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Ginzburg-Landau equation and the zero set of solutions

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§1. Introduction.

We consider the following energy functional (GL functional)

$$E(\Phi)=\int_{\Omega}\left(\frac{1}{2}|\nabla\Phi|^2+\frac{\lambda}{4}(1-|\Phi|^2)^2\right)dx \quad (\Phi\in H^1(\Omega;\mathbb{C})) \tag{1.1}$$

and its grandient flow equation

$$\begin{cases}\dfrac{\partial\Phi}{\partial t}=\Delta\Phi+\lambda(1-|\Phi|^2)\Phi & (t,x)\in(0,\infty)\times\Omega,\\ \dfrac{\partial\Phi}{\partial\nu}=0 \quad (t,x)\in(0,\infty)\times\partial\Omega & \text{(Neumann B.C.)}\end{cases} \tag{1.2}$$

$\lambda>0$ is a parameter and supposed to be large when we consider the "vortex motion phenomena". A zero point $x\in\Omega$ (i.e. $\Phi(t,x)=0$) is called a vortex at time t. Concerning these vortice, there have been many interesting studies recent 10 years. This point is an important part of the solution because the energy concentrates around it (for large $\lambda>0$) and behaves like a particle. Actcually one single vortex has energy $\pi\log(1/\epsilon)$ (cf. Bethuel-Brezis-Helein [1]). The situation of the solution is almost determined by the configuation of such points, which vary as time goes. In this way a system of ODE describing the orbits of vortices arise. We consider this dynamics in relation with problem of the existence of nontrivial stable equilibrium solutions of

$$\begin{cases}\Delta\Phi+\lambda(1-|\Phi|^2)\Phi=0 & x\in\Omega,\\ \dfrac{\partial\Phi}{\partial\nu}=0 & x\in\partial\Omega.\end{cases} \tag{1.3}$$

A stable solution of (1.3) is a local minimizer of the functional of (1.1). We take a small parameter $\epsilon>0$ by the relation $\lambda=1/\epsilon^2$ for the convenience of notation. For small $\epsilon>0$, the coefficient of the nonlinear term becomes large and $|\Phi(t,x)|$ goes close to 1 very quickly as t grows up, except for the small neighborhood of the zero point (vortex). So there arises a sharp layer around a vortex and we see from the expression of E, that a big contribution comes from the neighborhood of such vortices in the integration in the energy functional E. These vortices persist to exist because of the continuity of the solution and the invariance of the degreee around the zero and they move very slowly afterwards. The mathematical study of such phenomena were started in recent years while they had been studied by physicists earlier (cf. Neu [9]). The speed of this slow motion of vortices were studied by Rubinstein and Sternberg [10] and it turned out to be the order $O(1/\log(1/\epsilon))$. Therefore, by accelerrating this slow motion by the new time scale $s=t\log(1/\epsilon)$, we can see the motion of finite speed. These orbits of vortices were studied and described as a finite dimensional system of ODE's by Jerrard-Sonner [3], F.H.Lin [7,8] in the case of 2 dimensional domain Ω with the 1st kind boundary condition.

On the other hand, a qualitative study of the dynamics of (1.2) in relation with the geometry of the domain has been given (cf. Dancer [2], Jimbo-Morita [4,6], Jimbo-Morita-Zhai [5]). It was proved that in a simple domain such as a convex domain, there is no non-trivial stable equilibrium solution to (1.2), while there arise such a solution in a complicated domain. In this note, we want to understand about the relation between such situation of (1.2) and the limit ODE system.

§2. Vortex motion.

In this section we describe the ODE system of the motion of the vortices done in the work of F. H. Lin [7,8] and Jerrard-Sonner [3]. Applying their method, we can obtain the motion law of vortices for the case of Neumann B.C as well as 1st kind B.C.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. We assume throughout this note

(*) Ω : **contractible.**

Take any point $p \in \Omega$ and consider the following equation:

$$\text{(2.1)} \qquad \begin{cases} \Delta_x \varphi = 0 \quad \text{in} \quad \Omega, \\ \dfrac{\partial \varphi}{\partial \nu_x} = -\langle \nu_x, \nabla_x \mathrm{Arg}(x-p) \rangle \quad \text{in} \quad \partial\Omega, \end{cases}$$

where ν_x is the unit outward vector on $\partial\Omega$.

