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FIXED POINT THEOREMS AND THE EXISTENCE OF ECONOMIC EQUILIBRIA BASED ON CONDITIONS FOR LOCAL DIRECTIONS OF MAPPINGS

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Abstract

Fixed point theorems for set valued mappings are reexamined from a unified viewpoint on the local direction of mappings. Several important fixed point theorems are generalized so that we could apply them to game theoretic and economic equilibrium existence problems with non-ordered preferences having neither global continuity nor convexity conditions.

Keywords : General equilibrium, Excess demand, Nash equilibrium, Abstract economy, Gale-Nikaido-Debreu Theorem, Kakutani's Fixed point theorem, Browder's Fixed Point Theorem, Eaves' Theorem.

1 INTRODUCTION

In this paper, fixed point theorems for set valued mappings are reexamined from a unified viewpoint on the local directions of mappings, i.e., the sets, $\varphi(z) - z$, of a correspondence $\varphi : X \ni x \mapsto \varphi(x) \subset X$ for all z in a certain neighbourhood of x. Famous fixed point theorems such as the theorem of Kakutani (1941), Fan (1952), Glicksberg (1952), and Theorem 1 of Browder (1968), etc., may be considered as a special case of the main theorem, so that we could apply it to game theoretic and economic equilibrium existence problems with (possibly) non-ordered preferences having neither global continuity (such as lexicographic ordering preferences) nor convexity conditions, intrinsically (in the sense that we do not even assume $x \notin \mathrm{co}\,\varphi(x)$).

In section 2, the main fixed point theorem and its corollaries are proved. Amongst all, the case with condition (K*) in Theorem 1 gives a simple and powerful extension of Kakutani-Fan-Glicksberg's thorem and Browder's theorem (Browder (1968; Theorem 1)), and also gives a partial generalization of the concept of $\mathscr{L}$-majolized maps the notion frequently used in resent mathematical economics literature.

In section 3, the Nash equilibrium existence problem (c.f. Nash (1950), Nikaido (1959), Nishimura and Friedman (1981), etc.,) and the social equilibrium existence problem (c.f. Debreu (1952), Shafer and H.F.Sonnenschein (1975), Yannelis and Prabhakar (1983), etc.,) are reexamined. By applying the main theorem, we may obtain some of the most general results for these problems (e.g. see Theorem 5, Corollary 5.2). From the economic viewpoint, however, the most interesting result among these may be Corollary 5.1 of Theorem 5, which gives us a clear condition for the existence of economic equilibria with (intrinsically) non-convex non-ordered preferences.

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Section 4 is devoted to the market equilibrium existence theorems known as Gale-Nikaido-Debreu Theorem (c.f. Debreu (1956), Nikaido (1959), Mehta and Tarafdar (1987), etc.)

In this paper, all vector spaces are assumed to be over the real field R. The duality between two vector spaces E and F will be denoted by $\langle F, E\rangle$. Typically, F may be considered as the algebraic dual E^* or the topological dual E' of E when E is a locally convex space. All concepts and definitions for vector spaces will be used in the sense of Schaefer (1971).

2 FIXED POINT THEOREMS

Throughout this section, we denote by E a Hausdorff topological vector space over R. The algebraic dual of E is denoted by E^* and the topological dual of E is denoted by E' when E is a locally convex space. At first, we show the main fixed point theorem of this paper. (Case (K1) is a thorem of Urai and Hayashi (1997), and some special cases of (K2) and (K3) are shown in Urai (1998; Theorem 8.1).)

Theorem 1 : Let X be a non-empty compact convex subset of E, and let φ be a non-empty valued correspondence on X to X. Denote by K the set $\{x \in X \mid x \notin \varphi(x)\}$. Suppose that E and φ satisfy one of the following conditions:

(K1) E is a locally convex space, and for each $x \in K$, there exist a vector $p(x) \in E'$ and a neighbourhood $U(x)$ of x in X satisfying that $\forall z \in U(x)$, if $z \notin \varphi(z)$, then $\varphi(z) - z \subset \{v \in E \mid \langle p(x), v\rangle > 0\}$.

(K2) For each $x \in K$, we may define a vector $p(x) \in E^*$ such that $\varphi(x) - x \subset \{v \in E \mid \langle p(x), v\rangle > 0\}$. Moreover, for each $x \in K$, there are a point $y(x)$ in X and a neighbourhood $U(x)$ of x in X such that $\forall z \in U(x)$, if $z \in K$, then $\langle p(z), y(x) - z\rangle > 0$.

(K3) E is a locally convex space, and for each $x \in K$, we may define a vector $p(x) \in E^*$ such that $\varphi^i(x) - x \subset \{v \in E \mid \langle p(x), v\rangle > 0\}$. Moreover, for each $x \in K$, there are a vector $v(x)$ in E and a neighbourhood $U(x)$ of x in X such that $\forall z \in U(x)$, if $z \in K$, then $\exists \lambda(z) \in R_{++}$ satisfying $z + \lambda(z)v(x) \in X$, and $\langle p(z), v(x)\rangle > 0$.

(K*) There is a convex valued correspondence Φ such that for each $x \in K$, there exist a neighbourhood $U(x)$ of x in X and a point $y(x)$ such that for each $z \in U(x)$, $(z \in K) \Longrightarrow (\varphi(z) \subset \Phi(z)$ and $z \notin \Phi(z)$ and $y(x) \in \Phi(z))$.

Then, φ has a fixed point x^*, $x^* \in \varphi(x^*)$.

Proof : (Case: K1) Suppose that φ does not have a fixed point. Then, since $X = K$ is compact, we have $x_1, \cdots, x_n \in X$ and a finite open covering $U(x_1), \cdots, U(x_n)$ of X satisfying condition (K1). Let $\beta_t : X \to [0,1]$, $t = 1, \cdots, n$, be a partition of unity subordinated to $U(x_1), \cdots, U(x_n)$. Denote by f the continuous mapping $f : X \ni x \mapsto \sum_{t=1}^n \beta_t(x)p(x_t) \in E'$. Moreover, let ψ be a correspondence on E' to X such that $\psi(p) = \{x \in X \mid \langle p, x\rangle = \max_{y \in X}\langle p, y\rangle\}$. Since X is compact, and since each β_t, $p(x_t)$ are continuous, f is continuous and ψ is non-empty compact convex valued upper semi-continuous correspondence. Hence, $\psi \circ f$ has a fixed point $\hat{x} \in \psi(f(\hat{x}))$ under Fan-Glicksberg's fixed point theorem. By the definition of f and ψ, we have $\sum_{t=1}^n \beta_t(\hat{x})\langle p(x_t), \hat{x}\rangle \geq$

$\sum_{t=1}^{n} \beta_t(\hat{x})\langle p(x_t), z\rangle$ for all $z \in X$. On the other hand, since $\hat{x}$ belongs to at least one $U(x_t)$, we have for an arbitrary element z of $\varphi(\hat{x}) \subset \Phi(\hat{x})$, $\sum_{t=1}^{n} \beta_t(\hat{x})\langle p(x_t), z - \hat{x}\rangle > 0$, a contradiction.

(Case: K2) Suppose that φ does not have a fixed point. Then, since $X = K$ is compact, we have $x_1, \cdots, x_n \in X$ and a finite covering $\{U(x_1), \cdots, U(x_n)\}$ of X together with points $y(x^1), \cdots, y(x^n) \in X$ satisfying condition (K2). Let $\beta_t : X \to [0,1]$, $t = 1, \cdots, n$, be a partition of unity subordinated to $U(x_1), \cdots, U(x_n)$. Let us consider a function f on $D = \mathrm{co}\,\{y(x_1), \cdots, y(x_n)\}$ to itself such that $f(x) = \sum_{t=1}^{n} \beta_t(x) y(x_t)$. Then, f is a continuous function on the finite dimensional compact set D to itself. Hence, f has a fixed point z by Brouwer's fixed point theorem. On the other hand, for all t such that $z \in U(x_t)$, $y(x_t) - z$ satisfies $\langle p(z), y(x_t) - z\rangle > 0$, so that we have $\langle p(z), \sum_{t=1}^{n} \beta_t(z)(y(x_t) - z)\rangle > 0$. In other words, $\langle p(z), f(z) - z\rangle > 0$, so that we have $f(z) - z \neq 0$, a contradiction.

(Case: K3) Suppose that φ does not have a fixed point. Then, since $X = K$ is compact, we have $x_1, \cdots, x_n \in X$ and a finite covering $\{U(x_1), \cdots, U(x_n)\}$ of X together with vectors $v(x_1), \cdots, v(x_n) \in E$ satisfying (K3). Let $\beta_t : X \to [0,1]$, $t = 1, \cdots, n$, be a partition of unity subordinated to $U(x_1), \cdots, U(x_n)$. For each t and for each $z \in U(x_t)$, we may suppose that $\lambda(t,z)v(x_t) + z \in X$ for a certain $\alpha(t,z) \in R_{++}$. Denote by f the continuous mapping $f : X \ni x \mapsto x + \sum_{t=1}^{n} \beta_t(x)\lambda(t,x)v(x_t)$ and let z be a fixed point of f. Since for all t such that $z \in U(x_t)$, $\langle p(z), \lambda(t,z)v(x_t)\rangle > 0$, we have $\langle p(z), \beta_t(z)\lambda(t,z)v(x_t)\rangle > 0$. It follows that we have $\langle p(z), f(z) - z\rangle = \langle p(z), \sum_{t=1}^{n} \beta_t(z)\alpha(t,z)v(x_t)\rangle > 0$, which contradict the fact that $f(z) - z = 0$.

