \setminus \uparrow x$ generates a topology on L, which is called the *Lawson topology* and denoted by ΛL. If L is a continuous lattice or a CDP then ΛL is a compact Hausdorff(!) space. For a CDP L, let $L^* = L \cup \{*\}$ and order L^* such that $*$ is the last element and L is a subposet of L^*, then L^* is a continuous lattice. But such L^* is not necessarily a completely distributive lattice. However if L is a CDP, then for every $x \in L$, $\downarrow x$ is a completely distributive lattice. Thus CDP's enjoy some but not all properties of CDL's. For our purpose, we need the following facts.

Fact 1 *If L is a CDP, then $x = sup(\Downarrow x \cap M(L))$ for all $x \in L$.*

Fact 2 *If L is a CDL, then ΛL coincides with the interval topology generated by $\{\downarrow x : x \in L\} \cup \{\uparrow x : x \in L\}$ as a closed subbase. But this statement is not necessarily true for CDPs.*

We shall use I to denote the interval $[0, 1]$. For any set T, I^T, with the pointwise order, is a CDL and hence a CDP. ΛI^T is the usual product space and with the *canonical closed subbase* consisting of all forms of $\prod_{t \in T}[a_t, b_t]$, where $0 \leq a_t \leq b_t \leq 1$. (There is a slight difference between the definition here and the one in [11].)

Now for a NS space X, we fix a normal binary closed subbase $\mathcal{S}$. Let $\mathcal{S}^\cap$ be the family of all intersections of elements of $\mathcal{S}$. Then $\mathcal{S}^\cap$ is also a normal binary closed subbase and is closed under arbitrary intersection. We call such family a NS structure on a NS space. That is a *NS structure* on a NS space is a normal binary closed subbase which is closed under arbitrary intersection. By Hausdorff separation, every NS structure contains all singletone sets. In [7], on a NS space X with a fixed NS structure $\mathcal{S}$ and a fixed point $\perp \in X$ a partial order with the least element $\perp$ is defined as follows. For $a, b \in X$, we use $I(a, b)$ or $I_{\mathcal{S}}(a, b)$ (if necessary) to denote $\bigcap\{S \in \mathcal{S} : a, b \in S\}$. Define $a \leq_{\mathcal{S}} b$ if $a \in I_{\mathcal{S}}(\perp, b)$. In [12] the author proved the following facts :

Fact 3 *$(X, \leq_{\mathcal{S}})$ is a CDP and the original topology on X coincides with the Lawson topology of $(X, \leq_{\mathcal{S}})$.*

Fact 4 *For any CDP $(L, \leq)$, the family $\{\uparrow m : m \in M(L)\} \cup \{L \setminus \Uparrow m : m \in M(L)\}$ is a normal binary closed subbase for $\Lambda(X, \leq)$ and thus $\Lambda(L, \leq)$ is a NS space.*

For a CDP $(L, \leq)$, let $\mathcal{S}_{\leq}$ be the NS structure generated by the above normal binary closed subbase. The following facts proved in [6][12] are also needed in proving our results.

Fact 5 *For any $b \in X$ and $A \subset X$, $\bigcap\{S \in \mathcal{S} : S \supset A\} \cap \bigcap\{I(b,a) : a \in A\}$ is a single point set.*

Fact 6 *Every element of $\mathcal{S}$ is closed under arbitrary infs and arbitrary existing sups according to $\leq_{\mathcal{S}}$.*

Fact 7 *If X is connected then every element of $\mathcal{S}$ is connected.*

§3 The isomorphism theorem

By a *NS structural space* we mean a triplet $(X, \mathcal{S}, \perp)$, where X is a NS space, $\mathcal{S}$ is a NS structure on X and $\perp \in X$ is a point. For two NS structural spaces $(X, \mathcal{S}, \perp)$ and $(Y, \mathcal{T}, \perp)$, a mapping $f : X \longrightarrow Y$ is called a NS mapping if $f(\perp) = \perp$ and $f^{-1}(T) \in \mathcal{S}$ for any $T \in \mathcal{T}$. Let **NS** be the category consisting of all NS structural spaces and all NS mappings. This category was defined and studied in [10]. Let **CDP** be the category consisting of all CDP's and all mappings preserving existing infs and sups. Then we have the following theorem:

