

Pareto Optimum Allocations in the Economy with Clubs

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Abstract

Some necessary conditions for the Pareto optimality of the allocations in an economy with clubs are derived. Also, the price-supported allocations are defined and they are shown to be Pareto optimum. The usual definition of competitive equilibrium for economies only with private goods is extended for an economy with clubs, and it is proved that any allocation under the competitive equilibrium for the economy is Pareto optimum.

1. Introduction

Some commodities are shared and jointly consumed by people. Groups of people who are sharing goods are called "clubs", or consumption ownership-membership arrangements. Commodities consumed separately by a single person are purely private goods, whereas commodities consumed by all the people in the economy are purely public goods. Thus, commodities consumed by clubs are intermediate goods between the purely private good and the purely public good.

In this paper, we consider an economy with clubs and derive necessary conditions for Pareto optimum allocations in the economy. Also, we define price-supported allocations and show that they are Pareto optimum and satisfy the Pareto optimality conditions. Moreover, we define a competitive equilibrium and prove that any allocation under the competitive equilibrium for the economy is Pareto optimum. Our definition is a straight extension of the usual competitive equilibrium for economies only with private goods.

In his famous paper J. M. Buchanan (1965) obtained, as Pareto optimality conditions, the equilibrium conditions for an individual. Y.-K. Ng (1973) derived a more proper optimality condition directly from the definition of Pareto optimality. E. Berglas (1976) derived a condition for social optimality. E. Helpman and A. L. Hillman (1977) pointed out correctly a distinction between Buchanan's and Ng's analyses, and showed an optimality condition for club size. Also, A competitive equilibrium was defined by D. Foley (1967) and D. K. Richter (1974) for economies with public goods, and by S. Scotchmer and M. H. Wooders (1987) for economies with clubs.

2. Model

We consider a simple model of an economy in which there are two kinds of commodities. One of them is a private good and consumed by each single person. The other is a good shared and consumed in a club. The club is a group of people who share the good in consumption. We assume that there is only one club in the economy.

Let us denote a quantity of the good used for the club by " x ", which may be interpreted as the facilities of the club. The number of the members, people participating in the club, is denoted by " n ". We assume that people do not care about who are members of the club, but only about the number of its members. Therefore, the club is specified by pair (x, n) .

We assume that individuals are "divisible" and denote the set of all the persons in the economy by $A=[0, 1]$. The utility function of each person $a \in A$, when he (or she) is a member of club (x, n) , is denoted by

$$u^a = U^a((x, n), y),$$

where y is an amount of the private good.

The following assumption means that people prefer a larger and less crowded club.

Assumption 2.1: For each $a \in A$, $U^a((x, n), y)$ is increasing in both x and y , and decreasing in n .

If a club has no facilities, people can get nothing from belonging to the club. Therefore, people who do not belong to club (x, n) can be regarded as members of club $(0, n)$. Thus, by abusing notation, we denote the utility of person $a \in A$ who is not a member of the club by

$$u^a = U^a((0, n), y).$$

Assumption 2.2: For each $a \in A$, $U^a((0, n), y) = U^a((0, n'), y)$ for all n, y , and n' .