(Case: K*) Suppose that φ does not have a fixed point. Then, since $X = K$ is compact, we have $x_1, \cdots, x_n \in X$ and a finite covering $\{U(x_1), \cdots, U(x_n)\}$ of X together with points $y(x^1), \cdots, y(x^n) \in X$ satisfying condition (K*) for a certain correspondence Φ. Let $\beta_t : X \to [0,1]$, $t = 1, \cdots, n$, be a partition of unity subordinated to $U(x_1), \cdots, U(x_n)$. Let us consider a function f on $D = \mathrm{co}\,\{y(x_1), \cdots, y(x_n)\}$ to itself such that $f(x) = \sum_{t=1}^{n} \beta_t(x) y(x_t)$. Then, f is a continuous function on the finite dimensional compact set D to itself. Hence, f has a fixed point z by Brouwer's fixed point theorem. On the other hand, for all t such that $z \in U(x_t)$, $y(x_t) \in \Phi(z)$. Moreover, since Φ is convex valued, we have $z = \sum_{t=1}^{n} \beta_t(z) y(x_t) \in \Phi(z)$, which contradicts the condition $z \notin \Phi(z)$ stated in (K*). □

Corollary 1.1 : Let X be a non-empty compact convex subset of E, and let ψ be a (possibly empty valued) correspondence on X to X. Suppose that E and a correspondence $\varphi : X \to X$ such that $(x \notin \psi(x)) \Longrightarrow (\varphi(x) \neq \emptyset$ and $x \notin \varphi(x))$, (typically, φ may be taken as a selection of ψ if ψ is non-empty valued) satisfies one of the condition (K1), (K2), (K3), (K*) for $K = \{x \in X \,|\, x \notin \psi(x)\}$. Then, ψ has a fixed point.

Proof: Suppose that ψ does not have a fixed point. Then φ is non-empty valued and does not have a fixed point, either. Moreover, we have $X = \{x \in X \,|\, x \notin \psi(x)\} \subset \{x \in X \,|\, x \notin \varphi(x)\} \subset X$, i.e., φ satisfies one of the condition (K1), (K2), or (K3) even when we define K as $K = \{x \in X \,|\, x \notin \varphi(x)\}$. Hence, by applying Theorem 1 to the non-empty valued correspondence φ, we have a fixed point of φ, a contradiction. □

Theorem 1 and the corollary to Theorem 1 may be generalized for the product of mappings and may be reformulated as Nash equilibrium existence results in the following section.

Theorem 2 : For each $i \in I$, let X^i be a non-empty compact convex subset of E, and let φ^i be a non-empty valued correspondence on $X = \prod_{i \in I} X^i$ to X^i. Let $\varphi = \prod_{i \in I} \varphi^i : X \to X$ and $K = \{x \in X | x \notin \varphi(x)\}$. Suppose that E and φ satisfy one of the following conditions:

(NK1) E is a locally convex space. For each $x \in K$, there exist at least one $i \in I$, a vector $p^x \in E'$, and a neighbourhood $U(x)$ of x in X satisfying that $\forall z \in U(x)$, if $z \in K$, $\varphi^i(z) - z^i \subset \{v \in E | \langle p^x, v\rangle > 0\}$.

(NK2) For each i and for each x such that $x \notin \varphi^i(x)$, we may chose $p_i^x \in E^*$ such that $\varphi^i(x) - x^i \subset \{v \in E | \langle p_i^x, v\rangle > 0\}$. Moreover, for each $x \in K$, there exist at least one $i \in I$, an element $y^x \in X^i$, and a neighbourhood $U(x)$ of x in X satisfying that for all $z \in U(x) \cap K$, $\langle p_i^z, y^x - z^i\rangle > 0$.

(NK3) E is a locally convex space. For each i and for each x such that $x \notin \varphi^i(x)$, we may chose $p_i^x \in E^*$ such that $\varphi^i(x) - x^i \subset \{v \in E | \langle p_i^x, v\rangle > 0\}$. Moreover, for each $x \in K$, there exist at least one $i \in I$, a vector $v(x) \in E$, and a neighbourhood $U(x)$ of x in X, satisfying that $\forall z \in U(x) \cap K$, $\exists \lambda(z) \in R_{++}$, $z^i + \lambda(z)v^x \in X^i$ and $\langle p_i^z, v^x\rangle > 0$.

(NK*) For each i there is a convex valued correspondence $\Phi^i : X \to X^i$ such that $\forall x \in X, \varphi^i(x) \subset \Phi^i(x)$ and $(x^i \notin \varphi^i(x)) \Longrightarrow (x^i \notin \Phi^i(x))$. Moreover, for each $x \in K$, there exist at least one $i \in I$, an element $y^x \in X^i$, and a neighbourhood $U(x)$ of x in X satisfying that for all $z \in U(x) \cap K$, $y^x \in \Phi^i(z)$.

Then, φ has a fixed point x^*, $x^* \in \varphi(x^*)$.

Proof : (Case: NK1) Assume that φ does not have a fixed point. Then, since X is compact, we have a finite set $\{x^1, \cdots, x^k\} \subset X$, a covering $\{U(x^1), \cdots, U(x^k)\}$ of X, a finite sequence of indices $i^1, \cdots, i^k \in I$, and vectors $p^{x^1}, \cdots, p^{x^k} \in E'$, satisfying condition (NK1) for each $x^1, \cdots, x^k$. For each $x \in X$, let $J(x)$ be the set $\{i^m | x \in U(x^m)\} \subset I$, and let $N(x)$ be the set $\{n | x \in U(x^n)\} \subset \{1, \cdots, k\}$. Define for each $x \in X$, $p(x) \in (E')^{(I)}$ as $p(x) = (p^j)_{j \in I}$, where $p^j = p^{x^m}$ for a certain m such that $x \in U(x^m)$ for $j \in J(x)$, and $p^j = 0$ for $j \notin J(x)$. Then, the neighbourhood $V(x) = \bigcap_{m \in N(x)} U(x^m)$ satisfies that for all $z \in V(x)$, $\langle p(x), \varphi(z) - z\rangle = \sum_{j \in J(x)} \langle p^j, \varphi^j(z) - z^j\rangle \geq \frac{1}{k} \sum_{m \in N(x)} \langle p^{x^m}, \varphi^{i^m} - z^{i^m}\rangle > 0$. Hence, φ satisfies the condition (K1) in Theorem 1, so that it has a fixed point, a contradiction.

(Case: NK2) Suppose that φ has no fixed point. Then, since X is compact, we have a finite set $\{x^1, \cdots, x^k\} \subset X$, a covering $\{U(x^1), \cdots, U(x^k)\}$ of X, finite sequences of vectors $p_{i^1}^{x^1}, \cdots, p_{i^k}^{x^k}$, and $y_{i^1}^{x^1}, \cdots, y_{i^k}^{x^k}$ together with the sequence of indices $i^1, \cdots, i^k$, satisfying (NK2) for each non-fixed point $x^1, \cdots, x^k$ of φ. For each $x \in X$, let $J(x) = \{i^m | x \in U(x^m)\} \subset I$ and let $N(x) = \{m | x \in U(x^m)\} \subset \{1, \cdots, k\}$. Define for each $x \in X$, $p(x) \in (E')^{(I)}$ as $p(x) = (p^j)_{j \in I}$, where $p^j = p_{i^m}^x$ for a certain i^m such that $x \in U(x^m)$ for $j \in J(x)$ and $p^j = 0$ for $j \notin J(x)$. Moreover, for each $x \in X$, define $y(x) = (y^j)_{j \in I} \in X$ as $y^j = y_{i^m}^{x^m}$ for a certain m such that $x \in U(x^m)$ for $j \in J(x)$ and y^j is an arbitrary element of X^j for $j \notin J(x)$. Then, by considering the neighbourhood $\bigcap_{m \in N(x)} U(x^m)$ of x in X, the mapping φ satisfies (K2) of Theorem 1. (Indeed, for all $z \in \bigcap_{m \in N(x)} U(x^m)$ for a certain x, $\langle p(z), y(x) - z\rangle = \sum_{j \in J(x)} \langle p_j^z, y^j - z^j\rangle \geq \frac{1}{k} \sum_{m \in N(x)} \langle p_{i^m}^z, y_{i^m}^{x^m} - z^{i^m}\rangle > 0$.) Hence, φ has a fixed point, a contradiction.