Theorem 1 *There exist two functors Ψ : **NS** $\longrightarrow$ **CDP** and Φ : **CDP** $\longrightarrow$ **NS** such that $\Psi \circ \Phi = id_{\mathbf{CDP}}$ and $\Phi \circ \Psi = id_{\mathbf{NS}}$.*

Proof. For each $(X, \mathcal{S}, \perp) \in \mathbf{NS}$, we define

$$\Psi(X, \mathcal{S}, \perp) = (X, \leq_{\mathcal{S}}, \perp).$$

For a mapping f in **NS**, we define $\Psi(f) = f$. For $(X, \leq, \perp) \in \mathbf{CDP}$, we define

$$\Phi(X, \leq, \perp) = (X, \mathcal{S}_{\leq}, \perp).$$

For a mapping f in **CDP**, we define $\Phi(f) = f$. It follows from Fact 3 and Fact 4 that the two functors are well-defined for the objects. The remainder of proof of the theorem follows from the following lemmas:

Lemma 1 *If a mapping $f : (X, \mathcal{S}, \perp) \longrightarrow (Y, \mathcal{T}, \perp)$ is in* **NS**, *then $f : (X, \leq_{\mathcal{S}}, \perp) \longrightarrow (Y, \leq_{\mathcal{T}}, \perp)$ is in* **CDP**.

Proof As the topologies on X and Y are $\Lambda(X, \leq_{\mathcal{S}}, \perp)$ and $\Lambda(Y, \leq_{\mathcal{T}}, \perp)$ respectively, and f is continuous, we have that f preserves all down-directed infs and directed sups. It suffices to verify that f preserves finite infs and finite existing sups. Notice that f preserves order. Now for any $a, b \in X$ we have $f(a \wedge b) \leq f(a) \wedge f(b)$. If $f(a) \wedge f(b) \not\leq f(a \wedge b)$, then $f(a) \wedge f(b) \notin \downarrow f(a \wedge b)$. Thus there exist $T_1, T_2 \in \mathcal{T}$ such that $T_1 \cup T_2 = Y$ and $T_1 \cap \{f(a) \wedge f(b)\} = T_2 \cap \downarrow f(a \wedge b) = \emptyset$. Let $S_1 = f^{-1}(T_1), S_2 = f^{-1}(T_2)$. Then $S_1, S_2 \in \mathcal{S}$ and $S_1 \ni a \wedge b, \perp$, but $a \notin S_1$ nor $b \notin S_1$. Thus $a, b \in S_2$, but $a \wedge b \notin S_2$, which contradicts Fact 6. Similarly, f preserves finite existing sups.

Lemma 2 *If a mapping* $f : (X, \leq, \perp) \longrightarrow (Y, \leq, \perp)$ *is in* **CDP**, *then* $f : (X, \mathcal{S}_\leq, \perp) \longrightarrow (Y, \mathcal{S}_\leq, \perp)$ *is in* **NS**.

Proof. It suffices to verify that $A = f^{-1}(\uparrow y)$ and $B = f^{-1}(Y \setminus \Uparrow y)$ are in $\mathcal{S}_\leq$ for any $y \in M(Y)$. It is easy to show that $A = \emptyset$ or $A = \uparrow infA$. Thus $A \in \mathcal{S}_\leq$. In order to show $B \in \mathcal{S}_\leq$, suppose $x \notin B$. Then Fact 1 implies $y \ll f(x) = sup\{f(m) : m \ll x$ and $m \in M(X)\}$. Hence, it follows from $y \in M(Y)$ that there exists $m \in M(X)$ such that $m \ll x$ and $y \ll f(m)$. Thus $x \notin X \setminus \Uparrow m \supset B$. Moreover, $X \setminus \Uparrow m \in \mathcal{S}_\leq$. So, $B \in \mathcal{S}_\leq$. ∎

Lemma 3 $\Psi \circ \Phi = id_{\mathbf{CDP}}$ *and* $\Phi \circ \Psi = id_{\mathbf{NS}}$.