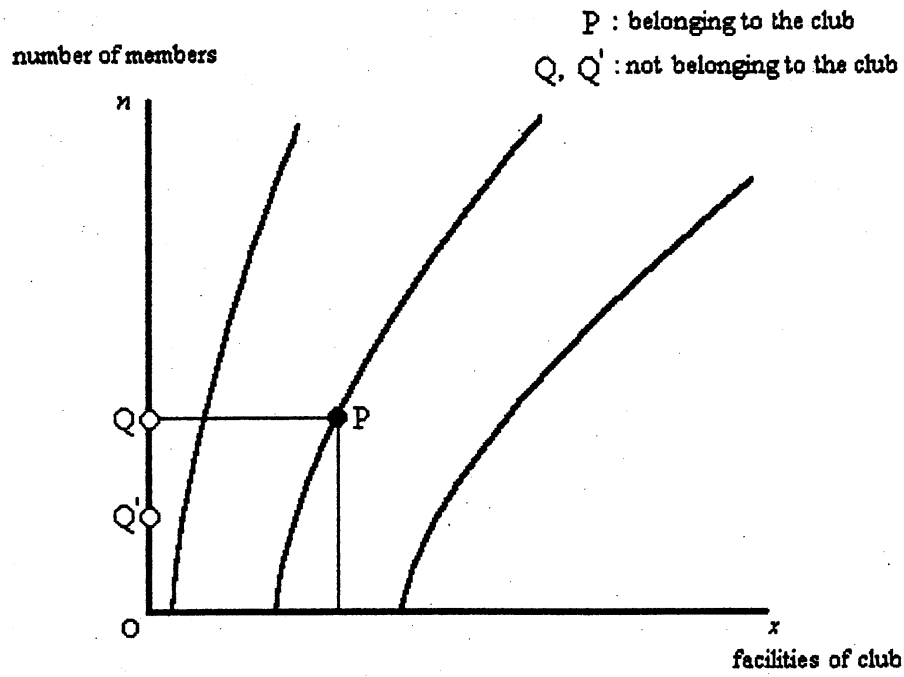


Fig 1: Indifference Curves in club (x, n)

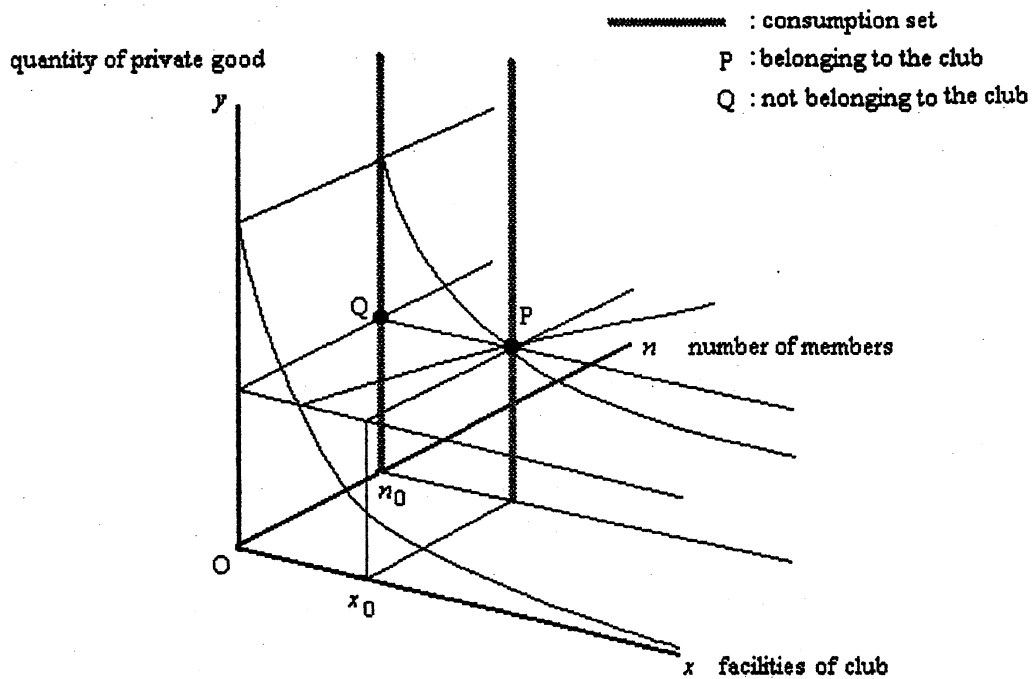


Fig 2: The consumption set for each person (given club (x_0, n_0))

