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$\mathcal{S}_4$ coverings of $\mathbf{P}^2$ and the topology of the complements of sextic curves

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Introduction

The main purpose of this article is to study the topology of the complements to reduced plane curves through a theory of branched Galois coverings. Let us start with introducing a few words on branched Galois coverings.

Let Y be a normal projective variety. Let X be a normal variety with finite surjective morphism $\pi : X \to Y$. Under this circumstance, the field of rational functions, $\mathbf{C}(X)$, of X is a finite extension of that of Y, $\mathbf{C}(Y)$, with $[\mathbf{C}(X) : \mathbf{C}(Y)] = \deg \pi$. We call X a Galois covering of Y, if $\mathbf{C}(X)$ is a Galois extension of $\mathbf{C}(Y)$. Let G be a finite group. We simply call X a G covering of Y if X is a Galois covering and $\mathrm{Gal}(\mathbf{C}(X)/\mathbf{C}(Y)) \cong G$. The branch locus of π is a subset of Y given by $\{y \in Y | \sharp(\pi^{-1}(y)) < \deg \pi\}$; and we denote it by $\Delta(X/Y)$ or Δ_π. If Y is smooth, then Δ_π is an algebraic subset of codimension 1 by the purity of the branch locus. Suppose that Y is smooth and let $\Delta_\pi = B_1 + \cdots + B_r$ be the decomposition into irreducible components. The ramification index of π along B_i is the one along the smooth part of B_i. If we say that a G covering $\pi : X \to Y$ is branched at $e_1 B_1 + \cdots + e_r B_r$, it means that (i) X is a G covering with $\Delta_\pi = B_1 + \cdots B_r$, and (ii) the ramification index along B_i is e_i.

Branched Galois coverings that play important roles in this article are those with Galois group $\mathcal{S}_4$ (the symmetric group of 4 letters). We make such coverings to understand the topology of the complements to plane sextic curves. In order to make our problem clear and to see the role of $\mathcal{S}_4$ coverings, let us review previous known results.

Let B be a reduced plane curve in $\mathbf{P}^2$. One of the fundamental questions that has been in our mind so far is as follows:

Question 0.1. What can one say about $\mathbf{P}^2 \setminus B$ just from the data of local topological type of singularities? To be more specific, can one determine whether the fundamental group $\pi_1(\mathbf{P}^2 \setminus B)$ is abelian or non-abelian just from such condition?

In what follows, we simply say the configuration of singularities instead of the data of local topological types of singularities.

From the viewpoint of Question 0.1, there do not seem to be many result on the non-commutativity on $\pi_1(\mathbf{P}^2 \setminus B)$, while there are several results on the commutativity (see [1], [3], [4], [8]).

In [14], the author gave a result on the non-commutativity. To explain it, we need some setting-ups.

Let B be as before and assume that B has at most simple singularities. We use the lower cases, a_n, d_n and e_n to describe the types of them. For $x \in \mathrm{Sing}(B)$, we denote its Milnor number by μ_x. We define the total Milnor number, μ_B, of B to be $\sum_{x\in \mathrm{Sing}(B)} \mu_x$. We next define a non-negative integer, l_p, for an odd prime p as follows:

if $p = 3$, $l_3 =$ the number of a_{3k-1} $(k \geq 1)$ and e_6, and

if $p \geq 5$, $l_p \geq 5$, $l_p =$ the number of a_{pk-1}.

Using these notations, we have

Theorem 0.2. ([14]) *Suppose that* $\deg B = d$ *is even. If there exists an odd prime* p *such that*

$$l_p + \mu_B > d^2 - 3d + 3,$$

then there exists a surjective homomorphism

$$\pi_1(\mathbf{P}^2 \setminus B) \to \mathcal{D}_{2p} = \langle \sigma, \tau | \sigma^2 = \tau^p = (\sigma\tau)^2 = 1\rangle.$$

In particular, $\pi_1(\mathbf{P}^2 \setminus B)$ *is non-abelian.*

Corollary 0.3. *The notations are the same as in Theorem 0.2. Suppose that B has only nodes and cusps and let a and b be the number of nodes and cusps, respectively. If* $a + 3b > d^2 - 3d + 3$, *then* $\pi_1(\mathbf{P}^2 \setminus B)$ *is non-abelian.*

The proof of Theorem 0.2 is based on an existence theorem on $\mathcal{D}_{2p}$ coverings branched at $2B$. Hence the inequality in Theorem 0.2 seems to give a very rough criterion. For sextic curves and $p = 3$, however, the inequality is sharp from the following result:

Theorem 0.4. ([2], [7], [11], [12]) *There exists a pair of irreducible sextic curves* (B_1, B_2) *as follows:*

(i) Both B_1 *and* B_2 *have the same configuration of singularities; and it is one of the following:*

$$3a_5 + 3a_1, \quad 6a_2 + 3a_1, \quad 3e_6, \quad e_6 + 4a_2 + 2a_1.$$

(ii) There exists a surjective homomorphism $\pi_1(\mathbf{P}^2 \setminus B_1) \to \mathcal{S}_3$ *for* B_1, *while there is no such homomorphism for* B_2.

On the other hand, it is known that there exist sextic curves, B_3, having the configurations of singularities: $3a_5 + 3a_1$, $6a_2 + 4a_1$, $e_6 + 4a_2 + 3a_1$, $3e_6 + a_1$ (see [15]). For B_3, the inequality in Theorem 0.2 is satisfied for $p = 3$. Hence there exists a surjective homomorphism $\pi_1(\mathbf{P}^2 \setminus B_3) \to \mathcal{S}_3$. In particular, $\pi_1(\mathbf{P}^2 \setminus B_3)$ is non-abelian.

These examples seem to be rather interesting, since the difference of the configurations of singularities between B_1 in Theorem 0.2 and B_3 is just the number of nodes. From observation from the commutativity statements as in [1], [3], [4], [8], the number of nodes does not seem to give much effect on the non-commutativity on the fundamental group of the residual space. In fact, by [2], the Alexander polynomials for B_1 in Theorem 0.4 and those for B_3 are $t^2 - t + 1$. This shows that one can not measure the difference of the non-commutativity between $\pi_1(\mathbf{P}^2 \setminus B_1)$ and $\pi_1(\mathbf{P}^2 \setminus B_3)$ by the Alexander polynomials, while they are likely to be different.