(Case: NK3) Assume that φ does not have a fixed point. Then, since X is compact, we have a finete set $\{x^1, \cdots, x^k\} \subset X$, a covering $\{U(x^1), \cdots, U(x^k)\}$ of X, a finite sequence of indices $i^1, \cdots, i^k$, vectors $p^{x^1}_{i1}, \cdots, p^{x^k}_{ik}$, in E^*, and vectors $v^{x^1}, \cdots, v^{x^k}$, satisfying (NK2) for each non-fixed point $x^1, \cdots, x^k$. For each $x \in X$, let $J(x)$ be the set $\{i(x^m) | x \in U(x^m)\}$, and let $N(x)$ be the set $\{n | x \in U(x^n)\}$. Define for each $x \in X$, $p(x) \in (E')^{(I)}$ as $p(x) = (p^j)_{j \in I}$, where $p^j = p^x_j$ for $j \in J(x)$ and $p^j = 0$ for $j \notin J(x)$. Moreover, for each $x \in X$, define $v(x) = (v^j)_{j \in I}$ as $v^j = v^{x^m}$ for a certain m such that $j = i(x^m)$ for $j \in J(x)$ and $v^j = 0$ for $j \notin J(x)$. Then, by considering the neighbourhood $\bigcap_{m \in N(x)} U(x^m)$ of x in X, the mapping φ satisfies (K2) of Theorem 1. (Indeed, for all $z \in \bigcap_{m \in N(x)} U(x^m)$ for a certain x, $\langle p(z), v(x) \rangle = \sum_{j \in J(x)} \langle p^z_j, v^j \rangle \geq \frac{1}{k} \sum_{m \in N(x)} \langle p^z_{i(x^m)}, v^{x^m} \rangle > 0$.) Hence, φ has a fixed point, a contradiction.

(Case: NK*) Suppose that φ has no fixed point. Then, since X is compact, we have a finite set $\{x^1, \cdots, x^k\} \subset X$, a covering $\{U(x^1), \cdots, U(x^k)\}$ of X, and a finite sequence $y^{x^1}_{i1}, \cdots, y^{x^k}_{ik}$ together with the sequence of indices $i^1, \cdots, i^k$, satisfying (NK*) for correspondences $\Phi^{i^1}, \cdots, \Phi^{i^k}$. For each $x \in X$, let $J(x) = \{i^m | x \in U(x^m)\} \subset I$ and let $N(x) = \{m | x \in U(x^m)\} \subset \{1, \cdots, k\}$. Denote by Φ the convex valued correspondence defined as $\Phi(x) = \prod_{i \in J(x)} \Phi^i(x) \times \prod_{i \in I, i \notin J(x)} X^i$. For each $x \in X$, define $y(x) = (y^j)_{j \in I} \in X$ by letting y^j be a $y^{x^m}_{i^m}$ for a certain $i^m = j$, $m \in N(x)$, for $j \in J(x)$ and y^j be an arbitrary element of $\varphi^j(x)$ for $j \notin J(x)$. Then, by considering the neighbourhood $\bigcap_{m \in N(x)} U(x^m)$ of x in X, the mapping φ satisfies (K*) of Theorem 1. (Indeed, for each $x \in X$, for each $z \in \bigcap_{m \in N(x)} U(x^m)$, and for each $j \in \{i^1, \cdots, i^k\}$, $y(x) = (y^j)_{j \in I}$ satisfies $y(x) \in \Phi(z)$ since for each $j \in J(x)$, $y^j \in \Phi^i(z)$ for all $z \in \bigcap_{m \in N(x)} U(x^m)$.) Hence, φ has a fixed point, a contradiction. □

Corollary 2.1 : For each $i \in I$, let X^i be a non-empty compact convex subset of E, and let ψ^i be a (possibly empty valued) correspondence on $X = \prod_{i \in I} X^i$ to X^i. Define a correspondence ψ as $\psi = \prod_{i \in I} \psi^i : X \to X$. Suppose that for each $i \in I$, we have a non-empty valued correspondence $\varphi^i : X \to X^i$, such that for each $x = (x^j)_{j \in I}$, $(x^i \notin \psi^i(x)) \Longrightarrow (x^i \notin \varphi^i(x))$, (typically, we may chose each φ^i as a selection of ψ^i when ψ^i is non-empty valued) and that E and φ^i, $i \in I$ satisfy one of the conditions (NK1), (NK2), (NK3), (NK*) in Theorem 2 for $K = \{x \in X | x \notin \psi(x)\}$. Then, ψ has a fixed point.

Proof : Suppose that ψ does not have a fixed point. Then, $\varphi = \prod_{i \in I} \varphi^i$ does not have a fixed point, either. Hence, we have $X = K = \{x \in X | x \notin \psi(x)\} \subset \{x \in X | x \notin \prod_{i \in I} \varphi^i(x)\} \subset X$, so that E and $\hat{\varphi}^i$, $i \in I$, satisfies one of the condition (NK1), (NK2), (NK3), (NK*) in Theorem 2 even when we define K as $K = \{x \in X | x \notin \varphi(x)\}$ instead of $K = \{x \in X | x \notin \psi(x)\}$. Therefore, since φ is non-empty valued, by Theorem 2, $\hat{\varphi}$ has a fixed point, a contradiction. □

3 NASH EQUILIBRIUM EXISTENCE THEOREMS

In this section, we apply theorems in the previous section to the existence of equilibrium problem for strategic form non-cooperative games (c.f. Nash (1950), Nash (1951), Nikaido (1959), etc).

Throughout this section, we denote by I the set of *players*. (The cardinal number of I is arbitrary.) For each $i \in I$, we denote by X^i the *strategy set* of player i. All strategy sets are assumed to be compact convex subsets of a Hausdorff topological vector space E. The payoff stracture for gemes will be given in the form of *preference (better set) correspondences* P^i, $i \in I$, which are defined as (possibly empty valuaed) correspondences on $X = \prod_{i\in I}$ to X^i, $i \in I$, satisfying that for each $x = (x^j)_{j\in I} \in X$, $x^i \notin P^i(x)$ (the irreflexivity) for all $i \in I$. For each $x = (x^j)_{j\in I} \in X$, the set $P^i(x)$ may be interpreted as the set of all strategies for player i which is better than x^i if the strategies of other players $(x^j)_{j\in I, j\neq i}$ are fixed. A *strategic form game* will be denoted by $(X^i, P^i)_{i\in I}$. For a strategic form game $(X^i, P^i)_{i\in I}$, a sequence of strategies, $(x^i)_{i\in I} \in X$, (a *strategy profie* for the game) is said to be a *Nash equilibrium* if $P^i((x^i)_{i\in I}) = \emptyset$ for all $i \in I$.

When $I = \{i\}$, the Nash equilibrium is nothing but a maximal element for the relation P^i on X^i. By applying the results in the previous section, we obtain the following maximal element existence theorem.

Theorem 3 : (Maximal Element Existence) Let X be a compact convex subset of a Hausdorff topological vector space E, and let P be a (possibly empty valued) correspondence on X to X such that for all $x \in X$, $x \notin P(x)$. Assume that there exists a correspondence $\varphi : X \to X$, satisfying that $\forall x \in X$, $(P(x) \neq \emptyset) \Longrightarrow (\varphi(x) \neq \emptyset$ and $P(x) \subset \varphi(x)$ and $x \notin \varphi(x))$, and that for φ together with E one of the conditions (K1), (K2), (K3), (K*) in Theorem 1 holds for $K = \{x \in X \,|\, P(x) \neq \emptyset\}$. Then there is a maximal element x^* of X with respect to P. ($P(x^*) = \emptyset$.)

Proof : Assume the contrary, i.e., assume that for all $x \in X$, $P(x) \neq \emptyset$. Then, we have $\{x \in X \,|\, x \notin P(x)\} = X = K = \{x \in X \,|\, P(x) \neq \emptyset\}$. Therefore, P satisfies all the conditions for ψ mentioned in Corollary 1.1, so that P has a fixed point, a contradiction. □

The above theorem shows that any types of convexity assumptions for P (including the weakest one, $x \notin \mathrm{co}\, P(x)$,) is unnecessary for assuring the existence of maximal elements even when the preference is non-ordered. The special case of Theorem 3 in which $P = \varphi$ satisfies condition (K*), gives us a generalization of the corollary on the maximal element existence in Yannelis and Prabhakar (1983; Corollary 5.1). (In the sense that if there is no maximal element, an $\mathscr{L}$-majolized map P satisfies the condition stated in Theorem 3 for (K*).)

As Theorem 1 (Corollary 1.1) gives the maximal element existence theorem, Theorem 2 (Corollary 2.1) gives the Nash equilibrium existence theorem.

Theorem 4 : (Nash Equilibrium Existence) For a strategic form game $(X^i, P^i)_{i\in I}$, the Nash equilibrium exists if the following conditions are satisfied.

(A1) For each $i \in I$, X^i is a non-empty compact convex subset of a Hausdorff topological vector space E.