quantity of private good

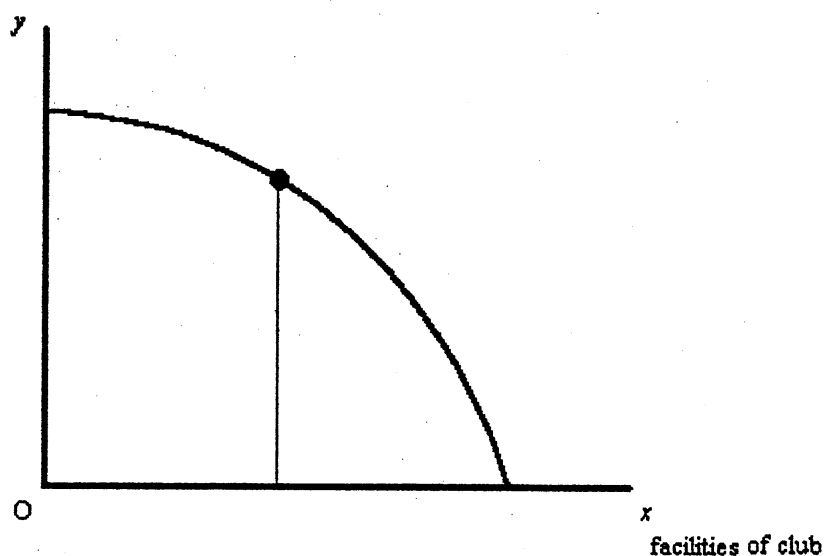


Fig 3: Production Possibility Frontier

Finally, we assume the measurability of the utility map and the continuity of the utility function of each person.

Assumption 2.3:

- (1) Map, $(a, (x, n), y) \rightarrow U^a((x, n), y)$, is measurable.
- (2) For each $a \in A$, $U^a((x, n), y)$ is continuous in $((x, n), y)$.

The production set of commodities is denoted by a set Y , which is described by a function F , i.e.,

$$Y = \{(x, y) \mid x \geq 0, y \geq 0, F(x, y) \leq 0\},$$

where x is a quantity used for the club and y is a quantity of the private good. The production possibility frontier of Y is the set of all the points (x, y) that satisfy

$$F(x, y) = 0.$$

Assumption 2.4: $F(x, y)$ is continuous and increasing in both x and y .

3. Pareto Optimum Allocations

To describe an allocation in the economy, we have to specify the facilities of the club, its members, and the distribution of the private good among people. Let us denote the facilities of the club by a number k and its members by a measurable subset M of A . Then, the club is denoted by (k, M) . Let $\lambda(M)$ be the Lebesgue measure of set M . We assume, without loss of generality, that $\lambda(M)$ is the number of the members of the club.

To denote the distribution of the private good, we use a real-valued measurable function f on A , where $f(a)$ is an quantity of the private good allocated to person $a \in A$. Thus, an allocation in the economy is indicated by these three elements, $\{(k, M), f\}$. An allocation $\{(k, M), f\}$ in the economy is said to be feasible if $(k, \int_A f da) \in Y$.

In allocation $\{(k, M), f\}$, the utility of member $a \in M$ is $U^a((k, \lambda(M)), f(a))$, whereas the utility of non-member $a \in A \setminus M$ is $U^a((0, \lambda(M)), f(a))$. Let χ_M be the indicator function of set M , that is, χ_M is a function such that $\chi_M(a) = 1$ for $a \in M$ and $\chi_M(a) = 0$ for $a \in A \setminus M$. Then, the utility of person $a \in A$ is denoted by $U^a((k\chi_M(a), \lambda(M)), f(a))$.

Definition 3.1: A feasible allocation $\{(k, M), f\}$ is said to be Pareto optimum if there is no other feasible allocation $\{(k', M'), f'\}$ such that

$$U^a((k\chi_M(a), \lambda(M)), f(a)) \leq U^a((k'\chi_{M'}(a), \lambda(M')), f'(a))$$

for all $a \in A$ and the strict inequality in the above holds for some $a \in A$.

In what follows, we confine ourselves to the case in which allocations are in the "interior". Namely, for any allocation $\{(k, M), f\}$, we assume that $k > 0$, $\lambda(M) > 0$, and $f(a) > 0$ for all $a \in A$. Also, we assume that function F and the utility functions of people are all differentiable in the interior of their domains.

The addition of members to the club affects the value of the club to any one member. The private good may be designated a numeraire good, and can be simply thought of as money. The value that each member $a \in M$ of the club loses from adding a member is denoted by

$$MRS_{yn}^a := - \frac{\partial U^a}{\partial n} \div \frac{\partial U^a}{\partial y}.$$

Thus, the total value that the members of the club lose for adding an additional member is $\int_M MRS_{yn}^a da$, which is the admission fee, or the price of membership of the club.

The following theorem corresponds to one of the Pareto optimality conditions asserted by Y.-K. Ng (1973) in an economy with clubs.