Now $\mathcal{S}_4$ coverings come in to our picture. We need them to see that the topology of $\mathbf{P}^2 \setminus B_1$ is different from $\mathbf{P}^2 \setminus B_3$; and it is the goal of this article.

Let B be a reduced sextic curve with at most simple singularities, and Let $f : Z' \to \mathbf{P}^2$ be a double covering branched along B and let $\mu : Z \to Z'$ be the canonical resolution. By the assumption it is minimal. Let $\mathrm{NS}(Z)$ be the Néron-Severi group of Z and let R be the subgroup of $\mathrm{NS}(Z)$ generated by all the irreducible components of the exceptional divisor of μ. Note that R has a natural decomposition $R = \oplus_{x \in \mathrm{Sing}(W)} R_x$, where R_x is the subgroup of $\mathrm{NS}(Z)$ generated by the exceptional divisor arising from x. As we assume that B has only simple singularities, R_x is isomorphic to one of the so-called A-D-E lattices. The graph of R_x is the dual graph of the exceptional set for x. We denote it by $G(R_x)$ and the graph, $G(R)$, of R is $\sum_{x \in \mathrm{Sing}(Z')} G(R_x)$. By our assumption, $G(R_x)$ is one of the Dynkin graphs, which we denote by the bold characters $\mathbf{A}_n$, $\mathbf{D}_n$ and $\mathbf{E}_n$. Note that these types correspond to those of lattices. Let G_1 be a subgraph of $G(R)$. We denote the subgroup (or lattice) of $\mathrm{NS}(Z)$ generated by the vertices of G_1 by $\mathcal{L}(G_1)$.

Now we are in position to state our main result.

Theorem 0.5. *Let $\pi : S \to \mathbf{P}^2$ be an $\mathcal{S}_4$ covering of $\mathbf{P}^2$ such that (i) π is branched at $2B$ and (ii) π factors $f : Z' \to \mathbf{P}^2$. Then $G(R)$ contains a subgraph either $\mathbf{A}_2^{\oplus 9}$ or $\mathbf{A}_2^{\oplus 6} \oplus \mathbf{A}_1^{\oplus 4}$.*

By Theorem 0.5, we can infer that here are no $\mathcal{S}_4$ coverings for the B_1's in Theorem 0.4.

Theorem 0.6. *Suppose that $G(R)$ contains $\mathbf{A}_2^{\oplus 6} \oplus \mathbf{A}_1^{\oplus 4}$ such that $\mathbf{A}_1^{\oplus 4}$ is a invariant block under the involution induced by the covering transformation. Then there exists an $\mathcal{S}_4$ covering of $\mathbf{P}^2$ such that (i) π is branched at $2B$ and (ii) π factors $f : Z' \to \mathbf{P}^2$.*

Theorem 0.6 shows that there exist $\mathcal{S}_4$ coverings for the B_3's as above.

§1 $\mathcal{S}_4$ coverings of algebraic varieties

We here give a short summary on $\mathcal{S}_4$ coverings of algebraic varieties that we need in the remaining section.

Let $\pi : X \to Y$ be an $\mathcal{S}_4$ covering. Let V_4 and $\mathcal{A}_4$ be Klein's four group and the alternating group of degree 4, respectively. We denote the invariant fields of V_4 and $\mathcal{A}_4$ by $\mathbf{C}(X)^{V_4}$ and $\mathbf{C}(X)^{\mathcal{A}_4}$, respectively. Let X^{V_4} (resp. $X^{\mathcal{A}_4}$) be the $\mathbf{C}(X)^{V_4}$ (resp. $\mathbf{C}(X)^{\mathcal{A}_4}$)-normalization of Y. Then:

(*i*) X is a V_4 covering of X^{V_4} and X^{V_4} is an $\mathcal{S}_3$ covering of Y. We denote the covering morphisms by $\nu_{V_4} : X \to X^{V_4}$ and $\pi_{V_4} : X^{V_4} \to Y$.

(*ii*) X is an $\mathcal{A}_4$ covering of $X^{\mathcal{A}_4}$ and $X^{\mathcal{A}_4}$ is a double covering of Y. We denote the covering morphisms by $\nu_{\mathcal{A}_4} : X \to X^{\mathcal{A}_4}$ and $\pi_{\mathcal{A}_4} : X^{\mathcal{A}_4} \to Y$.

(*iii*) X^{V_4} is a $\mathbf{Z}/3\mathbf{Z}$ covering of $X^{\mathcal{A}_4}$ and we denote the covering morphism by $\nu_{\mathcal{A}_4/V_4} : X^{V_4} \to X^{\mathcal{A}_4}$. Note that $\nu_{\mathcal{A}_4} = \nu_{\mathcal{A}_4/V_4} \circ \nu_{V_4}$.

Combining these all morphisms, we have the following diagram:

$$\begin{array}{ccc} X & \overset{\nu_{V_4}}{\to} & X^{V_4} \\ \pi\downarrow & & \downarrow \nu_{\mathcal{A}/V_4} \\ Y & \overset{\pi_{\mathcal{A}_4}}{\leftarrow} & Y^{\mathcal{A}_4}. \end{array}$$

We should notice that X^{V_4} and $X^{\mathcal{A}_4}$ are canonically determined once an $\mathcal{S}_4$ covering is given. Our idea to understand $\mathcal{S}_4$ coverings is a similar one to in [9]. Namely it may be formulated.

Problem 1.1. *Function field version* (i) Let $f : W \to Y$ be an $\mathcal{S}_3$ covering of Y. Find a condition for a bi-quadratic extension, K, of $\mathbf{C}(W)$ such that (*i*) K is a Galois extension of $\mathbf{C}(Y)$ with Galois group $\mathcal{S}_4$, and (*ii*) $K^{V_4} = \mathbf{C}(W)$. (Note that K-normalization of Y gives an $\mathcal{S}_4$ covering of Y.)