(A2) For each $i \in I$, P^i is a (possibly empty valued) correspondence on $X = \prod_{i\in I} X^i$ to X^i satisfying $\forall x = (x^j)_{j\in I} \in X$, $x^i \notin P^i(x)$.

(A3) For each P^i, we may define a non-empty valued correspondence $\varphi^i : X \to X^i$ satisfying that $\forall x = (x^j)_{j\in I} \in X$, $(P^i(x) \neq \emptyset) \Longrightarrow (x^i \notin \varphi^i(x))$.

(A4) E and φ^i, $i \in I$ fulfills one of the condition (NK1), (NK2), (NK3), (NK*) in Theorem 2 for $K = \{x \in X \mid \exists i, P^i(x) \neq \emptyset\}$.

Proof : Assume the contrary, that is, for each $x \in X$, there is at least one $i \in I$ such that $P^i(x) \neq \emptyset$. Then, we have $\{x \in X \mid x \notin \prod_{i \in I} P^i(x)\} = X = \{x \in X \mid \exists i, P^i(x) \neq \emptyset\} = K \subset X$. It follows that P^i, $i \in I$, satisfies all the conditions for ψ^i, $i \in I$, in Corollary 2.1, so that $P = \prod_{i \in I} P^i$ has a fixed point, which contradicts to the condition (A2). □

As in the maximal element existence theorem (Theorem 3), the convexity assumption for the preferences has been completely replaced in Theorem 4. Even in the special case of the theorem such that $P^i = \varphi^i$ for all $i \in I$, (in such cases, the condition "$\forall x, x^i \notin \mathrm{co}\, P^i(x)$" necessarily holds,) the theorem gives us a drastic improvement on the conditions assuring for the existence of Nash equilibria by replacing all types of continuity conditions for weeker conditions on the local direction of mappings (NK1), (NK2), or (NK3). We also note that the implication of the theorem contains the result of Nishimura and Friedman (1981) since the best response correspondences, if such exist, under the preferences P^i, $i \in I$, may typically be considered as examples of φ^i's satysfying the condition (NK1).

It is not difficult to extend our result to the existence of equilibrium problems for the abstract economy, a generalized non-cooperative strategic form games (c.f. Debreu (1952), Shafer and H.F.Sonnenschein (1975), etc). For the non-cooperative strategic form games, we add a structure of constraint correspondences describing the situation that for some reasons, an adequate outcome of the game should be restricted on a certain subset of the set of strategy profiles. That is, we consider a correspondence $K^i : \prod_{j \in I, j \neq i} \to X^i$ for each $i \in I$, and given other player's strategies, $(x^j)_{j \in I, j \neq i}$, restrict the choice of the strategy of player i, on the subset $K^i((x^j)_{j \in I, j \neq i})$ of X^i. We call a strategy profile $x_* = (x_*^i)_{i \in I}$ a *social equilibrium* (a generalized Nash equilibrium) if (1) $x_*^i \in K^i((x_*^j)_{j \in I, j \neq i})$ for each i, and (2) $P^i(x_*) = \emptyset$ for all $i \in I$. The generalized non-cooperative strategic form game (abstract economy) will be denoted by $(X^i, P^i, K^i)_{i \in I}$.

Theorem 5 : (Social Equilibrium Existence) An abstract economy $(X^i, P^i, K^i)_{i \in I}$ has a generalized Nash equilibrium if the following conditions are satisfied.

(B1) For each $i \in I$, X^i is a non-empty compact convex subset of a Hausdorff topological vector space E.

(B2) For each $i \in I$, P^i is a (possibly empty valued) correspondence on $X = \prod_{i \in I} X^i$ to X^i satisfying $\forall x = (x^j)_{j \in I} \in X$, $x^i \notin P^i(x)$, and K^i is a non-empty valued correspondence on X to X^i.

(B3) For each $i \in I$, we may define a non-empty valued correspondence $\varphi^i : X \to X^i$ satisfying that $\forall x = (x^j)_{j \in I} \in X$, $(x^i \in K^i(x)$ and $K^i(x) \cap P^i(x) \neq \emptyset) \Longrightarrow (x^i \notin \varphi^i(x))$, and that $\forall x = (x^j)_{j \in I} \in X$, $(x^i \notin K^i(x)) \Longrightarrow (x^i \notin \varphi^i(x))$.

(B4) E and φ^i, $i \in I$, satisfies one of the condition (NK1), (NK2), (NK3), (NK*) in Theorem 2 for $K = \{x = (x^j)_{j \in I} \in X \mid \exists i, (x^i \in K^i(x)$ and $K^i(x) \cap P^i(x) \neq \emptyset)$ or $(x^i \notin K^i(x))\}$.

Proof : For each $i \in I$, and $x = (x^j)_{j\in I} \in X$, if $x^i \notin K^i(x)$, let $B^i(x) = K^i(x)$, else if $P_i(x) \cap K_i(x) \neq \emptyset$, let $B^i = K^i(x) \cap P_i(x)$, else let $B^i(x) = \emptyset$. Then, $x^* \in X$ is a generalized Nash equilibrium point for $(X^i, P^i, K^i)_{i\in I}$ iff $x^* \in X$ is a Nash equilibrium point of (X^i, B^i). Since for each $x = (x^j)_{j\in I}$ in X, $B^i(x) \neq \emptyset$ necessarily implies that $\varphi^i(x) \neq \emptyset$ and $x^i \notin \varphi^i(x)$, and since $\{x \in X | \exists i, B^i(x) \neq \emptyset\}$ is clearly equal to K, conditions (A3) and (A4) in Theorem 4 is satisfied for the game (X^i, B^i). Hence, we have an equilibrium for (X^i, B^i). □

Corollary 5.1 : (Non-convex Social Equilibrium Existence) An abstract economy $(X^i, P^i, K^i)_{i\in I}$ has a generalized Nash equilibrium if the following conditions are satisfied.

(C1) For each $i \in I$, X^i is a non-empty compact convex subset of a Hausdorff topological vector space E.

(C2) For each $i \in I$, P^i is a (possibly empty valued) correspondence on $X = \prod_{i\in I} X^i$ to X^i satisfying $\forall x = (x^j)_{j\in I} \in X$, $x^i \notin P^i(x)$, and K^i is a non-empty valued correspondence on X to X^i.

(C3-1) For each $i \in I$, and for each $z = (z^j)_{j\in I} \in X$, such that $z^i \in K^i(z)$ and $P^i(z) \cap K^i(z) \neq \emptyset$, we may select a vector $p_i^z \in E^*$ representing (in a certain well defined sense) a direction of $P^i(z)$ from the point z^i.

(C4-1) For each $i \in I$, and for each $z = (z^j)_{j\in I} \in X$, such that $z^i \notin K^i(z)$, we may select a vector $p_i^z \in E^*$ representing (in a certain well defined sense) a direction of $K^i(z)$ from the point z^i.

(C5) If x is not an equilibrium point, then there exists at least one $i \in I$ such that there are a neighbourhood $U(x)$ of x in X and a point $y(x)$ satisfying that for every non-equilibrium point $z = (z^j)_{j\in I} \in U(x)$, $\langle p_i^z, y(x) - z^i \rangle > 0$.

Proof : Let $K = \{z = (z^j)_{j\in I} \in X | z^i \in K^i(z) \text{ and } P^i(z) \cap K^i(z) \neq \emptyset\} \cup \{z = (z^j)_{j\in I} | z^i \notin K^i(z)\}$. For each $i \in I$, let $\varphi^i(x) = \{z^i \in X^i | \langle p_i^x, z^i \rangle > 0\}$ for $x \in K$ and $\varphi^i(x) = \emptyset$ for $x \notin K$. Then, (C3-1) and (C4-1) implies that φ^i's satisfy (B3) in Theorem 5. Moreover, since x is not an equilibrium point iff $x \in K$, (C5) implies that E and φ^i, $i \in I$ satisfies (NK2) in Theorem 2 for K, so that (B4) in Theorem 5 is also satisfied. Hence, by Theorem 5, we have a generalized Nash equilibrium for the abstract economy $(X^i, P^i, K^i)_{i\in I}$. □

In order to clarify the relation of our results to resent reseraches such as Tan and Yuan (1994), Bagh (1998), we shall give the following special case of Theorem 5 as another corollary. By considering the fact (i) that in pseudo-metric locally convex space, compact convex valued upper semi-continuous correspondences K^i, $i \in I$ satisfies the condition (NK*) on the open set $\{x = (x^j)_{j\in I} \in X | x^i \notin K^i(x)\}$, and (ii) that $\mathscr{L}$-majolized correspondences[1] P^i, $i \in I$, satisfies the condition (NK*) on $\{x \in X | P^i(x) \neq \emptyset\}$, we can see that the following corollary generalize their results in many applications.

[1] In the sense of Bagh (1998). For the definition, see also Yannelis and Prabhakar (1983) and Tan and Yuan (1994). Note that Bagh's definition of $\mathscr{L}$-majolized map is slightly different from that of Yannelis-Prabhakar-Tan-Yuan's.