We note that

$$\nabla_x \mathrm{Arg}(x-p) = \left(\frac{-(x_2-p_2)}{|x-p|^2}, \frac{x_1-p_1}{|x-p|^2} \right).$$

Proposition. (2.1) has a solution $\varphi = \varphi(x,p)$ which is a function in x (with parameter p). This solution is unique up to additive constants.

Remark. The above fact follows from the integral condition

$$\int_{\partial\Omega} \langle \nu_x, \nabla_x \mathrm{Arg}(x-p) \rangle \, dS = 0.$$

We should note that $\nabla_x \varphi(x)$ is uniquely determined only by $p \in \Omega$ in spite of the ambiguity of solution.

From now we use the following notation for 2 vector.

Notation. $\begin{pmatrix} a \\ b \end{pmatrix}^{\perp} = \begin{pmatrix} -b \\ a \end{pmatrix}.$

We denote the configuration of m vortices by

$$\mathbf{y}(t) = (y^{(1)}(t), y^{(2)}(t), ..., y^{(m)}(t)) \in \widehat{\Omega} = \Omega \times \Omega \times \cdots \times \Omega.$$

$y^{(j)}(t)$ denotes the position of the j-th vortex.

Proposition. The time variation of the configuration $\mathbf{y}(t)$ is given by the following system of the ODEs

$$\frac{d}{dt}y^{(j)} = -2\left\{\sum_{k=1}^{m}\nabla_x\varphi(y^{(j)}(t),y^{(k)}(t))^{\perp} + \sum_{k\neq j}\frac{y^{(k)}(t)-y^{(j)}(t)}{|y^{(k)}(t)-y^{(j)}(t)|^2}\right\} \tag{2.2}$$

for $j = 1, 2, ..., m$. See also [3], [7], [8]. Let us make sure that

$$\nabla_x\varphi(y^{(j)}(t),y^{(k)}(t)) = \nabla_x\varphi(x,p)_{|x=y^{(j)}(t),p=y^{(k)}(t)}.$$

We rewrite the above system in a simpler form. Consider

$$H(x) = \mathrm{Arc}(x-p) + \varphi(x,p) \tag{2.3}$$

which is multi-valued function. It is easy to see that $H(x)$ is harmonic and satisfies the Neumann boundary condition on $\partial\Omega$. We are naturally lead to take the conjugate harmonic function Ψ in Ω. From the Cauchy-Riemann equation, $\nabla H \perp \nabla G$ in Ω. As we are assuming Ω is contractible, G is constant on $\partial\Omega$ from the Neumann B.C. of H. So we can assume G satisfies the Dirichlet B.C. on $\partial\Omega$. Of course G has a log-singularity at p. Precisely, G is defined by the following system of equations.

Let $q \in \Omega$ and a function $G = G(x,q)$ such that

$$\begin{cases} \Delta_x G = 0 \quad \text{in} \quad \Omega\setminus\{q\}, \\ G = 0 \quad \text{on} \quad \partial\Omega, \\ G(x) \sim \log|x-q| + O(1) \quad \text{near} \quad q. \end{cases} \tag{2.4}$$

Note that the solution G in (2.4) is unique. Actually it is proved by the aid of the maximum principle and Riemann's removable singularity theorem.

From (2.3) and the Cauchy-Riemann equation $\nabla G^{\perp} = \nabla H$,

$$\nabla G^{\perp} = \nabla H = \left(\frac{-(x_2-p_2)}{|x-p|^2}, \frac{x_1-p_1}{|x-p|^2}\right) + \nabla\varphi(x,p).$$

Consequently,

$$-\nabla G = \frac{-(x-p)}{|x-p|^2} + \nabla\varphi(x,p)^{\perp}.$$

By the aid of this function, we express the right hand of the ODE system (2.2), as follows,

Proposition.

$$\frac{d}{dt}y^{(j)} = -2\left\{\nabla_x\varphi(y^{(j)}(t),y^{(j)}(t))^{\perp} + \sum_{k\neq j}\nabla_x G(y^{(j)}(t),y^{(k)}(t))\right\} \tag{2.5}$$

for $j = 1, 2, 3, \cdots, m$. This form is useful for the analysis on the special case in §4.