Geometric version (ii) Let $f : W \to Y$ be a smooth $\mathcal{S}_3$ covering of Y, and let D be an effective divisor divisor on W. Find a condition on D for the existence of an $\mathcal{S}_4$ covering, X, of Y satisfying (*a*) $X_{V_4} = W$ and (*b*) $\Delta(X/W) \subset \mathrm{Supp}(D)$

For (i), we have the following result:

Proposition 1.2. *Let $f : W \to Y$ be an $\mathcal{S}_3$ covering of Y. Suppose that there exist three rational functions, φ_1, φ_2 and φ_3 with the following properties:*
(*i*) *$\varphi_i \notin (\mathbf{C}(X)^\times)^2$ for each i.*
(*ii*) *If we denote $\mathrm{Gal}(\mathbf{C}(W)/\mathbf{C}(Y)) = \langle \sigma, \tau | \sigma^2 = \tau^3 (\sigma\tau)^2 = 1\rangle$, then*
(*iia*) *$\varphi_1^\sigma = \varphi_2$, $\varphi_3^\sigma = \varphi_3$,*

and

(*iib*) $\varphi_1^\tau = \varphi_2$, $\varphi_2^\sigma = \varphi_3$, $\varphi_3^\sigma = \varphi_1$.

(*iii*) $\varphi_1\varphi_2\varphi_3 \in (f^*\mathbf{C}(Y)^\times)^2$.

Then the bi-quadratic extension $K = \mathbf{C}(W)(\sqrt{\varphi_1}, \sqrt{\varphi_2})$ *is an* S_4 *extension of* $\mathbf{C}(Y)$ *such that* $K^{V_4} = \mathbf{C}(W)$. *In particular, the* K*-normalization,* X*, of* Y *is an* S_4 *covering of* Y *with* $X^{V_4} = W$.

Conversely, if there exists an S_4 *covering* $\pi : X \to Y$ *with* $X^{V_4} = W$*, there there exist three rational functions* φ_1*,* φ_2*, and* φ_3 *in* $\mathbf{C}(W)$ *satisfying the three properties* (*i*), (*ii*) *and* (*iii*) *as above.*

Our proof is based on the Galois theory and Lagrange's method in solving a quartic equation. For details, see [16]

Proposition 1.2 gives an answer to Problem 1.1 (i). We now go on to the second question.

Proposition 1.3. *Let* $f : W \to Y$ *be a smooth* S_3 *covering of* Y. *Suppose that there exist three different reduced divisors,* D_1*,* D_2 *and* D_3 *on* W *such that*

(*i*) *With the same notation on* Gal(W/Y) *as those in Proposition 1.2,*

(*ia*) $D_1^\sigma = D_2$ *and* $D_3^\sigma = D_3$*, and*

(*ib*) $D_1^\tau = D_2$, $D_2^\tau = D_3$, $D_3^\tau = D_1$.

(*ii*) *There exists a line bundle,* $\mathbf{L}$*, such that* $D_1 \sim 2\,\mathbf{L}$.

Then there exists an S_4 *covering* $\pi : X \to Y$ *satisfying* (*i*) $X_{V_4} = W$ *and* (*ii*) $\Delta(X/W) = \mathrm{Supp}(D_1 + D_2 + D_3)$.

Proof. Choose effective divisors D_0 and D_∞ so that $\mathbf{L} \sim D_\infty - D_0$. Then we have $D_1 + 2D_0 \sim 2D_\infty$. Hence there exists a rational function, ψ,on W such that

$$(\psi) = (D_1 + 2D_0) - 2D_\infty.$$

Define three rational functions, φ_1, φ_2 and φ_3 as follows:

$$\varphi_1 = \psi\psi^\sigma\psi^{\tau^2}\psi^{\sigma\tau^2}, \quad \varphi_2 = \psi\psi^\sigma\psi^\tau\psi^{\sigma\tau}, \quad \varphi_3 = \psi^\tau\psi^{\tau^2}\psi^{\sigma\tau}\psi^{\sigma\tau^2}.$$

Then one can easily check the following:

(i) $\varphi_1^\sigma = \varphi_2$, $\varphi_3^\sigma = \varphi_3$, $\varphi_1^\tau = \varphi_2$, $\varphi_2^\tau = \varphi_3$, $\varphi_3^\tau = \varphi_1$.

(ii) $\varphi_1\varphi_2\varphi_3 = (\psi\psi^\sigma\psi^\tau\psi^{\sigma\tau}\psi^{\tau^2}\psi^{\sigma\tau^2})^2 \in (f^*\mathbf{C}(Y)^\times)^2$.

(iii)

$$\begin{aligned}
(\varphi_1) &= D_2 + D_3 + 2(D_1 + D_0 + D_0^\sigma + D_0^{\tau^2} + D_0^{\sigma\tau^2}) - 2(D_\infty + D_\infty^\sigma + D_\infty^{\tau^2} + D_\infty^{\sigma\tau^2}) \\
(\varphi_2) &= D_1 + D_3 + 2(D_2 + D_0 + D_0^\sigma + D_0^\tau + D_0^{\sigma\tau}) - 2(D_\infty + D_\infty^\sigma + D_\infty^\tau + D_\infty^{\sigma\tau}) \\
(\varphi_3) &= D_1 + D_2 + 2(D_3 + D_0^\tau + D_0^{\tau^2} + D_0^{\sigma\tau} + D_0^{\sigma\tau^2}) - 2(D_\infty^\tau + D_\infty^{\tau^2} + D_\infty^{\sigma\tau} + D_\infty^{\sigma\tau^2}).
\end{aligned}$$

In particular, $\varphi_i \notin (\mathbf{C}(W))^2$ $(i = 1, 2, 3)$.

Now the existence for an $\mathcal{S}_4$ covering with property (i) follows from Proposition 2.2. The assertion on $\Delta(X/W)$ follows from (iii).

In Proposition 1.3, we assume W to be smooth. This assumption, however, seems to be too strong, as one can easily see that such coverings are singular in many cases (*cf.* [9]). In the case of when $\dim Y = 2$, we can avoid this inconvenience.

Let $f : W \to \Sigma$ be an $\mathcal{S}_3$ covering of a smooth algebraic surface Σ. Let $\mu : \tilde{W} \to W$ be the minimal resolution. By the uniqueness of the minimal resolution, $\mathcal{S}_3$ is also considered as a finite automorphism group of $\tilde{W}$ on Σ, and we have the following version:

Proposition 1.4. *Let D_1, D_2 and D_3 be three reduced divisor on $\tilde{W}$ such that*

(i) With the same notation on $\mathrm{Gal}(W/Y)$ *as those in Proposition 1.2,*

(ia) $D_1^\sigma = D_2$ and $D_3^\sigma = D_3$, and

(ib) $D_1^\tau = D_2$, $D_2^\tau = D_3$, $D_3^\tau = D_1$.