Corollary 5.2 : (Social Equilibrium Existence) An abstract economy $(X^i, P^i, K^i)_{i\in I}$ has a generalized Nash equilibrium if the following conditions are satisfied.

(C1) For each $i \in I$, X^i is a non-empty compact convex subset of a Hausdorff topological vector space E.

(C2) For each $i \in I$, P^i is a (possibly empty valued) correspondence on $X = \prod_{i\in I} X^i$ to X^i satisfying $\forall x = (x^j)_{j\in I} \in X$, $x^i \notin P^i(x)$, and K^i is a non-empty valued correspondence on X to X^i.

(C3-2) For each $i \in I$, the pair P^i and E satisfies condition (NK*) for $K = \{x \in X \,|\, P^i(x) \neq \emptyset\}$.

(C4-2) For each $i \in I$, the pair K^i and E satisfies condition (NK*) for $K = \{x = (x^j)_{j\in I} \in X \,|\, x^i \notin K^i(x)\}$.

(C5-2) For each $i \in I$, $\{x \in X \,|\, K^i(x) \cap P^i(x) \neq \emptyset\}$ is open.

Proof : For each $i \in I$, let $\hat{P}^i$ and $\hat{K}^i$ be extensions of P^i and K^i, respectively, satisfying the condition in (NK*). Moreover, let us define a non-empty valued correspondence $\varphi^i : X \to X^i$ as $\varphi^i(x) = \hat{K}^i(x)$ for $x \in \{x = (x^j)_{j\in I} \in X \,|\, x^i \notin K^i(x)\}$, $\varphi^i(x) = \hat{P}^i(x)$ for $x \in \{x = (x^j)_{j\in I} \in X \,|\, x^i \in K^i(x)$ and $P^i(x) \cap K^i(x) \neq \emptyset\}$, and $\varphi^i(x) = X^i$ for $\{x = (x^j)_{j\in I} \in X \,|\, x^i \in K^i(x)$ and $P^i(x) \cap K^i(x) = \emptyset\}$. Clearly, each φ^i satisfies the condition stated in (B3) in Theorem 5. Furthermore, since each pair of K^i and E satisfies (NK*) for $\{x = (x^j)_{j\in I} \in X \,|\, x^i \notin K^i(x)\}$, we have for each i the set $\{x = (x^j)_{j\in I} \in X \,|\, x^i \notin K^i(x)\}$ is open. Moreover, since by (C5-2), the set $\{x \in X \,|\, K^i(x) \cap P^i(x) \neq \emptyset\}$ is also open, E and φ^i's satisfy (B4) in Theorem 5 for $K = \{x = (x^j)_{j\in I} \in X \,|\, (x^i \in K^i(x)$ and $K^i(x) \cap P^i(x) \neq \emptyset)$ or $(x^i \notin K^i(x))\}$. Hence, by Theorem 5, the abstract economy $(X^i, P^i, K^i)_{i\in I}$ has a generalized Nash equilibrium. □

4 GALE-NIKAIDO-DEBREU THEOREM

The purpose of this section is to apply our results in previous sections to the market equilibrium existence problem of Gale-Nikaido-Debreu type (Gale (1955), Nikaido (1956a), Debreu (1956)). We can find one of the most general form of results for this problem in Nikaido (1956b), Nikaido (1957), or Nikaido (1959). After 1980's, essentially the same problem (with some varieties in topologies, boundary conditions, and so on,) has been treated by many authors (e.g., Aliprantis and Brown (1983), Florenzano (1983), Mehta and Tarafdar (1987), etc).

Let E be a vector space, and assume that there is a duality $\langle E, F\rangle$ between E and a certain vector space F. Denote by $P \subset E$ a non-empty closed convex cone with vertex 0 such that $P \cap -P \neq P$, and by P^* the polar cone of $-P$ with respect to the duality $\langle E, F\rangle$. Moreover, denote by P_0^* the set $P^* \setminus \{0\}$. At first we apply Theorem 1 to the setting given in Nikaido (1959).

Theorem 6 : (Market Equilibrium Existence: with Compact Range) Suppose that there is a non-empty valued correspondence ζ defined on a convex $\sigma(F, E)$-dense subset D of P_0^* to E satisfying the following conditions.

(D1-1) For each convex hull A of a finite subset of D and the cone $L_A \subset P_0^*$ spanned by A, and for each $p \in A$ such that $\zeta(p) \cap L_A^\circ = \emptyset$, there are a neighbourhood $U(p)$ of p in $(F, \sigma(F,E))$ and a point $\bar{p}$ in A such that $\forall q \in A \cap U(p)$, $\forall z \in \zeta(q)$, $(\zeta(q) \cap L_A^\circ = \emptyset \Longrightarrow \langle \bar{p}, z\rangle > 0)$, where L_A° denotes the polar of L_A.

(D2-1) Compact Range: The range of ζ, $\bigcup_{p\in D} \zeta(p)$, is $\sigma(E,F)$-compact.

(D3) Walras' Law: $\forall p \in D, \langle p, z\rangle \leq 0$ for all $z \in \zeta(p)$.

Then, $\exists p^*$, $\zeta(p^*) \cap -P \neq \emptyset$.

Proof : Let us divide the proof in three steps.

(STEP1: We use only (D1-1) and (D3)) Let A be a convex hull of a finite subset of D, and let $L_A \subset P_0^*$ be the convex cone spanned by A. Then, $\forall p \in A$, $\zeta(p) \cap L_A^\circ = \emptyset$ means, by (D1-1), that there are a neighbourhood $U(p) \subset (F, \sigma(F,E))$ of p and a point $\bar{p}$ in A such that $\forall q \in A \cap U(p)$, $\forall z \in \zeta(q)$, $(\zeta(q) \cap L_A^\circ = \emptyset \Longrightarrow \langle \bar{p}, z\rangle > 0$. Since A is a compact subset of $(F, \sigma(F,E))$, by letting $K = \{p \in A | \zeta(p) \cap L_A^\circ = \emptyset\}$, $\varphi(p) = \{q \in A | \forall z \in \zeta(q), \langle q, z\rangle > 0\}$ for $p \in K$, and $\varphi(p) = A$ for $p \notin K$, we see that $K = \{p \in A | p \notin \varphi(p)\}$ by (D3) and that A and φ satisfies the condition (K2) in Theorem 1, so that φ has a fixed point p_A. By the definition of φ, we have $\zeta(p_A) \cap L_A^\circ \neq \emptyset$.

(STEP2: We use only (D2-1) and the definition of p_A.) Denote by $\mathscr{A}$ the set of all convex hull of finite subset of D directed by the inclusion. By (D2), an arbitrarily fixed net $\{z_A \in \zeta(p_A) \cap L_A^\circ, A \in \mathscr{A}\}$ has a subnet $\{z_{A_\mu} \in \zeta(p_{A_\mu}) \cap L_A^\circ, \mu \in \mathscr{M}\}$ converging to a point z_* in the range of ζ under the topology $\sigma(E,F)$.

(STEP3: We use (D1-1), the definition of p_A and p_*, and the fact $p_* \in D$.) Now, assume that $z_* \notin -P$. Then, since P is closed, there is a vector $\bar{p} \in D$ such that $\langle \bar{p}, z_*\rangle > 0$. On the other hand, since for all $\mu \in \mathscr{M}$ sufficiently large, we have $\bar{p} \in A_\mu$, we have $\langle \bar{p}, z_{A_\mu}\rangle \leq 0$ for all $\mu \in \mathscr{M}$ sufficiently large, so that we have $\langle \bar{p}, z_*\rangle \leq 0$, a contradiction. Hence, $z_* \in -P$, and it follows that there exists a $p \in D$, $\zeta(p) \cap -P = \emptyset$. □

We may also obtain the following theorem which may be considered as a generalization of the result given in Aliprantis and Brown (1983), the Gale-Nikaido-Debreu Theorem with a boundary condition.

Theorem 7 : (Market Equilibrium Existence: with Boundary Condition) Suppose that P^* is spanned by a $\sigma(F,E)$-compact subset Δ of P^*, and that there is a non-empty valued correspondence ζ defined on a convex $\sigma(F,E)$-dense subset D of $\Delta \setminus \{0\}$ to E satisfying the following conditions.

(D1-1) For each convex hull A of a finite subset of D and the cone L_A spanned by A, and for each $p \in A$ such that $\zeta(p) \cap L_A^\circ = \emptyset$, there are a neighbourhood $U(p)$ of p in $(F, \sigma(F,E))$ and a point $\bar{p}$ in A such that $\forall q \in A \cap U(p)$, $\forall z \in \zeta(q)$, $(\zeta(q) \cap L_A^\circ = \emptyset \Longrightarrow \langle \bar{p}, z\rangle > 0)$, where L_A° denotes the polar of L_A.