Theorem 3.1: Let $\{(k, M), f\}$ be a Pareto optimum allocation and put $q = \int_M MRS_{ym}^a da$. Then the following holds:

$$U^a((0, \lambda(M)), f(a) + q) \leq U^a((k, \lambda(M)), f(a)) \quad \text{for all } a \in M$$

and

$$U^a((k, \lambda(M)), f(a) - q) \leq U^a((0, \lambda(M)), f(a)) \quad \text{for all } a \in A \setminus M.$$

Proof: Suppose that $U^a((0, \lambda(M)), f(a) + q) > U^a((k, \lambda(M)), f(a))$ for some $a \in M$. Then, there exist $\varepsilon > 0$ and $E \subset M$ with $\lambda(E) > 0$ such that

$$U^a((0, \lambda(M) - \lambda(E)), f(a) + q - \int_E MRS_{ym}^a da - \varepsilon) > U^a((k, \lambda(M)), f(a))$$

for all $a \in E$.

Define $g: A \rightarrow \mathcal{R}_+$ by

$$g(a) = \begin{cases} f(a) - (MRS_{ym}^a - \frac{\varepsilon}{\lambda(M) - \lambda(E)})\lambda(E) & \text{for } a \in M \setminus E \\ f(a) + q - \int_E MRS_{ym}^a da - \varepsilon & \text{for } a \in E \\ f(a) & \text{for } a \in A \setminus M \end{cases}$$

Then, clearly, $\int_M g da = \int_M f da$. Also, if we choose E so that $\lambda(E)$ is sufficiently small, then we

have

$$\begin{aligned} U^a((k, \lambda(M) - \lambda(E)), g(a)) &> U^a((k, \lambda(M) - \lambda(E)), f(a) - MRS_{ym}^a \lambda(E)) \\ &\doteq U^a((k, \lambda(M)), f(a)) \quad \text{for all } a \in M \setminus E. \end{aligned}$$

This shows that allocation $\{(k, M \setminus E), g\}$ improves allocation $\{(k, M), f\}$, which contradicts the Pareto optimality of $\{(k, M), f\}$.

On the other hand, suppose that $U^a((k, \lambda(M)), f(a) - q) > U^a((0, \lambda(M)), f(a))$ for some $a \in A \setminus M$. Then, there exist $\varepsilon > 0$ and $E \subset A \setminus M$ with $\lambda(E) > 0$ such that

$$U^a((k, \lambda(M) + \lambda(E)), f(a) - q - \varepsilon) > U^a((0, \lambda(M)), f(a)) \quad \text{for all } a \in E.$$

Define $g: A \rightarrow \mathcal{R}_+$ by

$$g(a) = \begin{cases} f(a) + (MRS_{ym}^a + \frac{\varepsilon}{\lambda(M)})\lambda(E) & \text{for } a \in M \\ f(a) - q - \varepsilon & \text{for } a \in E \\ f(a) & \text{for } a \in A \setminus (M \cup E) \end{cases}$$

Then, clearly, $\int_A g da = \int_A f da$. Also, if we choose E so that $\lambda(E)$ is sufficiently small, then we have

$$U^a((k, \lambda(M) + \lambda(E)), g(a)) > U^a((k, \lambda(M) + \lambda(E)), f(a) + MRS_{ym}^a \lambda(E)) \\ \doteq U^a((k, \lambda(M)), f(a)) \quad \text{for all } a \in M.$$

This shows that allocation $\{(k, M \cup E), g\}$ improves allocation $\{(k, M), f\}$, which contradicts the Pareto optimality of $\{(k, M), f\}$. ■

The condition in the above theorem says that any member of the club wants to have more than q for leaving the club, whereas any non-member of the club will not pay more than q for entering the club. Thus, no Pareto improvement can be made by any contract between any member and any non-member of the club.