(ii) There exists a line bundle, $\mathbf{L}$, on $\tilde{W}$ such that $D_1 \sim 2\,\mathbf{L}$.

Then there exists an $\mathcal{S}_4$ covering $\pi : S \to \Sigma$ satisfying (i) $S^{V_4} \doteq W$ and (ii) $\Delta(S/\Sigma) = \Delta_f \cup f \circ \mu(\mathrm{Supp}(D_1 + D_2 + D_3))$.

Proof. Likewise the proof of Proposition 1.3, we have a V_4 covering, $\tilde{S}$, of $\tilde{W}$ such that

(*i*) $\mathbf{C}(\tilde{S})$ is an $\mathcal{S}_4$ extension of Σ, and

(*ii*) $\Delta(\tilde{S}/\tilde{W}) = \mathrm{Supp}(D_1 + D_2 + D_3)$.

Then the Stein factorization, S, of $\tilde{S}$ is the desired one.

Corollary 1.5. *Under the same assumption and notations as in Proposition 1.4, if* $\mathrm{Supp}(D_1 + D_2 + D_3)$ *is a subset of the exceptional divisor of μ, then there exists an $\mathcal{S}_4$ covering of Σ with branch locus Δ_f.*

§2 Lattices

Definition 2.1. A lattice is a free $\mathbf{Z}$ module of finite rank equipped with $\mathbf{Z}$ valued symmetric bilinear form $(\ ,\ )$.

Let L_1 and L_2 be two lattice. We denote the orthogonal direct sum of them by $L_1 \oplus L_2$; and L^n denotes $L \oplus \cdots \oplus L$ (n copies). The discriminant, $\mathrm{disc}\, L$, of a lattice L is the determinant of the intersection matrix of L. A lattice is called unimodular if $\mathrm{disc}\, L = \pm 1$.

A sublattice, M, of L is called primitive if L/M is torsion-free.

Example 2.2. Let X be an algebraic surface and let $H^2(X,\mathbf{Z})$ be the second cohomology group. If $H^2(X,\mathbf{Z})$ is torsion-free, then $H^2(X,\mathbf{Z})$ is unimodular lattice with respect to the intersection product by Poincaré duality. The Néron-Severi group of X is a primitive sublattice of $H^2(X,\mathbf{Z})$.

§3. Automorphisms of order 2 or 3 and the rational quotients by them

Let X be a surface, and let σ be an automorphism of order 2 or 3 of X with only isolated fixed points, $Q_1,\ldots,Q_k$. Let G be the group of generated by σ. Let $\overline{Y} = X/G$ and let $\pi : X \to \overline{Y}$ be the quotient map. $\overline{Y}$ has quotient singularities at the points $P_i = \pi(Q_i)$. Let $\mu : Y \to \overline{Y}$ be the minimal resolution of $\overline{Y}$. We call the induced rational map $X \cdots \to Y$ the rational quotient map and call Y the rational quotient of X by G. Let $\tilde{X}$ be the $\mathbf{C}(X)$-normalization of Y. It is a cyclic covering of degree $\sharp(G)$ branched along at most the exceptional set of $Y \to \overline{Y}$. In what follows, we look into the relation among X, $\tilde{X}$ and Y.

Case 1. $\sharp(G) = 2$. One obtains $\tilde{X}$ from X by blowing-up at $Q_1,\ldots,Q_k$. For details, see [5], §3.

Case 2. $\sharp(G) = 3$. In this case, the action of G around each fixed point is divided into two types. Namely, if we choose a small neighborhood, $U : (x,y) \subset \mathbf{C}^2$, $Q_i = (0,0)$ appropriately, then we may assume that the action of σ is given either (i) $(x,y) \mapsto (\epsilon x, \epsilon y)$, or (ii) $(x,y) \mapsto (\epsilon x, \epsilon^2 y)$, where $\epsilon = \exp(2\pi i/3)$. Hence P_i is a cyclic quotient singularity of type $(1,3)$ for (i), while it is one of type $(2,3)$ for (ii), i.e., a rational double point of type A_2. We relabel the Q_i's so that $P_1,\ldots,P_t$ are type $(1,3)$ and $P_{t+1},\ldots,P_k$ are type $(2,3)$. To obtain $\tilde{X}$ from X, we first consider a successive blowing-ups of X in the following way:

(i) Blow up at Q_i one time for $i = 1,\ldots,t$, and

(ii) Blow up at Q_i three times for $i = t+1,\ldots,k$ so that the induced automorphism from σ has no isolated fixed point. One can easily see that the exceptional set is tree of three $\mathbf{P}^1$ and that the self intersection number of the middle component is -3, while those of the remaining two is -1

We next contract the $k-t$ (-3) curves arising from (ii). Then we obtain $\tilde{X}$.

We next consider how we obtain $\tilde{X}$ from Y. Let $C_1,\ldots,$ C_t be the exceptional curves for $P_1,\ldots,$ P_t and let $C_{i,1}$ and $C_{i,2}$ $(t+1 \geq i \geq k)$ be the exceptional curves for $P_{t+1},\ldots,$ P_k. Since $\tilde{X} \to Y$ is a cyclic triple covering of Y, and the branch locus is the exceptional set of $\mu : Y \to \overline{Y}$, one can find a line bundle L on Y such that

$$3L \sim \sum_{i=1}^{t} C_i + \sum_{i=t+1}^{k} (C_{i,1} + 2C_{i,2}).$$

Remark 3.3. A divisor in the form of $C_{i,1} + C_{i,2}$ $(i \geq t+1)$ does not appear in the right hand side, since $C_{i,1}(C_{i,1} + C_{i,2}) = -1$ is not divisible by 3.

From the linear equivalence as above, one can obtain a cyclic triple covering, Z, of Y branched along $\mathrm{Supp}(\sum_{i=1}^{t} C_i + \sum_{i=t+1}^{k}(C_{i,1} + 2C_{i,2})$. If we choose L in an appropriate way, $Z = \tilde{X}$. In particular, if $\mathrm{Pic}(Y)$ has no 3-torsion, then $Z = \tilde{X}$

§4 $K3$ surfaces and their rational cyclic quotients of degree 2 and 3

A $K3$ surface is a simply connected compact complex manifold of dimension 2 with trivial canonical bundle. Throughout this article, we only consider *algebraic* $K3$ surfaces.