(D1-2) For each $p \in D$ such that $\zeta(p) \cap -P \neq \emptyset$, there exist a neighbourhood $U(p)$ of p in $(F, \sigma(F,E))$ and a vector $\bar{p} \in D$ such that $\forall q \in U(p) \cap D$, $\forall z \in \zeta(q)$, $(\zeta(q) \cap -P = \emptyset \Longrightarrow \langle \bar{p}, z\rangle > 0)$.

(D2-2) Boundary Condition: For each net $\{p^\nu, \nu \in \mathscr{N}\}$ in D converging to a point $\hat{p} \in \Delta \setminus D$, there is a vector $\bar{\hat{p}} \in D$ such that for a certain subnet $\{p^\mu, \mathscr{M}\}$ of $\{p^\nu, \mathscr{N}\}$, $\langle \bar{\hat{p}}, z \rangle > 0$ for all $z \in \varphi(p^\mu)$ for all $\mu \in \mathscr{M}$.

(D3) Walras' Law: $\forall p \in D, \langle p, z \rangle \leq 0$ for all $z \in \zeta(p)$.

Then, $\exists p^*$, $\zeta(p^*) \cap -P \neq \emptyset$.

Proof : Let us divide the proof in three steps.

(STEP1: We use only (D1-1) and (D3)) Let A be a convex hull of a finite subset of D, and let L_A be the convex cone spanned by A. Then, $\forall p \in A \subset D$, $\zeta(p) \cap L_A^\circ = \emptyset$ means that, by (D1-1), there are a neighbourhood $U(p) \subset (F, \sigma(F, E))$ of p and a point $\bar{p}$ in A such that $\forall q \in A \cap U(p)$, $\forall z \in \zeta(q)$, $(\zeta(q) \cap L_A^\circ = \emptyset \Longrightarrow \langle \bar{p}, z \rangle > 0$. Since A is a compact subset of $(F, \sigma(F, E))$, by letting $K = \{p \in A | \zeta(p) \cap L_A^\circ = \emptyset\}$, $\varphi(p) = \{q \in A | \forall z \in \zeta(q), \langle q, z \rangle > 0\}$ for $p \in K$, and $\varphi(p) = A$ for $p \notin K$, we see that $K = \{p \in A | p \notin \varphi(p)\}$ by (D3) and that A and φ satisfies the condition (K2) in Theorem 1, so that φ has a fixed point p_A. By the definition of φ, we have $\zeta(p_A) \cap L_A^\circ \neq \emptyset$.

(STEP2: We use only (D2-2) and the definition of p_A.) Denote by $\mathscr{A}$ the set of all convex hull of finite subset of D directed by the inclusion. Since $\{p_A, A \in \mathscr{A}\}$ is a net in the compact set Δ, it has a subnet $\{p_{A_\mu}, \mu \in \mathscr{M}\}$ converging to a point $p_* \in \Delta$. If $p_* \in \Delta \setminus D$, then by (D2-2), there is a subnet $\{p_{A_{\mu(\nu)}}, \nu \in \mathscr{N}\}$ of $\{p_{A_\mu}, \mu \in \mathscr{M}\}$ and $\bar{p}_* \in D$ such that $\langle \bar{p}_*, z \rangle > 0$ for all $z \in \varphi(p_{A_{\mu(\nu)}})$ for all $\nu \in \mathscr{N}$, which is impossible since for all A sufficiently large, $\bar{p}_* \in A$ and each one of such a p_A (which may be considered as equal to a $p_{A_{\mu(\nu)}}$ for a ν sufficiently large) satisfies $\zeta(p_A) \cap L_A^\circ \neq \emptyset$ i.e., $\exists z \in \zeta(p_{A_{\mu(\nu)}})$ such that $\langle \bar{p}_*, z \rangle \leq 0$. Therefore, we have $p_* \in D$.

(STEP3: We use (D1-2), the definition of p_A and p_*, and the fact $p_* \in D$.) Now assume that for all $p \in D$, $\zeta(p) \cap -P = \emptyset$. Then, by (D1-2),

there exist a neighbourhood $U(p_*)$ of p_* in $(F, \sigma(F, E))$ and a vector $\bar{p}_* \in D$ such that for all convex hull A of a finite subset of D satisfying that $\{p_*, \bar{p}_*\} \subset A$, we have $\forall q \in U(p_*) \cap A$, $\forall z \in \zeta(q)$, $\langle \bar{p}_*, z \rangle > 0$. On the other hand, the subnet $\{p_{A_\mu}, \mu \in \mathscr{M}\}$ converges to p_* so that for all $\mu \in \mathscr{M}$ sufficiently large, $A_\mu \supset \{p_*, \bar{p}_*\}$ and $p_{A_\mu} \in U(p_*)$. Of course, by the definition of such a p_{A_μ}, $\exists z_\mu \in \zeta(p_{A_\mu})$ such that $\langle \bar{p}_*, z_\mu \rangle \leq 0$, a contradiction. Therefore, there exists a $p \in D$, $\zeta(p) \cap -P = \emptyset$. □

In the above setting, if we use a slightly more stringent boundary condition (D2-3) in the next theorem, we may perfectly drop the condition (D1-1). Note that in the following theorem, the condition (D2-3) is stronger than the boundary condition (D2-2) of, so called, Grandmont (1977) type, but is weaker than the boundary condition of Neuefeind (1980) type.

Theorem 8 : (Market Equilibrium Existence: with Strong Boundary Condition) Suppose that P^* is spanned by a $\sigma(F, E)$-compact subset Δ of P^*, and that there is a non-empty valued correspondence ζ defined on a convex $\sigma(F, E)$-dense subset D of $\Delta \setminus \{0\}$ to E satisfying the following conditions.

(D1-2) For each $p \in D$ such that $\zeta(p) \cap -P \neq \emptyset$, there exist a neighbourhood $U(p)$ of p in $(F, \sigma(F, E))$ and a vector $\bar{p} \in D$ such that $\forall q \in U(p) \cap D$, $\forall z \in \zeta(q)$, $(\zeta(q) \cap -P = \emptyset \Longrightarrow \langle \bar{p}, z \rangle > 0)$.

(D2-3) Strong Boundary Condition: For each point $\hat{p} \in \Delta \setminus D$, there exist a neighbourhood $U(\hat{p})$ of $\hat{p}$ in $(F, \sigma(F, E))$ and a vector $\bar{\hat{p}} \in D$ such that $\forall q \in D \cap U(\hat{p}), \forall z \in \varphi(q), (\varphi(q) \cap -P = \emptyset \Longrightarrow \langle \bar{\hat{p}}, z \rangle > 0)$.

(D3) Walras' Law: $\forall p \in D, \langle p, z \rangle \leq 0$ for all $z \in \zeta(p)$.

Then, $\exists p^*, \zeta(p^*) \cap -P \neq \emptyset$.

Proof : The argument is essentially the same with the (STEP1) in the proof of the previous theorem. Since Δ is a compact subset of $(F, \sigma(F, E))$, by letting $K = \{p \in D | \zeta(p) \cap -P = \emptyset\} \cup (\Delta \setminus D)$, $\varphi(p) = \{q \in D | \forall z \in \zeta(q), \langle q, z \rangle > 0\}$ for $p \in K \cap D$, $\varphi(\hat{p}) = \{\bar{\hat{p}}\}$ for $p \in K \setminus D$, and $\varphi(p) = \Delta$ for $p \notin K$, we see that $K = \{p \in \Delta | p \notin \varphi(p)\}$ by (D3), and that Δ and φ satisfies the condition (K2) in Theorem 1, so that φ has a fixed point p^*. By the definition of φ, we have $\zeta(p^*) \cap -P \neq \emptyset$. □

In Theorem 8, if we consider the special case $\Delta = D$, i.e., the mapping φ (the excess demand correspondence) is defined on the whole Δ, then the above theorem gives the result in Urai and Hayashi (1997). (Of course, in such a case, condition (D2-3) can be dropped.) Even in such a special case, the result is one of the most general form of Gale-Nikaido-Debreu Theorem. (See, e.g., Mehta and Tarafdar (1987; Theorem 8). We do not assume the value of φ to be compact and/or convex.)

Note also that in all preceeding theorems of this section, the condition (D3: Walras' Law) may be replaced by the following weak version of Walras' Law (used in Yannelis (1985), Mehta and Tarafdar (1987),) without any changing in the proofs.

(D3-1) Weak Walras' Law: $\forall p \in D, \langle p, z \rangle \leq 0$ for a certain $z \in \zeta(p)$.

I think that such a generalization is unnecessary since Walras' law from an economic viewpoint has an important meaning representing the fact that the circulation of income is closed in a model.

5 RELATIONS TO OTHER MATHEMATICAL RESULTS

5.1 Kakutani's Fixed Point Theorem

In locally convex spaces, the following fixed point theorem is known as a generalization of the fixed point theorem of Kakutani (1941).

Theorem 9 : (Fan (1952), Glicksberg (1952)) Let X be a compact convex subset of a locally convex Hausdorff topological vector space over R, and let φ be a non-empty closed convex valued upper semi-continuous correspondence on X to itself. Then, φ has a fixed point.