§3. Non-existence of Pattern formation

Let us consider the original equation (1.2). The relation between the geometric property of the domain and the structure of the solutions is studied recently. One of important insights is the observation that if the geometrical situation is very simple, then the structure of the stable solutions will be very simple. Actually one result proved from such point of view is the following.

Theorem (Jimbo-Morita [4]) If Ω is convex, there is no-nonconstant stable solutions in (1.3) for any $\lambda > 0$.

This result suggests that the dynamics of the "limit system" in $\widehat{\Omega}$ may not have any stable equilibrium point provided that Ω is convex. We want to investigate this problem in more details about the special cases.

§4. Special Case $\Omega =$ Disk

In this section we deal with a very special case

$$\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid |x| < 1\}.$$

The functions $\varphi(x,p), G(x,q)$ can be seen better and more information of the dynamics of (2.2) can be obtained, because we can discuss the situation more explicitly.

Proposition.

$$G(x,q) = \log|x-q| - \log|q||x-q^*|, \quad H(x,p) = \mathrm{Arc}(x-p) - \mathrm{Arc}|p|(x-p^*)$$

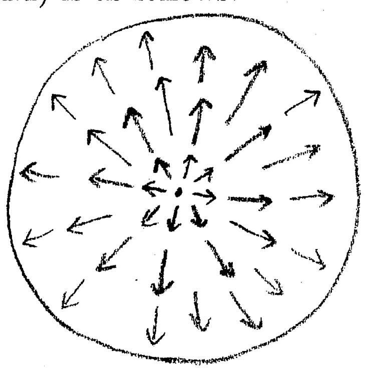
where z^* is the Kelvin transform about the unit circle $\partial\Omega$, that is

$$z^* = \begin{cases} z/|z|^2 & \text{for} \quad z \in \mathbb{R}^2 \setminus \{0\} \\ \infty & \text{for} \quad z = 0 \end{cases}$$

The case of 1 vortex ($m = 1$). We put $y(t) = y^{(1)}(t)$ for simplicity of notation. The position of the vortex is described by the equation,

$$\frac{d}{dt}y(t) = -2\frac{y(t) - y(t)^*}{|y(t) - y(t)^*|^2} \tag{4.1}$$

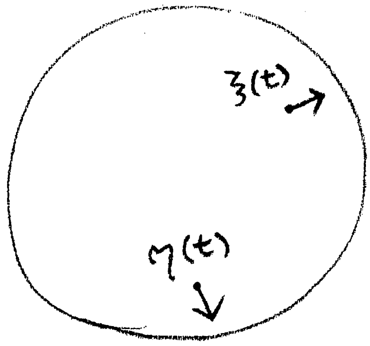
The vector field defined by (4.1) is as follows.



The case 2 vortices ($m = 2$). We put $\xi(t) = y^{(1)}(t)$, $\eta(t) = y^{(2)}(t)$ for simplicity of notation.

$$
(4.2) \qquad \begin{cases} \dfrac{d}{dt}\xi(t) = -2\left(\dfrac{\xi(t)-\xi(t)^*}{|\xi(t)-\xi(t)^*|^2} - \nabla_x G(\xi(t),\eta(t))\right) \\ \dfrac{d}{dt}\eta(t) = -2\left(\dfrac{\eta(t)-\eta(t)^*}{|\eta(t)-\eta(t)^*|^2} - \nabla_x G(\eta(t),\xi(t))\right) \end{cases}
$$

From the geometric consideration on the right hand side of (4.2), we see that the vortex which is close to the circle will be pushed out to the boundary.



By summing up the above two case, we have the following result.

Theorem . For Ω =Disk and $m = 1$ or 2, there is no stable equilibrium configuration to (2.2).

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