By Example 2.2, for a $K3$ surface X, $H^2(X, \mathbf{Z})$ is a unimodular lattice; and by the Noether formula, $\mathrm{rank}\, H^2(X, \mathbf{Z}) = 22$. Let $\mathrm{NS}(X)$ be the Néron-Severi group of X. As X is simply connected, $\mathrm{NS}(X) = \mathrm{Pic}(X)$

Definition 4.1. We call $\sum_{i=1}^{k} C_i$ p-divisible if $1/p\,(\sum_{i=1}^{k} C_i) \in \mathrm{NS}(Y)$, i.e., there exists L in $\mathrm{NS}(Y)$ $pL \approx \sum_{i=1}^{k} C_i$.

Lemma 4.2. **([6], Lemma 3, [5], Lemma 3.3)** *Let $C_1, \ldots, C_k$ be disjoint (-2) curves on a $K3$ surface Y, and suppose $1/2 \sum_{i=1}^{k} C_i \in \mathrm{NS}(Y)$. Then $k = 0$, 8 or 16*

For a proof, see [5].

Corollary 4.3. *Let $C_1, \ldots, C_l$ be disjoint (-2) curves on a $K3$ surface Y, and let L be the sublattice generated by $C_1, \ldots, C_l$. Then:*
(i) If $(\mathrm{NS}(Y)/L)_{tor} \supset \mathbf{Z}/2\mathbf{Z}$, then $l \geq 8$, and
(ii) If $(\mathrm{NS}(Y)/L)_{tor} \supset (\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$, then $l \geq 12$.

Proof. (i) Let $D = \sum_{i=1}^{l} a_i C_i$ be an element of L such that $\frac{1}{2}D \notin L$ but $\frac{1}{2}D \in \mathrm{NS}(Y)$. By replacing D by $D_1 = \sum_{i=1}^{l}(a_i - 2[a_i/2])C_i$, $[x]$ being the maximal integer not exceeding x, we may assume that D is a non-zero reduced effective divisor. Hence by Lemma 4.1, the number of irreducible component of D is either 8 or 16.

(ii) Suppose that $k \leq 11$ and $(\mathrm{NS}(Y)/L)_{tor} \supset (\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$. Let D_1 and D_2 be elements of L such that $\frac{1}{2}D_1$ and $\frac{1}{2}D_2$ give rise to distinct elements in $(\mathrm{NS}(Y)/L)_{tor}$. Then, by

Lemma 4.2 and the assumption, both D_1 and D_2 have 8 irreducible components. Hence by relabeling C_i if necessary, one may assume

$$\begin{aligned} D_1 &= C_1 + \cdots + C_t + C_{t+1} + \cdots + C_8 \\ D_2 &= C_1 + \cdots + C_t + C_9 + \cdots + C_l, \end{aligned}$$

where $1 \leq t \leq 7, 9 \leq l \leq 11$. Since $l \leq 11$, $t \geq 5$. Let $X_1 \cdots \to Y$ be the rational quotient map with respect to D_1 as in §3. Then the divisor $\tilde{D}_2$ on X_1 coming from D_2 is in the form of $(C_9' + C_9'') + \cdots + (C_l' + C_l'')$; and $1/2\tilde{D}_2 \in \mathrm{NS}(X_1)$. Hence the number of irreducible components $\tilde{D}_2$ is either 8 or 16 by Lemma 4.1. But this is impossible as $9 \leq l \leq 11$.

Remark 4.4. I. Shimada recently studied embedding of L into $H^2(Y, \mathbf{Z})$. Corollary 4.3 is straight forward from his result.

Lemma 4.5. *Let $(C_{i,1}, C_{i,2})$ $(i = 1, ..., k)$ be pairs of (-2) curves on a $K3$ surface Y such that*

(i) $C_{i,1}C_{i,2} = 1$ and the divisors $C_{1,1} + C_{1,2}, ..., C_{k,1} + C_{k,2}$ are disjoint.

(ii) $1/3 \sum_{i=1}^{k}(C_{i,1} + 2C_{i,2}) \in \mathrm{NS}(Y)$,

then $k = 0, 6$ or 9.

Proof. Suppose that $k > 0$ and let $X \cdots \to Y$ be the rational quotient map of degree 3 as in §3 and let Q_i $(i = 1, ..., k)$ be the points lying over $C_{i,1} + 2C_{i,2}$, $(i = 1, ..., k)$, respectively. Then

$$\begin{aligned} \chi_{top}(X) &= \chi_{top}(X \setminus \{Q_1, ..., Q_k\}) + k \\ &= 3\chi_{top}(Y \setminus \cup_{i=1}^{k}(C_{i,1} \cup C_{i,2}) + k \\ &= 72 - 8k. \end{aligned}$$

As $K_X \sim 0$, X is either a $K3$ surface or an abelian surface. Hence $k = 6$ for the first case and $k = 9$ for the second.

§5. Cyclic coverings of rational double points

Let (X, x) be a 2-dimensional normal singularity, i.e., X is a normal irreducible complex space having a unique singularity at x. Let (Y, y) be another 2-dimensional normal singularity and let $f : (Y, y) \to (X, x)$ be a finite morphism such that (i) $Y \setminus y \to X \setminus x$ is unramified, and (ii) $f^{-1}(x) = y$.

Such f is determined by a subgroup of finite index of the local fundamental group, $\pi_1^{loc}(X, x)$, of (X, x). For rational double points, the following result is well-known:

Type of (X,x)	Local equation	$\pi_1^{loc}(X,x)$	$\sharp\left(\pi_1^{loc}(X,x)\right)$
A_n	$z^2+y^2+x^{n+1}=0$	cyclic group	$n+1$
D_n $(n\geq 4)$	$z^2+x(y^2+x^{n-2})=0$	binary dihedral group	$4(n-2)$
E_6	$z^2+y^3+x^4=0$	binary tetrahedral group	24
E_7	$z^2+y(y^2+x^3)=0$	binary octahedral group	48
E_8	$z^2+y^3+x^5=0$	binary icosahedral group	120

(Note that A_0 is nothing but a smooth point.)