Proof. We only prove the second equation, that is, for any NS structural space $(X, \mathcal{S}, \perp)$, $\mathcal{S}$ is the smallest NS structure including $\mathcal{A} = \{X \setminus \Uparrow m : m \in M(X)\} \bigcup \{\uparrow m : m \in M(X)\}$, where the partial order on X is $\leq_\mathcal{S}$. At first $\mathcal{A} \subset \mathcal{S}$. For any $x, y \in X$, if $y \notin \uparrow x$, there exists $S \in \mathcal{S}$ such that $y, \perp \in S$ but $x \notin S$. The normality of $\mathcal{S}$ implies that there exist $A, B \in \mathcal{S}$ such that $A \bigcup B = X$ and $A \bigcap S = B \bigcap \{x\} = \emptyset$. Then $\uparrow x \subset A \not\ni y$. In fact, otherwise, there exists $z \in \uparrow x \cap B$. It follows from $\perp \in B$ that $x \in B$, a contradiction. Thus $\uparrow x$ is an intersection of elements of $\mathcal{S}$ and hence it is in $\mathcal{S}$. Now for any $m \in M(X)$ and $y \notin X \setminus \Uparrow m$, let $x = sup(\downarrow y \cap (X \setminus \Uparrow m))$, which exists since the set is included in $\downarrow y$. Then it follows from $m \in M(X)$ that $x \in X \setminus \Uparrow m$. Thus $\downarrow x \cap \{y\} = \emptyset$. By the normality of $\mathcal{S}$ there exist $A, B \in \mathcal{S}$ such that $A \cup B = X$ and $A \cap \downarrow x = B \cap \{y\} = \emptyset$. Then $y \notin B \supset X \setminus \Uparrow m$. In fact, otherwise, choose $z \in (X \setminus \Uparrow m) \cap A$ then $y \wedge z \in X \setminus \Uparrow m$ and hence $y \wedge z \notin A$, a contradiction. Thus $X \setminus \Uparrow m$ is also an intersection of elements of $\mathcal{S}$ and hence it is in $\mathcal{S}$. Secondly, for any $S \in \mathcal{S}$ and any $x \notin S$, there exists $A \in \mathcal{A}$ such that $A \supset S$ and $x \notin A$. In fact, let $y = infS$. If $y \not\leq x$, then $A = \uparrow y$ satisfies the conditions. If $y \leq x$, then, by the normality of $\mathcal{S}$, there exist $C, D \in \mathcal{S}$ such that $C \cup D = X$ and $C \cap S = D \cap \{x\} = \emptyset$. Then $D \ni \perp$. It follows from Fact 6 and $x = sup(M(X) \cap \Downarrow x)$ that $M(X) \cap \Downarrow x \not\subset D$. Thus there exists $m \in M(X) \cap \Downarrow x \cap C$. Let $A = X \setminus \Uparrow m$. Then A satisfies the conditions. So, we have proved that S is an intersection of elements of $\mathcal{A}$.

From above theorem and its proof it is natural to wonder whether the NS structures of a space are all identical. At first we note that for any NS structure $\mathcal{S}$ on a space X and any homeomorphism $h : X \longrightarrow X$, $h(\mathcal{S}) = \{h(S) : S \in \mathcal{S}\}$ is also a NS structure on X and $(X, \leq_\mathcal{S}, \perp)$ is a CDL if and only if so is $(X, \leq_{h(\mathcal{S})}, h(\perp))$. Moreover, for $X = I \times I$, it is easy to give a homeomorphism $h : X \longrightarrow X$ such that $h(\mathcal{S}) \neq \mathcal{S}$, where $\mathcal{S}$ is the canonical closed subbase on $I \times I$. NS structures on $I \times I$ are not unique. Thus we ask whether every NS structure on a space may be a homeomorphic image of a fixed NS structure. The answer is no. We give two counterexamples. One is a linearly ordered space and the other is $I \times I$.

Example 1 Let $X = \{*\} \cup L$, where $* \notin L$ and $L = \omega_1 \times [0, 1) \cup \{\omega_1\}$ is ordered in such a way that $\omega_1 \times [0, 1)$ is in the lexicographical order and ω_1 is the last element. Topologize X such that $*$ is an isolated point and L has the ordering topology. Then X is a NS space. We extend the order on L into two linear orders on X such that $*$ is the last element and the least element, respectively. The two linear orders produce two NS structures on X. It is easy to see that these two NS structures are not homeomorphic. On the other hand, by the following Corollary 1, these are the only NS structures on X.

