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On the Formation and Stability of Networks

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1 Introduction

Recently properties of networks formed by economic agents have attracted the attention of many researchers. There are various economic networks: interconnections of airlines, telephone lines and Internet service providers. Administrative organization of a firm and the industry structure describing relation of firms are considered as information processing networks. Informal cooperate structure of free software developers (as in Linux) is another example.

The rest of the paper is organized as follows. Section 2 introduces a basic model of network formation. Section 3 discusses potential games and refines a concept of Nash equilibrium. Section 4 analyzes an evolutionary network formation process based on bounded rationality of players. Section 5 concludes.

2 Network Formation Game

In this section we describe the key notions of economic networks. Let $\mathcal{N} = \{1, 2, ...N\}$ be the set of players. A network is represented by a graph g in the set $\mathcal{G}$ of all possible graphs. Every player i simultaneously chooses a strategy, which specifies names of other players with whom he wishes to form links. Formally this is described by the vector x_i's with components

$$x_{ij} = \begin{cases} 1 & \text{if } i \text{ wishes to form a link with } j, \\ 0 & \text{otherwise} \end{cases}.$$

We assume that a link between two players are actually formed iff they both wish to form it. So we may describe the network γ realized by a strategy profile x as follows:

$$\gamma(x) = \{ij : \exists i, \exists j \text{ s.t. } x_{ij} = x_{ji} = 1\}.$$

When a network is formed, players get payoffs according to their position in the network and their own strategies. Qin (1996) has also assumed the same rules of network formation. Formally, player i's payoff function π_i is expressed as

$$\pi_i(x) = V_i(\gamma(x)) - C_i(x_i),$$

where V_i is player i's revenue function and C_i is player i's cost function. We assume that $C_i(x_i) \geq 0$ for all x_i and that $C_i(x_i)$ is non-decreasing with respect to x_i. Let $N(g) = \{i \in \mathcal{N} : \exists j \text{ s.t } ij \in g\}$. We assume that $V_i(g) = V_i(g')$ if g' is a component of g such that $i \in N(g')$.

We first observe the essential multiplicity of Nash equilibria (NE) of the network formation games. Various network situations which have few common properties may be realized as Nash equilibria. In fact the following proposition holds true.

Proposition 1.

(A) If g is a NE network, its component g' is also a NE network.

(B) If g and g' are NE networks and $N(g) \cap N(g') = \emptyset$, $g \cup g'$ is also a NE network.

3 Potential Games

We describe two distinct ways of selecting equilibria; one is to choose potential maximizing strategy profiles and the other is to choose evolutionary equilibrium.

Potential games have been introduced by Monderer and Shapley (1996). In a potential game, the set of strategy profiles which locally maximize its potential function coincides with the set of all Nash equilibrium strategy profiles of the game. Thus the strategy profiles which globally maximize its potential function are the most robust Nash equilibria and so we adopt them as the representatives.

Definition 1. Monderer and Shapley (1996) Let $\Gamma(\mathcal{N}, \{X_i\}_{i\in\mathcal{N}}, \{\pi_i\}_{i\in\mathcal{N}})$ be a game in strategic form with finite players. The set of players is $\mathcal{N} = \{1, 2, ..., N\}$, the set of strategies of player i is X_i, and the payoff function of player i is $\pi_i : X \to R$. A function $P : X \to R$ is a potential for Γ if for every $y_{-i} \in X_{-i}$[1]

$$\pi_i(x_i, y_{-i}) - \pi_i(z_i, y_{-i}) = P(x_i, y_{-i}) - P(z_i, y_{-i}) \quad \text{for every } x_i,\ z_i \in X_i.$$

Γ is called a potential game if it admits a potential.

Proposition 2. Monderer and Shapley (1996) LEMMA 2.1 Let P be a potential function for $\Gamma(\mathcal{N}, \{X_i\}_{i\in\mathcal{N}}, \{\pi_i\}_{i\in\mathcal{N}})$. Then the equilibrium set of $\Gamma(\mathcal{N}, \{X_i\}_{i\in\mathcal{N}}, \{\pi_i\}_{i\in\mathcal{N}})$ coincides with the equilibrium set $\Gamma(\mathcal{N}, \{X_i\}_{i\in\mathcal{N}}, \{P\}_{i\in\mathcal{N}})$. That is, $x \in X$[2] is an equilibrium point for Γ if and only if for every $i \in \mathcal{N}$

$$P(x) \geq P(z_i, x_{-i}) \quad \text{for every } z_i \in X_i.$$

With this concept, we analyze a unanimous game.