We now consider the case when f is a p-cyclic (p: odd prime) covering

Lemma 5.1. *If f is a p-cyclic covering, then the pair (X,x) and (Y,y) is one of the following:*

p	(X,x)	(Y,y)
2	A_n $(n\equiv 1 \bmod 2)$	$A_{\frac{n-1}{2}}$
2	D_n $(n:\text{even})$	A_{2n-5} or $D_{\frac{n}{2}+1}$
2	D_n $(n:\text{odd})$	A_{2n-5}
2	E_7	E_6
3	A_n $(n\equiv 2 \bmod 3)$	$A_{\frac{n-2}{3}}$
3	E_6	D_4
$p\geq 5$	A_n $(n+1\equiv 0 \bmod p)$	$A_{\frac{n+1}{p}-1}$

Proof. If $f:(Y,y)\to(X,x)$ is a p-cyclic covering, then it corresponds to a normal subgroup of $\pi_1^{loc}(X,x)$ of index p. Our statement easily follows from the case-by-case checking.

5. Local structure of an $\mathcal{S}_4$ covering of a surface.

We go on to study the local structure of an $\mathcal{S}_4$ covering. Let $\pi: S\to\Sigma$ be an $\mathcal{S}_4$ covering. As we introduced §1, we have the commutative diagram as follows:

$$\begin{array}{ccc} S & \xrightarrow{\nu_{V_4}} & S^{V_4} \\ \pi\downarrow & & \downarrow\nu_{\mathcal{A}_4/V_4} \\ \Sigma & \xleftarrow{\pi_{\mathcal{A}_4}} & S^{\mathcal{A}_4} \end{array}$$

where $\nu_{V_4}: S\to S^{V_4}$ is a $(\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$ covering, $\nu_{\mathcal{A}_4/V_4}: S^{V_4}\to S^{\mathcal{A}_4}$ is a $\mathbf{Z}/3\mathbf{Z}$ covering, $\nu_{\mathcal{A}_4} := \nu_{\mathcal{A}_4/V_4}\circ\nu_{V_4}: S\to S^{\mathcal{A}_4}$ is an $\mathcal{A}_4$ covering, $\pi_{V_4} := \pi_{\mathcal{A}_4}\circ\nu_{\mathcal{A}_4/V_4}: S^{V_4}\to\Sigma$ is an $\mathcal{S}_3$ covering, and $\pi_{\mathcal{A}_4}: S^{\mathcal{A}_4}\to\Sigma$ is a $\mathbf{Z}/2\mathbf{Z}$ covering.

In the following, we always assume:
(i) the branch locus $B := \Delta(S/\Sigma)$ has at most simple singularities,
(ii) π is branched at $2B$, and
(iii) $\pi_{\mathcal{A}_4}$ is branched along B.
Under these three conditions, one can conclude that ν_{V_4}, $\nu_{\mathcal{A}_4/V_4}$ and $\nu_{\mathcal{A}_4}$ are branched at most singular points of the base surfaces; and all of these singularities are rational double points by Lemma 5.1. We next consider what kinds of singularities we have on S and S^{V_4}. Choose $x \in \mathrm{Sing}(S^{\mathcal{A}_4})$. Then $\nu_{\mathcal{A}_4}^{-1}(x)$ and $\nu_{\mathcal{A}_4/V_4}^{-1}(x)$ consists of some rational double points. The table below explains what type appears.

Lemma 6.1.

Type of x	$\nu_{\mathcal{A}_4/V_4}^{-1}(x)$	$\nu_{\mathcal{A}_4}^{-1}(x)$	*Condition for n*
A_n	$A_{\frac{n-2}{3}}$	$4A_{\frac{n-2}{3}}$	$n \equiv 2 \bmod 3$
A_n	$3A_n$	$6A_{\frac{n-1}{2}}$	$n \equiv 1 \bmod 2$
A_n	$3A_n$	$12A_n$	*No condition*
D_n	$3D_n$	$6D_{\frac{n}{2}+1}$	$n \equiv 0 \bmod 2$
D_n	$3D_n$	$6A_{2n-5}$	*No condition*
D_n	$3D_n$	$12D_n$	*No condition*
E_6	D_4	A_1	
E_6	D_4	$2A_3$	
E_6	D_4	$4D_4$	
E_6	$3E_6$	$12E_6$	
E_7	$3E_7$	$6E_6$	
E_7	$3E_7$	$12E_7$	
E_8	$3E_8$	$12E_8$	

Here the coefficients of the types of singularities mean the number of singularities, e.g., $3A_n$ means three A_n singularities. All the statements easily follows from Lemma 6.1.

§7 Proof of Theorem 0.5

We keep the notations as before. Theorem 0.5 is straight forward from the following:

Proposition 7.1. *Suppose that* $\deg B = 6$ *and there exists an* $\mathcal{S}_4$ *covering* $\pi : S \to \mathbf{P}^2$ *such that*
(i) π *is branched at* $2B$, *and (ii)* $S^{\mathcal{A}_4} = W'$.

Then the minimal resolution, $\tilde{S}$, of S is either an abelian surface or a $K3$ surface. Moreover, if $\tilde{S}$ is an abelian surface (resp. $K3$ surface), then $G(R)$ contains $\mathbf{A}_2^{\oplus 9}$ (resp. $\mathbf{A}_2^{\oplus 6} \oplus \mathbf{A}_1^{\oplus 4}$).

We need several lemmas to prove Proposition 7.1. Let us start with the following lemma:

Lemma 7.2. *Let $\tilde{S}$ is as above. Then $\tilde{S}$ is either an abelian surface or a $K3$ surface. Moreover,* $\mathrm{Sing}(S) \neq \emptyset$, *then $\tilde{S}$ is a $K3$ surface.*

Proof. Let $K_{W'}$ be the canonical bundle of W' (Note that one can define $K_{W'}$ as we assume that W' has only rational double singularities). By the assumption, $\nu_{\mathcal{A}_4} : S \to S^{\mathcal{A}_4}$ is branched at at most $\mathrm{Sing}(S^{\mathcal{A}_4})$. Also, by Lemma 5.1, S has again at most rational double points as its singularities. Hence we have $K_{\tilde{S}} = \mu_1^* K_S = \mu_1^* \nu_{\mathcal{A}_4}^* K_{\mathcal{A}_4} = 0$, where $\mu_1 : \tilde{S} \to S$ denotes the minimal resolution. Hence, by the classification for algebraic surfaces, $\tilde{S}$ is either an abelian surface or a $K3$ surface. Moreover, if $\mathrm{Sing}(S) \neq \emptyset$, $\tilde{S}$ contains at least one smooth rational curve. This implies the last assumption.