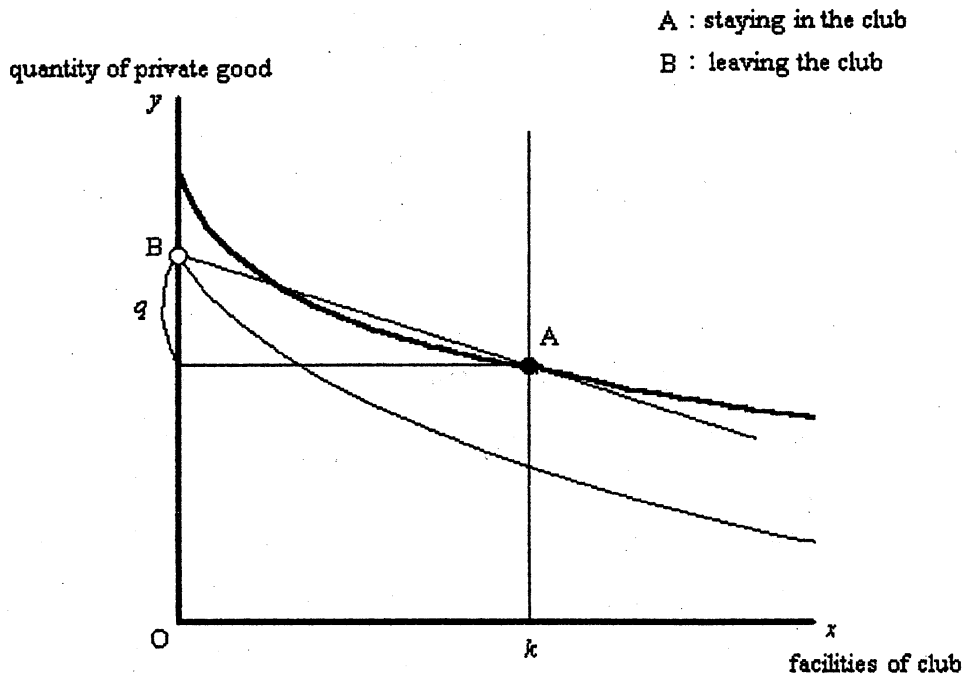


Fig. 5: Indifference curves in (x, y) for a member

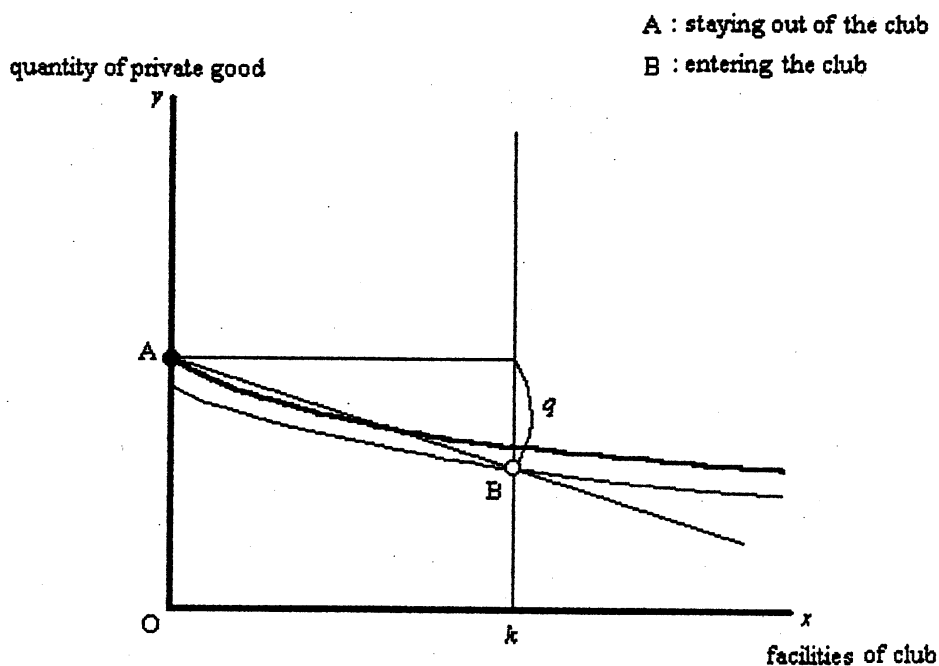


Fig. 6: Indifference curves (x, y) for a non-member

The value that each member $a \in M$ of the club gains from increasing the facilities of the club by one unit is denoted by

$$MRS_{yx}^a := \frac{\partial U^a}{\partial x} \div \frac{\partial U^a}{\partial y}.$$

The marginal cost to increase the facilities of the club is denoted by

$$MRT_{yx} := \frac{\partial F}{\partial x} \div \frac{\partial F}{\partial y}.$$

The following theorem is one of the Pareto optimality conditions derived by J. M. Buchanan (1965) and Y.-K. Ng (1973), which is a generalization of the Pareto optimality condition for allocations of purely public goods proved by P. A. Samuelson (1954).