Example 2 Let $X = I \times I$ and $\mathcal{S}$ the canonical closed subbase on X. Let $\mathcal{T}$ be the family of all polygons whose sides are sides of $I \times I$ or straight lines with the inclinations 1 or -1. Then $\mathcal{T}$ is a NS structure on X (cf. [11]). It is trivial to check that $(X, \leq_{\mathcal{S}}, (0,0))$ is a CDL but $(X, \leq_{\mathcal{T}}, (a,b))$ is not a CDL for any $(a,b) \in X$. Thus $\mathcal{S}$ and $\mathcal{T}$ are two NS structures on X and each of them is not any homeomorphic image of the other.

But we have the following theorem.

Theorem 2 *Let X be a NS space and $\mathcal{S}, \mathcal{T}$ two NS structures on X. If $\mathcal{S} \subset \mathcal{T}$, then $\mathcal{S} = \mathcal{T}$.*

Proof. Suppose that there exists $A \in \mathcal{T} \setminus \mathcal{S}$. Let $B = \bigcap\{S \in \mathcal{S} : S \supset A\}$. Choose $b \in B \setminus A$. Then, by Fact 5, $B \cap \bigcap_{x \in A} I_{\mathcal{S}}(x,b) = \{b\}$. Since $\mathcal{S} \subset \mathcal{T}$, we have $I_{\mathcal{T}}(x,b) \subset I_{\mathcal{S}}(x,b)$ for all $x \in A$. Moreover, $A \subset B$ and $b \notin A$. Thus $A \cap \bigcap_{x \in A} I_{\mathcal{T}}(x,b) = \emptyset$. This contradicts the assumption that $\mathcal{T}$ is binary.

Corollary 1 *For a connected compact linearly ordered space X, $\{[a,b] : a, b \in X\}$ is the unique NS structure on X.*

Proof. Let X be a connected compact linearly ordered space and $\mathcal{S}$ a NS structure on X. It follows from Fact 7 that $\mathcal{S} \subset \{[a,b] : a,b \in X\}$. Thus $\mathcal{S} = \{[a,b] : a,b \in X\}$. ∎

Corollary 2 *For a connected compact linearly ordered space $(X, \leq)$, $\leq$ is the unique partial order R such that (X, R) is a CDP and has the same least element with $(X, \leq)$ and $\Lambda(X, \leq) = \Lambda(X, R)$.*

Proof. This follows directly from Theorem 1 and the above corollary. ∎

Another application of the above theorems is to show that there exists a NS space which is not homeomorphic to any CDL with the Lawson topology or equivalent with the interval topology. In fact, Let $X = \{(a,b,c) \in I^3 : a = b = 0 \text{ or } b = c = 0 \text{ or } c = a = 0\}$. Using the same method as the proof of Corollary 1, it may be proved that $\mathcal{S}|X$, where $\mathcal{S}$ is the canonical closed subbase for I^3, is the unique NS structure on X and hence X is a NS space. However, for any $\perp \in X$, $(X, \leq_{\mathcal{S}|X}, \perp)$ is not a CDL. Thus, by Theorem 1, X is not homeomorphic to any CDL with the Lawson topology.

§4 Zero-dimensional NS spaces

In this section we at first show that the two functors defined in the last section restrict to isomorphisms between the category of all zero-dimensional NS structural spaces and that of all algebraic CDP's. Then we prove that every zero-dimensional NS space has a normal binary closed subbase consisting of clopen sets. Let **ONS** be the full subcategory of **NS** consisting of all zero-dimensional NS structural spaces and **ACDP** the full subcategory of **CDP** consisting of all algebraic CDP's.

Theorem 3 Ψ : **ONS** $\longrightarrow$ **ACDP** *and* Φ : **ACDP** $\longrightarrow$ **ONS** *are isomorphic.*

Proof. It is well-known that ΛL is zero-dimensional if and only if L is algebraic for any continuous lattice L (see, for example, [4]). Thus the proof follows from Theorem 1. ∎

Theorem 4 *For a space X, the following statements are equivalent:*
(1) X is zero-dimensional NS;
(2) X has a normal binary closed subbase consisting of clopen sets;
(3) X has binary closed subbase which is closed under complements;
(4) X is homeomorphic to ΛL for some algebraic CDP L.