Lemma 7.3. *If S is an abelian surface, then:*
(i) S^{V_4} is an abelian surface,
(ii) $G(R) = \mathbf{A}_2^{\oplus 9}$, and
(iii) B is a nine cuspidal sextic curve.

Proof. Suppose that S^{V_4} is not abelian surface. Then, by [17], S^{V_4} is a $K3$ surface with $16A_1$ singularities. Hence, by Lemma 5.1, singularities of $S^{\mathcal{A}_4}$ are of types either A_1, A_2 or A_5. Let n_1, n_2 and n_5 be the number of singularities of types A_1, A_2, and A_5, respectively. Then we have

$$\begin{aligned} 3n_1 + n_5 &= 16 \\ n_1 + 2n_2 + 5n_5 &\leq 19. \end{aligned}$$

Moreover, since $\nu_{V_4/\mathcal{A}_4} : S^{V_4} \to S^{\mathcal{A}_4}$ is branched at some singularities of $S^{\mathcal{A}_4}$, by Lemmas 4.5 and 6.1, the corresponding graph $\mathbf{A}_1^{\oplus n_1} \oplus \mathbf{A}_2^{\oplus n_2} \oplus \mathbf{A}_5^{\oplus n_5}$ contains a subgraph $\mathbf{A}^{\oplus 6}$. Hence the only possible triplet (n_1, n_2, n_5) is $(5, 4, 1)$.

Hence one can conclude the singularities of $S^{\mathcal{A}_4}$ is $5A_1 + 4A_2 + A_5$. Therefore the singularities of the branch locus are $5a_1 + 4a_2 + a_5$.

Claim There exists no reduced sextic curve, B, with singularities $5a_1 + 4a_2 + a_5$.

Proof of Claim Taking contribution of the genus drop from each singularity into account, we infer that B is reducible. As B has $4a_2$ singularities, it must have an irreducible component of degree 5. Put $B = B_1 + L$, where $\deg B_1 = 5$ and L is a line. Then:

Either B_1 has $4a_2 + a_5$ and B_1 meets L_1 transeversely at five distinct points,

or

B_1 has $4a_2 + 3a_1$ and B_1 meets L_1 at 3 distinct points; L_1 is the tangent line at an inflection point of B_1.

In both cases, however, there is no such quintic curve by considering the contribution of the genus drop from singularities.

By Claim, we have the first assertion for Lemma 6.3. We now go on to the second. By (i), we may assume that S^{V_4} is an abelian surface. By Lemma 4.5, $S^{\mathcal{A}_4}$ has just $9A_2$ singularities. This implies that B has nine cusps.

Lemma 7.4. *If S is a $K3$ surface with rational double points, then $G(R)$ contains a subgraph $\mathbf{A}_2^{\oplus 6} \oplus \mathbf{A}_1^{\oplus 4}$.*

Proof. Let $\mu_1 : \tilde{S} \to S$ and $\mu_2 : Z \to S^{\mathcal{A}_4} = Z'$ be the minimal resolution of S and $S^{\mathcal{A}_4}$, respectively. By the uniqueness of the minimal resolution, $\mathrm{Gal}(S/\mathbf{P}^2) \cong \mathcal{S}_4$ is also considered as a finite automorpism group of $\tilde{S}$ and μ_1 is $\mathcal{S}_4$-equivalent. Let $\tilde{S}^{\mathcal{A}_4}$ be the quotient surface by $\mathcal{A}_4$. $\tilde{S}^{\mathcal{A}_4}$ is again a $K3$ surface with rational double points, and there exists a morphism $\overline{\mu}_1 : \tilde{S}^{\mathcal{A}_4} \to S^{\mathcal{A}_4}$ such that the following diagram commutes.

$$\begin{array}{ccc} S & \overset{\mu_1}{\leftarrow} & \tilde{S} \\ \downarrow & & \downarrow \\ S^{\mathcal{A}_4} & \overset{\overline{\mu}_1}{\leftarrow} & \tilde{S}^{\mathcal{A}_4} \end{array}$$

The minimal resolution of $\tilde{S}^{\mathcal{A}_4}$ is Z by the uniqueness of the minimal model. Hence One may assume that μ_2 factors $\tilde{S}^{\mathcal{A}_4}$; and the exceptional set for $W \to S^{\mathcal{A}_4}$ contains that of $W \to \tilde{S}^{\mathcal{A}_4}$. By [17] $S^{\mathcal{A}_4}$ has singularities $6A_2 + 4A_1$; and we have the assertion.

By Lemmas 7.2, 7.3 and 7.4, we have Proposition 7.1.

An easy but interesting corollary to Proposition 7.1 is as follows:

Corollary 7.5. *Under the same notation as before, let B be a plane sextic curve with singularities $\sum_l \alpha_l a_l + \sum_m \beta_m d_m + \sum_n \gamma_n e_n$, $(\alpha_l, \beta_m, \gamma_n \in \mathbf{Z}_{\geq 0})$. Then we have*

$$G(R) = \bigoplus_l \mathbf{A}_l^{\oplus \alpha_l} \oplus \bigoplus_m \mathbf{D}_m^{\oplus \beta_m} \oplus \bigoplus_n \mathbf{E}_n^{\oplus \gamma_n}.$$

If $G(R)$ contains a subgraph neither $\mathbf{A}_2^{\oplus 9}$ nor $\mathbf{A}_2^{\oplus 6} \oplus \mathbf{A}_1^{\oplus 4}$, there is no S_4 covering of $\mathbf{P}^2$ branched at $2B$.

§8 Proof of Theorem 0.6

The goal of this section is to prove Theorem 0.6. Let us start with some setting-ups.

Let B be a reduced plane sextic curve with at most simple singularities. Let $f' : Z' \to \mathbf{P}^2$ be a double covering with $\Delta'_f = B$, and let $\mu : Z \to Z'$ be the canonical resolution of Z'. We denote the the subgroup of $\mathrm{NS}(Z)$ generated by the pull-back of a line of $\mathbf{P}^2$ and the irreducible components of the exceptional divisor of μ by T. As one can easily see, it has an orthogonal decomposition with respect to the intersection pairing:

$$T = \mathbf{Z}\, L \oplus \bigoplus_{x \in \mathrm{Sing}(Z')} R_x,$$

where L denotes the pull-back of a line, and R_x denotes the subgroup generated by all the irreducible components of the exceptional divisor for $x \in \mathrm{Sing}(Z')$.