Theorem 3.2: Let $\{(k, M), f\}$ be a Pareto optimum allocation. Then, we have

$$\int_M MRS_{yx}^a da = MRT_{yx}$$

Proof: Suppose $\int_M MRS_{yx}^a da < MRT_{yx}$. Then, there exists $\varepsilon > 0$ such that $\int_M MRS_{yx}^a da + \varepsilon <$

MRT_{yx} , and for all sufficiently small $\delta > 0$,

$$(k - \delta, \int_A f da + (\int_M MRS_{yx}^a da + \varepsilon) \delta) \in Y.$$

Define $g: A \rightarrow \mathcal{R}_+$ by

$$g(a) = \begin{cases} f(a) + (MRS_{yx}^a + \frac{\varepsilon}{\lambda(M)})\delta & \text{for } a \in M \\ f(a) & \text{for } a \in A \setminus M \end{cases}$$

Then, clearly, $\int_A g da = \int_A f da + (\int_M MRS_{yx}^a da + \varepsilon) \delta$, and therefore $(k - \delta, \int_A g da) \in Y$, which

implies that allocation $\{(k - \delta, M), g\}$ is feasible. Also, if we choose a small δ , then

$$\begin{aligned} U^a((k - \delta, \lambda(M)), g(a)) &> U^a((k - \delta, \lambda(M)), f(a) + MRS_{yx}^a \delta) \\ &\doteq U^a((k, \lambda(M)), f(a)) \quad \text{for all } a \in M. \end{aligned}$$

This shows that allocation $\{(k - \delta, M), g\}$ improves allocation $\{(k, M), f\}$, which contradicts the Pareto optimality of $\{(k, M), f\}$.

In case of $\int_M MRS_{yx}^a da > MRT_{yx}$, choose $\varepsilon < 0$ and $\delta < 0$ such that $\int_M MRS_{yx}^a da + \varepsilon > MRT_{yx}$ and $(k - \delta, \int_A f da + (\int_M MRS_{yx}^a da + \varepsilon) \delta) \in Y$. Then, we can have the same contradiction. ■

4. Supporting Prices

The conditions in Theorems 3.1 and 3.2 are necessary conditions for Pareto optimality, but not sufficient conditions. In what follows, we will show a sufficient condition for Pareto optimality.

We assume that the price of the private good is unity. Let us denote the price of the commodity used for the club by p .

Definition 4.1: A feasible allocation $\{(k, M), f\}$ is said to be supported by a price p if the following conditions are satisfied:

(1) If $\{(k', M'), f'\}$ is an allocation such that

$$U^a((k \chi_M(a), \lambda(M)), f(a)) \leq U^a((k' \chi_{M'}(a), \lambda(M')), f'(a)) \quad \text{for all } a \in E,$$

then we have

$$\frac{pk}{\lambda(M)} \lambda(M \cap E) + \int_E f da \leq \frac{pk'}{\lambda(M')} \lambda(M' \cap E) + \int_E f da.$$

$$(2) \quad pk + \int_A f da \geq px + y \quad \text{for all } (x, y) \in Y.$$

The above definition means that, if a feasible allocation is supported by a price, people are minimizing their costs and the value of produced goods is maximized.

The following theorem shows that being supported by a price is a sufficient condition for feasible allocations to be Pareto optimum.

Theorem 4.1: If a feasible allocation is supported by a price, then it is Pareto optimum.

Proof: Suppose that a feasible allocation $\{(k, M), f\}$ supported by a price p were not Pareto optimum. Then there is a feasible allocation $\{(k', M'), f'\}$ such that

$$U^a((k \chi_M(a), \lambda(M)), f(a)) \leq U^a((k' \chi_{M'}(a), \lambda(M')), f'(a))$$

for all $a \in A$ and the strict inequality holds for some $a \in A$. Therefore, by (1) of Definition 4.1, we can show that

$$pk + \int_A f da < pk' + \int_A f' da,$$

which contradicts (2) of Definition 4.1. ■

Lemma 4.1: If a feasible allocation $\{(k, M), f\}$ is supported by a price p , then

$$\int_M MRS_{ym}^a da = \frac{pk}{\lambda(M)} \quad \text{and} \quad MRT_{yx} = p.$$