The following lemma shows: (i) that we may consider the above result as a special case of (K1) of Theorem 1, and (ii) that in a pseudo-metrizable topological vector space, the above result may also be seen as a special case of (K*) of Theorem 1.

Lemma 10 : Let φ be a non-empty closed convex valued upper semi-continuous correspondence on a compact convex subset X of a locally convex Hausdorff topological vector space E over R to itself. Then, the following conditions are satisfied.

(i) For each $x \in K = \{z \in X | z \notin \varphi(z)\}$, there are a vector $p^x \in E'$ and an open neighbourhood U^x of x in X such that for all $z \in U^x$, $w \in \varphi(z)$, $(z \in K) \Longrightarrow (\langle p^x, w-z\rangle > 0)$. (That is, φ satisfies (K1).)

(ii) If E is pseudo-metrizable, then there is a correspondence $\Phi : X \to X$, satisfying that for each $x \in K = \{z \in X | z \notin \varphi(z)\}$, $\varphi(x) \subset \Phi(x)$, $\Phi(x)$ is convex, and there are an open neighbourhood $U(x)$ of x in X and a point $y^x \in X$ such that $\forall z \in U(x) \cap K$, $y^x \in \Phi(z)$. (That is, φ satisfies (K*).)

Proof:

(i) For each $x \in K$, let p^x be the normal vector of a hyper plane which separates x and $\varphi(x)$. Then, by the upper semi-continuity of φ, we have an open neighbourhood U^x of x in X satisfying the condition.

(ii) For each $x \in K$, let p^x be the normal vector of a hyper plane which separates x and $\varphi(x)$. Then, by the upper semi-continuity of φ, we have an open neighbourhood U^x of x in X satisfying the condition stated in (i). If E is pseudo-metrizable, K is also pseudo-metrizable. Hence, K is paracompact and we may suppose that the open cover $\{V(x)\}_{x\in K}$ has a locally finite refinement $\{V(x)\}_{x\in J}$. For each $z \in K$, let $\Phi(z) = \{w \in X | \langle p^x, (w-z)\rangle > 0 \text{ for all } x \in J \text{ such that } z \in V(x)\}$. Moreover, let $\Phi(x) = X$ for each $x \notin K$. Then, for each $z \in K$, by letting $U(z)$ be the intersection $\bigcap_{x\in J, z\in V(x)} V(x)$ and y^z be an arbitrary element of $\varphi(z)$, the correspondence $\Phi : X \to X$ satisfies all of the condition stated in (ii). □

5.2 $\mathscr{L}$-majolized Maps

Let I be a non-empty index set, and let $X = \prod_{i\in I}$ be the product of subsets of a topological vector space E. Moreover, let $\phi : X \to X^i$ be a correspondence on X to a certain X^i. At first, we shall give the following definitions.[2]

(1) We say that ϕ is *of class* $\mathscr{L}$ if $\forall x = (x_j)_{j\in I} \in X$, $x_i \notin \operatorname{co}\phi(x)$ and $\forall y \in X^i$, $\phi^{-1}(y)$ is open in X.

(2) A correspondence $\Phi_x : X \to X^i$ is said to be an $\mathscr{L}$-majorant of ϕ at x if Φ_x is of class $\mathscr{L}$ and there is an open neighbourhood U_x of x in X such that $\phi(z) \subset \Phi_x(z)$ for all $z \in U_x$.

(3) ϕ is said to be $\mathscr{L}$-majolized if for all $x \in X$ such that $\phi(x) \neq \emptyset$, there is an $\mathscr{L}$-majorant of ϕ at x.

For the special case $I = \{i\}$, the following result is known.

Theorem 11: (Yannelis-Prabhakar (1984) Corollary 5.1) Let X be a non-empty, compact, convex subset of a Hausdorff topological vextor space and $P : X \to X$ be an $\mathscr{L}$-majolized correspondence. Then there exists an x^* such that $P(x^*) = \emptyset$.

As stated before, our Theorem 3 essentially generalize the above result as a maximal element existence theorem in the sense that if we assume that there are no maximal elements, then we

[2] More generally, see, e.g., Tan and Yuan (1994).

have $X = K = \{x \in X | P(x) \neq \emptyset\}$ and that P satisfies the condition in Theorem 3 for (K*). If X is a subset of pseudo-metrizable space, we can see that the above Theorem 11 is indeed a special case of our Theorem 3.

Lemma 12 : Let X be a non-empty, compact, convex subset of a pseudo-metrizable topological vector space and $P : X \twoheadrightarrow X^i$ be an $\mathscr{L}$-majolized correspondence. Then, there is a convex non-empty valued correspondence $\Phi : X \twoheadrightarrow X$ such that $\forall x \in K = \{z \in X | P(x) \neq \emptyset\}$, $\Phi(x) \neq \emptyset$, $P(x) \subset \Phi(x)$, $x \notin \Phi(x)$, and for all $x \in K$, there exist a neighbourhood $U(x)$ of x in X and a point $y^x \in X^i$ such that for each $z \in U(x) \cap K$, $y^x \in \Phi(z)$. (That is, for Φ, condition (K*) in Theorem 1 is satisfied.)

Proof : Since P is $\mathscr{L}$-majolized, for each $x \in K$, there are an $\mathscr{L}$-majorant Φ_x of P at x and an open neighbourhood U_x of x in X such that $\forall z \in U_x$, $\phi(z) \subset \Phi_x(z)$. Since X is a subset of pseudo-metrizable space, K is also pseudo-metrizable. Hence, K is paracompact and we may suppose that the open cover $\{U_x\}_{x \in K}$ has a locally finite refinement $\{U_x\}_{x \in J}$. For each $z \in K$, let $\Phi(z) = \bigcap_{x \in J, z \in U_x} \Phi_x(z)$. Moreover, for each $z \notin K$, let $\Phi(z) = X$. Then, for each $z \in K$, by letting $U(z)$ be the intersection $\bigcap_{x \in J, z \in U_x} U_x$ and y^z be an arbitrary element of $P(z)$, the correspondence $\Phi : X \twoheadrightarrow X$ satisfies all of the condition stated above. □

5.3 Eaves' Theorem

The following theorem is known as Eaves' theorem.

Theorem 13 : (Eaves (1974)) *Let S be a simplex of full dimension in R^ℓ and v be a function on S to R^ℓ such that $x + v(x) \in \operatorname{int} S$ for all $x \in S \setminus \operatorname{int} S$. Then, there is a point $x^0 \in S$ such that for all neighbourhood U of x^0 in S, $0 \in \operatorname{co} v[U]$.*

In the theorem, int denotes the interior in R^ℓ and co denotes the convex hull. As we can see in Nishimura and Friedman (1981), Eaves' theorem enables us to constract economic equilibrium arguments without referring to the convexity and/or continuity of individual preferences or best reply correspondences. Here, it is shown that Eaves' theorem may easily be generalized through our Theorem 1.

At first, we see the following lemma which is an immediate consequence of case (K1) of Theorem 1.

Lemma 14 : *Let X be a non-empty compact convex subset of R^ℓ, and f be a function on X to X. Then, there is a point $x^0 \in X$ such that for all neighbourhood U of x^0 in X, $\varphi(x) = f(x) - x$ satisfies $0 \in \operatorname{co} \varphi[U]$.*

Proof : Suppose that for all x in X, there is a neighbourhood U^x of x such that $0 \notin \operatorname{co} \varphi[U^x]$. Then, there is a vector p^x in the topological dual of R^ℓ such that $p^x(\varphi(z)) = p^x(f(z) - z) > 0$ for all $z \in U^x$. Hence, f satisfies the condition (K1) of Theorem 1, so that f has a fixed point x^0, which is contradictory since $0 \neq \varphi(x) = f(x) - x$ for all $x \in X$. □

In the above proof, the separation argument crucially depends on the fact that the dimension of the total space is finite. Now, we prove the main theorem.