Put $\overline{R} = \mathcal{L}(\mathbf{A}_2^{\oplus 6} \oplus \mathbf{A}_1^{\oplus 4})$ and $\overline{T} = \mathbf{Z}\, L \oplus \overline{R}$. Let $T^\sharp$ and $\overline{T}^\sharp$ be the primitive hull of T and $\overline{T}$ in $\mathrm{NS}(Z)$, respectively. Now let us start with the following lemma.

Lemma 8.1. *$\overline{T}^\sharp/\overline{T}$ has a 3-torsion. In particular, $T^\sharp/T$ has a 3-torsion.*

Proof. By Nikulin's theory used in [17] §1 or [11], $\overline{T}^\sharp/\overline{T}$ has a 3-torsion. As $\overline{T}^\sharp \subset T^\sharp \subset \mathrm{NS}(Z)$, we can find D in $T^\sharp$ which gives a 3-torsion in $\overline{T}^\sharp/\overline{T}$. We now show that this D gives a 3-torsion in $T^\sharp/T$, too. To see this, it is enough to show $D \notin T$. Suppose that $D \in T$ and write

$$D \sim aL + \sum_{x \in \mathrm{Sing}(Z')} \sum_i b_{i,x} \Theta_{i,x},$$

where $\Theta_{i,x}$'s denote the exceptional (-2) curves which form a basis of R_x. On the other hand, as $3D \in \overline{T}$ and $D \notin \overline{T}$, we have

$$3D \sim a'L + \sum_{x \in \mathrm{Sing}(Z')} \sum_i b'_{i,x} \Theta_{i,x},$$

where all $\Theta_{i,x} \in \overline{T}$, and at least one of $b_{i,x}$'s is not divisible by 3. Combining these two relations, we obtain a non-trivial linear relation among L and the $\Theta_{i,x}$'s, but this is impossible as they form a basis in T.

By Theorem 0.3 [14], and Lemma 3.2, we have an $\mathcal{S}_3$ covering, W', of $\mathbf{P}^2$ such that $D(W'/\mathbf{P}^2) = Z'$. Let W be the minimal resolution. $\mathrm{Gal}(W'/\mathbf{P}^2) \cong \mathcal{S}_3$ also acts W and let τ be an element of order 3. Then we have a commutative diagram

$$\begin{array}{ccc} W' & \leftarrow & W \\ \downarrow & & \downarrow \\ Z' & \leftarrow & W/\langle\tau\rangle. \end{array}$$

Since W' is a $K3$ surface with rational double points, τ has only isolated fixed points. Hence, by Lemma 4.5, $W/\langle\tau\rangle$ has singularities $6A_2$, and its minimal resolution is Z. Let $\Theta_{i,1}, \Theta_{i,2}$ $(i = 1, ..., 6)$ be the exceptional curves. By our construction of W', these 12 curves give rise to $\mathbf{A}_2^{\oplus 6}$ in the assumption in Theorem 0.6 Hence $\mathrm{NS}(W)$ contains 12 disjoint (-2) curves C_j $(j = 1, ..., 12)$ that come from $\mathbf{A}_1^{\oplus 4}$ in the assumption in Theorem 0.6. Note that $\mathcal{S}_3$ acts in such a way that, for any element, τ, of order 3, τ fixes no C_j. By Lemma 4.1, if we choose 8 of the 12 C_j's, say $C_{j_1}, ..., C_{j_8}$, appropriately, then $\sum_{k=1}^{8} C_{j_k}$ is 2-divisible in $\mathrm{NS}(W)$. Conversely, any 2-divisible member in $\oplus_j \mathbf{Z}\, C_j$ is represented in this form.

Lemma 8.2 *Let $D_1 = \sum_{k=1}^{8} C_{j_k}$ and $D_2 = \sum_{l=1}^{8} C_{j_l}$ be reduced divisor representing 2-divisible member of $\oplus_j \mathbf{Z}\, C_j$. Then either $D_1 = D_2$ or D_1 and D_2 have 4 exact common components.*

Proof. Suppose that $D_1 \neq D_2$ and let h be the number of common components. As $D_1 \neq D_2$, $D_1 + D_2 -$ (*common components*) is also 2-divisible. Hence, by Lemma 4.2, the number of irreducible components, $16 - 2h$, is equal to 8, i.e., $h = 4$.

Corollary 8.3. *Let τ be as above and let D be a 2-divisible reduced divisor in $\oplus_j \mathbf{Z}\, C_j$. Then D, $\tau^* D$ and $(\tau^2)^* D$ are distinct divisors. Moreover, any two of these three divisors have 4 common components.*

Proof. Suppose that $D = \tau^* D$. Then $(\tau^2)^* D = D$, and D is a τ-invariant divisor. On the other hand, as τ fixes no C_j, the number of irreducible components of any τ-invariant divisor is 3-divisible. This is impossible as the number of irreducible components of D is 2. Hence D, $\tau^* D$ and $(\tau^2)^* D$ are different each other. The last assertion easily follows from Lemma 8.2.

We now construct three effective reduced divisors D_1, D_2 and D_3 on W such that
(i) $\mathrm{Supp}(D_1 + D_2 + D_3) \subset \mathrm{Supp}(C_1 + \cdots + C_{12})$, and
(ii) D_1, D_2 and D_3 satisfy the conditions in Proposition 1.3.

Let τ be as before and let σ be an element of order 2 in $\mathcal{S}_3$. Let D be any 2-divisible reduced divisor in $\oplus_j \mathbf{Z}\, C_j$. There are two possibilities: 1. $D = \sigma^* D$ and 2. $D \neq \sigma^* D$.

Case 1. $D = \sigma^* D$.

Put $D_1 = \tau^* D$, $D_2 = (\tau^2)^* D$, and $D_3 = D$. Then these three divisors are distinct by Corollary 3.3.4, and satisfy

(i) $D_1^\sigma = D_2$, $D_1^\tau = D_2$, and $D_2^\tau = D_3$, and

(ii) D_1 is 2-divisible.

Case 2. $D \neq \sigma^* D$.

Consider the divisor $D + \sigma^* D$. It is another 2-divisible divisor and is written in the form of $D' + 2D''$, where both D' and D'' are reduced and σ-invariant. Hence D' is 2-divisible as well as σ-invariant. Thus we can reduce our problem to Case 1. This finishes our proof of Theorem 0.6.

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