Proof: Let $E \subset M$ and $\lambda(E) > 0$. If $\lambda(E)$ is sufficiently small, then for each $a \in M$ there exists $\varepsilon(a) > 0$ such that

$$U^a((k, \lambda(M) - \lambda(E)), f(a) - (MRS_{ym}^a - \varepsilon(a))\lambda(E)) > U^a((k, \lambda(M)), f(a)).$$

Now, let $\{(k', M'), f'\}$ be an allocation such that $k' = k$, $M' = M \setminus E$, and

$$f'(a) = \begin{cases} f(a) & \text{for each } a \in A \setminus M' \\ f(a) - (MRS_{ym}^a - \varepsilon(a))\lambda(E) & \text{for each } a \in M'. \end{cases}$$

Then, $U^a((k, \lambda(M')), f'(a)) > U^a((k, \lambda(M)), f(a))$ for each $a \in M'$. Therefore, by (1) of Definition 4.1, we have

$$\begin{aligned} \frac{pk}{\lambda(M)} \lambda(M') + \int_{M'} f da &\leq pk + \int_{M'} f da \\ &= pk + \int_M f da + \lambda(E) \left(- \int_M MRS_{ym}^a da + \int_M \varepsilon(a) da \right), \end{aligned}$$

$$\text{i.e., } \int_{M'} MRS_{ym}^a da \leq \frac{pk}{\lambda(M)} + \int_M \varepsilon(a) da.$$

For each $a \in M$, when $\lambda(E)$ goes to 0, $\varepsilon(a)$ also goes to 0. Thus, the above inequality implies that

$$\int_M MRS_{ym}^a da \leq \frac{pk}{\lambda(M)}.$$

In order to get the opposite inequality, let $E \subset A \setminus M$ and $\lambda(E) > 0$. If $\lambda(E)$ is sufficiently small, then for each $a \in M$ there exists $\varepsilon(a) > 0$ such that

$$U^a((k, \lambda(M) + \lambda(E)), f(a) + (MRS_{ym}^a + \varepsilon(a))\lambda(E)) > U^a((k, \lambda(M)), f(a)).$$

Now, let $\{(k', M'), f'\}$ be an allocation such that $k' = k$, $M' = M \cup E$, and

$$f'(a) = \begin{cases} f(a) & \text{for each } a \in A \setminus M \\ f(a) + (MRS_{ym}^a + \varepsilon(a))\lambda(E) & \text{for each } a \in M. \end{cases}$$

Then, $U^a((k, \lambda(M')), f'(a)) > U^a((k, \lambda(M)), f(a))$ for each $a \in M$. Therefore, by (1) of Definition 4.1, we have

$$\begin{aligned} pk + \int_M f da &\leq \frac{pk}{\lambda(M')} \lambda(M) + \int_M f da \\ &= \frac{pk}{\lambda(M')} \lambda(M) + \int_M f da + \lambda(E) \left(\int_M MRS_{ym}^a da + \int_M \varepsilon(a) da \right), \end{aligned}$$

$$\text{i.e., } \frac{pk}{\lambda(M')} \leq \int_M MRS_{ym}^a da + \int_M \varepsilon(a) da.$$

For each $a \in M$, when $\lambda(E)$ goes to 0, $\varepsilon(a)$ also goes to 0. Thus, the above inequality implies that

$$\frac{pk}{\lambda(M)} \leq \int_M MRS_{ym}^a da.$$

Finally, by (2) of Definition 4.1, we can easily show that $MRT_{yx} = p$. ■

Now we can show that the membership fee of the club is $\frac{pk}{\lambda(M)}$ and the demand price for the club is p .

Theorem 4.2: Let $\{(k, M), f\}$ be a feasible allocation supported by a price p . Then, we have the following:

$$(1) \quad U^a\left((0, \lambda(M)), f(a) + \frac{pk}{\lambda(M)}\right) \leq U^a((k, \lambda(M)), f(a)) \quad \text{for all } a \in M$$

and

$$U^a\left((k, \lambda(M)), f(a) - \frac{pk}{\lambda(M)}\right) \leq U^a((0, \lambda(M)), f(a)) \quad \text{for all } a \in A \setminus M.$$

$$(2) \quad \int_M MRS_{yx}^a da = p$$

Proof: By Theorem 4.1, any feasible allocation supported by a price is Pareto optimum, and therefore this theorem immediately follows from Lemma 4.1, Theorems 3.1 and 3.2. ■

We do not know under what assumptions the converse of Theorem 4.1 holds, that is, if an allocation is Pareto optimum, then it is supported by a price. The characterization, by using prices, of Pareto optimum allocations in the economy with clubs is an open problem.