Theorem 15 : (Generalization of Eaves' Theorem) *Let X be a non-empty compact convex subset of R^ℓ, and v be a function on X to R^ℓ such that $x+v(x) \in X$ for all $x \in X \setminus \operatorname{int} X$. Then, there is a point $x^0 \in X$ such that for all neighbourhood U of x^0 in X, $0 \in \operatorname{co} v[U]$.*

Proof : For each $x \in \operatorname{int} X$, let λ_x be a positive real number such that $x + \lambda_x v(x) \in X$ and for each $x \in X \setminus \operatorname{int} X$, let $\lambda_x = 1$. Let us define a function $f : X \to X$ as

$$f(x) = x + \lambda_x v(x).$$

By lemma 14, there is $x^0 \in X$ such that for all neighbourhood U of x^0, $0 \in \operatorname{co}\{f(x) - x | x \in U\}$. That is, for a certain natural number n, there are $x^1, \cdots, x^n \in X$ and $\alpha^1, \cdots, \alpha^n \in R_+$, $\sum_{i=1}^n \alpha^i = 1$, such that $0 = \sum_{i=1}^n \alpha^i \lambda_{x^i} v(x^i)$. Hence, if we define λ_0 as $\min\{\lambda_{x^1}, \cdots, \lambda x^n\}$ and λ_i as $\frac{\lambda_{x^i}}{\lambda_0}$ for each $i = 1, \cdots, n$, we have

$$0 \in \operatorname{co}\{\lambda_1 v(x^1), \cdots, \lambda_n v(x^n)\},$$

$\lambda_i \geq 1$ for all $i = 1, \cdots, n$. On the other hand, if $0 \notin \operatorname{co}\{v(x^1), \cdots, v(x^n)\}$, there exists a p in the topological dual of R^ℓ such that $p(v(x^i)) > 0$ for all $i = 1, \cdots, n$. Hence, we have $0 \notin \{x \in R^\ell | p(x) > 0\} \supset \operatorname{co}\{\lambda_1 v(x^1), \cdots, \lambda_n v(x^n)\}$, a contradiction. Therefore, we have $0 \in \operatorname{co}\{v(x^1), \cdots, v(x^n)\}$, and x^0 satisfies the condition stated in the theorem. □

Note that Theorem 15 generalize Theorem 13 in three ways, i.e., in Theorem 15, (i) X may not be a simplex, (ii) X may not be full dimensional, and (iii) $x+v(x)$ may not be an element of $\operatorname{int} X$.

5.4 Further Generalization

Let X be a subset of a topological vector space E. Suppose that for a certain pair (x, y) of elements of X, we may define a convex subset $V(x, y)$ of X satisfying

(i) $x \notin V(x,y)$,

(ii) $y \in V(x,y)$,

(iii) $(z \in V(x,y)) \Longrightarrow (y \in V(x,z))$.

The set $V(x,y)$ may be interpreted as a set representing the direction of y at x. By considering a space X equipped with such a structure, we may obtain the following fixed point theorem, which may considered as a further generalization of Theorem 1. (By taking such a structure appropriately, each condition in Theorem 1 may be considered as a special case of condition (K) in Theorem 16.)

Theorem 16 : (A Generalization of Threorem 1) Let X be a non-empty compact convex subset of a Hausdorff topological vector space E, and let φ be a non-empty valued correspondence on X to X. Suppose that for a certain subset $S \subset X \times X$ and for each $(x,y) \in S$, a convex subset $V(x,y) \subset X$ is defined so that $x \notin V(x,y)$, $y \in V(x,y)$, and for each $z \in X$, $(z \in V(x,y))$ iff $(y \in V(x,z))$. Suppose that φ satisfies the following condition:

(K) For each x such that $x \notin \varphi(x)$, there exist a point $y^x \in X$ and a neighbourhood $U(x)$ of x in X satisfying that $\forall z \in U(x)$, if $z \notin \varphi(z)$, then $\varphi(z) \subset V(z, y^x)$.

Then, φ has a fixed point.

Proof : Assume that φ does not have a fixed point. Then, since $X = \{x \in X | x \notin \varphi(x)\}$ is compact, we have points $x^1, \cdots, x^n \in X$, open neighbourhoods $U(x^1), \cdots, U(x^n)$ of each $x^1, \cdots, x^n$ in X such that $\bigcup_{t=1}^n U(x^t) \supset X$, together with points $y^{x^1}, \cdots, y^{x^n} \in X$ satisfying for each x^t, $t = 1, \cdots, n$, the point y^{x^t} and the neighbourhood $U(x^t)$ satisfies condition (K). Let $\beta_t : X \to [0,1]$, $t = 1, \cdots, n$, be a partition of unity subordinated to $U(x^1), \cdots, U(x^n)$. Let us consider a function f on $D = \mathrm{co}\{y(x^1), \cdots, y(x^n)\}$ to itself such that $f(x) = \sum_{t=1}^n \beta_t(x) y(x^t)$. Then, f is a continuous function on the finite dimensional compact set D to itself. Hence, f has a fixed point z by Brouwer's fixed point theorem. On the other hand, for all t such that $z \in U(x^t)$, $\varphi(z) \subset V(z, y^{x^t})$, hence, for an arbitrary element y of $\varphi(z)$, $y^{x^t} \in V(z, y)$. Since $V(z, y)$ is convex, we have $z = \sum_{t=1}^n \beta_t(z) y(x^t) \in V(z, y)$, which contradicts the condition $z \notin V(z, y)$. □

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REFERENCES

Aliprantis, C. D. and Brown, D. J. (1983): "Equilibria in markets with a Riesz space of commodities," *Journal of Mathematical Economics 11*, 189–207.

Bagh, A. (1998): "Equilibrium in abstract economies without the lower semi-continuity of the constraint maps," *Journal of Mathematical Economics 30*(2), 175–185.

Browder, F. (1968): "The fixed point theory of multi-valued mappings in topological vector spaces," *Mathematical Annals 177*, 283–301.

Debreu, G. (1952): "A social equilibrium existence theorem," *Proceedings of the National Academy of Sciences of the U.S.A. 38*, 886–893. Reprinted as Chapter 2 in G. Debreu, *Mathematical Economics*, Cambridge University Press, Cambridge, 1983.

Debreu, G. (1956): "Market equilibrium," *Proceedings of the National Academy of Sciences of the U.S.A. 42*, 876–878. Reprinted as Chapter 7 in G. Debreu, *Mathematical Economics*, Cambridge University Press, Cambridge, 1983.

Eaves, B. C. (1974): "Properly labeled simplexes," in *Studies in Optimization, Volume II*, (Dantzig, G. B. and Eaves, B. C. ed) , vol. 10 of *Studies in Mathematics*, Mathematical Association of America.

Fan, K. (1952): "Fixed-point and minimax theorems in locally convex topological linear spaces," *Proceedings of the National Academy of Sciences of the U.S.A. 38*, 121–126.

Florenzano, M. (1983): "On the existence of equilibria in economies with an infinite dimensional commodity space," *Journal of Mathematical Economics 12*, 207–219.

Gale, D. (1955): "The law of supply and demand," *Math. Scad. 3*, 155–169.

Glicksberg, K. K. (1952): "A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points," *Proceedings in the American Mathematical Society 3*, 170–174.

Grandmont, J. M. (1977): "Temporary general equilibrium theory," *Econometrica 45*(3), 535–572.

Kakutani, S. (1941): "A generalization of Brouwer's Fixed Point Theorem," *Duke Math. J. 8*(3).

Mehta, G. and Tarafdar, E. (1987): "Infinite-Dimensional Gale-Nikaido-Debreu theorem and a Fixed-point Theorem of Tarafdar," *Journal of Economic Theory 41*, 333–339.

Nash, J. (1950): "Equilibrium states in N-person games," *Proceedings of the National Academy of Sciences of the U.S.A. 36.*

Nash, J. (1951): "Non-cooperative games," *Annals of Mathematics 54*, 289–295.

Neuefeind, W. (1980): "Notes on existence of equilibrium proofs and the boundary behavior of supply," *Econometrica 48*(7), 1831–1837.

Nikaido, H. (1956a): "On the classical multilateral exchange problem," *Metroeconomica 8*, 135–145. A supplementary note: Metroeconomica, 8, 209–210, 1957.

Nikaido, H. (1956b): "On the existence of competitive equilibrium for infinitely many commodities," Tech. Report, Dept. of Econ. No. 34, Stanford University.

Nikaido, H. (1957): "Existence of equilibrium based on the Walras' law," ISER Discussion Paper No. 2, Institution of Social and Economic Research.

Nikaido, H. (1959): "Coincidence and some systems of inequalities," *Journal of The Mathematical Society of Japan 11*(4), 354–373.

Nishimura, K. and Friedman, J. (1981): "Existence of Nash equilibrium in n-person games without quasi-concavity," *International Economic Review 22*, 637–648.

Schaefer, H. H. (1971): *Topological Vector Spaces.* Springer-Verlag, New York/Berlin.

Shafer, W. and H.F.Sonnenschein, (1975): "Equilibrium in abstract economies without ordered preferences," *Journal of Mathematical Economics 2*, 345–348.

Tan, K.-K. and Yuan, X.-Z. (1994): "Existence of equilibrium for abstract economies," *Journal of Mathematical Economics 23*, 243–251.

Urai, K. (1998): "Incomplete markets and temporary equilibria, II: Firms' objectives," in *Ippan Kinkou Riron no Shin Tenkai (New Developments in the Theory of General Equilibrium)*, Japanese, (Kuga, K. ed) , Chapter 8, pp. 233–262, Taga Publishing House, Tokyo.

Urai, K. and Hayashi, T. (1997): "A generalization of continuity and convexity conditions for correspondences in the economic equilibrium theory," Discussion Paper, Faculty of Economics and Osaka School of International Public Policy, Osaka University.

Yannelis, N. (1985): "On a market equilibrium theorem with an infinite number of commodities," *J.Math.Anal.Appl. 108*, 595–599.

Yannelis, N. and Prabhakar, N. (1983): "Existence of maximal elements and equilibria in linear topological spaces," *Journal of Mathematical Economics 12*, 233–245.