Example: Unanimous Game Every player i must pay connection cost $k(|x_i|)$ in all situations but can gain common revenue V only when all players are connected by a network. Formally, player i's payoff function is described by a

$$\pi_i(x) = I(\gamma(x))V - k(|x_i|),$$

where

$$I(g) = \begin{cases} 1 & \text{if all players are connected in g,} \\ 0 & \text{otherwise.} \end{cases}$$

We show that if the connection cost function is concave (resp. convex) with respect to the number of links, only a star[3] (resp. line) network is realized as potential maximizing behavior. A star network and a line network are depicted in figure 1.

Definition 2. A network $g \in \mathcal{G}$ is called a star if there exists $i \in \mathcal{N}$ such that

$$g = \{ij : j \in \mathcal{N} \text{ and } j \neq i\}.$$

[1] $X_{-i} := \times_{j\in\mathcal{N}\setminus\{i\}} X_j$.
[2] $X := \times_{j\in\mathcal{N}} X_j$.
[3] We often call it a hub.

Definition 3. A network $g \in \mathcal{G}$ is called a line if there exists a sequence $i_1, i_2, ..., i_N$ of distinct players and g can be described as follows.

$$g = \{i_t i_{t+1} : t = 1, 2, ..., N-1\}.$$

Proposition 3. (A) If V is large enough and $k(n) - k(n-1) < k(n-1) - k(n-2)$ for all $n = 2, 3, ..., N-1$, only star networks are realized by the potential maximizing strategy profiles. (B) If V is large enough and $k(n) - k(n-1) > k(n-1) - k(n-2)$ for all $n = 2, 3, ..., N-1$, only line networks are realized by the potential maximizing strategy profiles.

4 Evolutionary Games

Evolutionary game theory presupposes bounded rationality of players. As in Forster and Young (1990), Young (1993), Kandori, Mailth and Rob (1993), and Nöldeke and Samuelson (1993), we employ a stochastic mutation model. In such a model, only strategy profiles which are most robust against mutants are observed in the long run. In this paper, we additionally assume that every player can observe only those players with whom he forms links. This assumption describes the situation where the attentions of players are limited; player's search behaviors are restricted by his link strategies. In such a case, we can establish the following proposition.

Definition 4. A non-empty network $g \in \mathcal{G}$ is called a hub if there exists a star $f \in \mathcal{G}$ such that $g \subset f$.

Proposition 4.

(A) The empty network is always a long-run equilibrium (LRE) network.

(B) The hub networks which are realized as NE are also contained in the set of LRE. If no star network can be realized as a NE, the empty network is the only LRE.

(C) If a maximal network g is observed in a LRE, there exist a NE network f and a player i which satisfy following conditions.

(C.1) $g \not\subset f$.

(C.2) $g \setminus \cup_{ij \in g}\{ij\} \subset f \setminus \cup_{ik \in g}\{ik\}$.

(C.1) says that there is a player i such that if we delete all links of player i from g and f, the former set is a subset of the latter.

5 Conclusions

We analyzed endogenous network formation problems where all players could freely choose partners.

First we discussed the property of the essential multiplicity of Nash equilibria in the network formation games.

Second we refined the concept of Nash equilibrium in games with potentials. A unanimous games was analyzed and showed that if the connection cost function is concave (resp. convex), only a star (resp. line) network is supported as a robust network.

Finally we studied an evolutionary network formation process where players have bounded rationality and pay limited attention. Three results were obtained. (A) The empty network is always a LRE network. (B) NE hub networks are always LRE networks, and their existence is necessary to the existence of the other non-empty LRE networks. (C) Maximal LRE networks should satisfy a certain selection condition as stated in Proposition 4.

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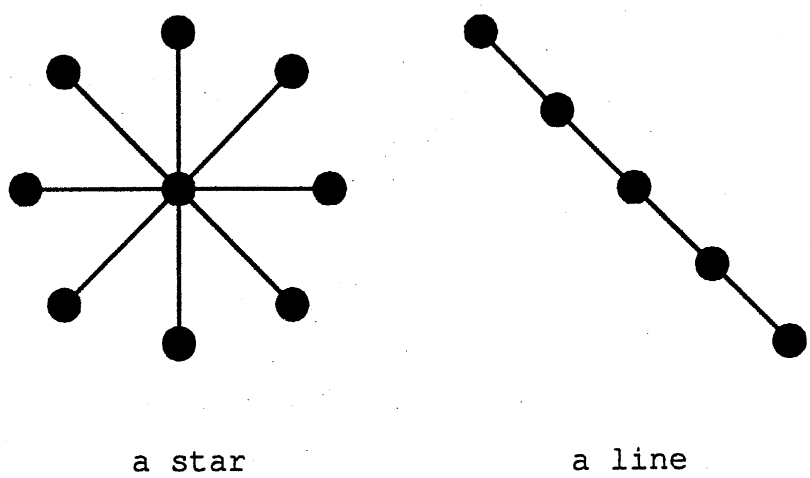

Figure 1: a star and a line