5. Competitive Equilibrium

Let us denote the price of the good used for the club by p and the price of membership of the club by q .

Definition 5.1: A feasible allocation $\{(k, M), f\}$ is said to be competitive if there exist prices p and q such that the following conditions are satisfied:

$$(1) \quad U^a((0, \lambda(M)), f(a) + q) \leq U^a((k, \lambda(M)), f(a)) \quad \text{for all } a \in M$$

and

$$U^a((k, \lambda(M)), f(a) - q) \leq U^a((0, \lambda(M)), f(a)) \quad \text{for all } a \in A \setminus M.$$

(2) If $\{(k', M'), f'\}$ is an allocation such that

$$U^a((k\chi_M(a), \lambda(M)), f(a)) \leq U^a((k', \lambda(M')), f'(a)) \quad \text{for all } a \in M',$$

then

$$q\lambda(M \cap M') + \int_{M'} f da \leq pk' + \int_{M'} f da.$$

$$(3) \quad q\lambda(M) - pk = 0$$

$$(4) \quad pk + \int_A f da \geq px + y \quad \text{for all } (x, y) \in Y.$$

In the above definition, condition (1) means that each person is maximizing utility under a budget constraint. Condition (2) means that the club can't change its members by making better offers to new members at the same cost. Therefore, conditions (1) and (2) imply that the market of membership is in equilibrium. Condition (3) means that the market of membership is competitive and the club gets no profits in equilibrium. Condition (4) means the producers of commodities are maximizing profits.

In (1) of the above definition, it is assumed that each person decide whether he (or she) should join the existing club, or not. Therefore, our definition of competitive equilibrium is different from that of S. Scotchmer, S. and M. H. Wooders (1987), in which people choose one club to join among many potentially existing clubs.

Now we can prove the basic theorem of welfare economics for economies with clubs.

Theorem 5.1: Any competitive allocation is Pareto optimum.

Proof: Suppose that a competitive allocation $\{(k, M), f\}$ were not Pareto optimum. Then there is a feasible allocation $\{(k', M'), f'\}$ such that

$$U^a((k\chi_M(a), \lambda(M)), f(a)) \leq U^a((k'\chi_{M'}(a), \lambda(M')), f'(a)) \quad \text{for all } a \in A,$$

where the strict inequality holds for some $a \in A$.

For each $a \in M'$, we have

$$U^a((k\chi_M(a), \lambda(M)), f(a)) \leq U^a((k', \lambda(M')), f'(a)).$$

Therefore, by (2) of Definition 5.1, we have

$$q\lambda(M \cap M') + \int_{M'} f da \leq pk' + \int_{M'} f' da.$$

For each $a \in M \setminus M'$, we have

$$U^a((k, \lambda(M)), f(a)) \leq U^a((0, \lambda(M')), f'(a)) = U^a((0, \lambda(M)), f'(a)),$$

which implies, by (1) of Definition 5.1, that $f(a) + q \leq f'(a)$. Thus, we have

$$q\lambda(M \setminus M') + \int_{M \setminus M'} f da \leq \int_{M \setminus M'} f' da.$$

For each $a \in A \setminus (M \cup M')$, we have

$$U^a((0, \lambda(M)), f(a)) \leq U^a((0, \lambda(M')), f'(a)) = U^a((0, \lambda(M)), f'(a)),$$

which implies that $f(a) \leq f(a)$. Thus, we have

$$\int_{A \setminus (M, M')} f da \leq \int_{A \setminus (M, M')} f da .$$

In one of the above three inequalities, the strict inequality holds. Therefore, by adding them up, we have

$$q \lambda(M) + \int_A f da < pk' + \int_A f da ,$$

which, by (3) of Definition 5.1, contradicts (4) of Definition 5.1. ■

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