Proof. (1)$\Rightarrow$(4) has been proved in the above theorem. (3)$\Rightarrow$(2) and (2)$\Rightarrow$(1) are trivial. Now we show (4)$\Rightarrow$ (3). Suppose that X is homeomorphic to $\Lambda(L,\leq)$, where L is a algebraic CDP. Let $B = C(L) \cap M(L)$. We show that B satisfies the following condition:
(SB) For every $x \in L, x = sup\{b \in B : b \ll x\}$.
In fact, for any $c \in C(L)$, since $c = sup\{m \in M(L) : m \leq c\}$, there exists a finite set $A \subset M(L)$ such that $c = supA$. Without loss of generality, we may assume A is an anti-chain. Then $A \subset B$. In fact, for any $a \in A$ and any directed D with $a = supD$. Then $sup((A\setminus\{a\}) \cup D) = supA = c$. Hence there exists a finite subset $F \subset (A\setminus\{a\}) \cup D$ such that $c \leq supF$. By $a \in M(L)$ and $a \leq c$ we have $a \leq f$ for some $f \in F$. Then $f \in D$ since elements of A are not comparable. This shows $a \in C(L)$. Hence $a \in B$. It follows that B satisfies (SB) since L is algebraic. By Lemma 2.8 in [12] we have

$$\mathcal{S}_B = \{L\setminus \uparrow b : b \in B\} \cup \{\uparrow b : b \in B\}$$

is a closed subbase for ΛL. Moreover, by $B \subset M(L)$ and Fact 4 we have $\mathcal{S}_B$ is binary. Thus ΛL has a binary closed subbase which is closed under complements. ∎

Remark 1 *In [1], Bell and Ginsburg gave an example to show that not every zero-dimensional supercompact space has a binary closed subbase consisting of clopen sets.*

Remark 2 *In [8] a pair $(X, \mathcal{S})$ is called an orthopair if X is a Hausdorff space and $\mathcal{S}$ is a subbase for X satisfying (3) in the above theorem. Ovchinnikov proved in [8] that there is a bijective correspondence between the orthoposets (see [2]) and orthopairs. Hence, he set up an exact topological analogs to orthoposets which is similar to the Stone Representation Theorem for Boolean algebra.*

Every NS space can be embedded into a Hilbert cube I^T as a special subspace. In 1978, van Mill and Wattel in [7] showed that a space is a NS space if and only if it can be embedded into I^T as a closed and triple convex subspace. (A subset A of I^T is called *triple convex* if $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x) \in A$ for all $x, y, z \in A$.) In 1992, Szymanski in [11] showed that this is equivalent to ask that the restriction to this subspace of the canonical closed subbase of I^T is binary. In 1993, the author in [12] showed that a space is NS if and only if it can be embedded into a Hilbert cube as a subspace which is closed with arbitrary infs and arbitrary relatively directed sups. It is trivial that such subspace is closed and triple convex. But the converse is not true. For example, the anti-diagonal in $I \times I$ is closed and triple convex but not closed with finite infs. Now we consider embedding of zero-dimensional NS space into a Cantor cube 2^T.

Theorem 5 *For a topological space X the following conditions are equivalent:*
(1) X is a zero-dimensional NS space;
(2) X can be embedded into a Cantor cube 2^T as a closed and triple convex subspace;
(3) X can be embedded into a Cantor cube 2^T as a subspace which is closed with arbitrary infs and arbitrary relatively directed sups.

Proof. (3)⇒(2) and (2)⇒(1) are trivial (cf. [11][12]). We have only to show (1)⇒(3). We suppose that L is a algebraic CDL and $X = \Lambda(L, \leq)$. By the proof of Theorem 4 we have that $\mathcal{S}_B$ is a closed subbase for the space X. For every $b \in B$, let $f_b : X \longrightarrow 2 = \{0, 1\}$ by $f_b(x) = 1$ if and only if $b \leq x$. This generates a continuous one-to-one mapping $F : X \longrightarrow 2^B$. It is not difficult to verify the image of X is closed with any finite infs and finite relatively directed sups in 2^B. Moreover, the image of X is compact subspace of 2^B. Thus, it is closed with arbitrary infs and relatively directed sups. ∎

It is well-known that every zero-dimensional compact space of the countable weight can be embedded into the Cantor set 2^N as a closed subspace and hence it is homeomorphic to a CDL with the interval topology. But this statement is not true for zero-dimensional NS spaces of larger weights. In fact, let $A(m)$ be the one-point compactification of the discrete space of weight m. Then $A(m)$ is a zero-dimensional NS space but $A(m)$ is not homeomorphic to any CDL with the interval topology unless m is countable.

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