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Introduction to the viscosity solution theory of first order PDEs

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1. Introduction

In this survey note, we consider the Hamilton-Jacobi (HJ in short) equations:

$$\text{(HJ)} \qquad u_t + H(t,x,u,Du) = 0 \quad \text{in } \mathbf{R}^n \times (0,T),$$

where $n \geq 1$ is an integer and $T \in (0,\infty]$. Here, Du stands for the (spatial) gradient of u and u_t means $\frac{\partial u}{\partial t}$. We will assume that H is (locally Lipschitz) continuous.

We will consider the Cauchy problem of (HJ) under the initial condition:

$$\text{(IC)} \qquad u(0,x) = \psi(x) \quad \text{for } x \in \mathbf{R}^n,$$

where $\psi : \mathbf{R}^n \to \mathbf{R}$ is a given Lipschitz continuous function.

Since we treat the nonlinear PDEs, we can not expect classical solutions of (HJ) in general. Also, since we mainly treat the nondivergent type PDEs, the notion of "distribution" weak solutions does not fit our setting. Moreover, "almost everywhere" solutions are not unique in many cases.

The "correct" notion of weak solutions is that of viscosity solutions which was presented by Crandall and P.-L. Lions. The original derivation of it is as follows: Approximate (HJ) by

$$\text{(HJ)}_\epsilon \qquad u_t^\epsilon - \epsilon\Delta u^\epsilon + H(t,x,u,Du^\epsilon) = 0, \qquad (\epsilon > 0),$$

find the solutions u^ϵ of (HJ)_ϵ, suppose the (local) uniform convergence of u^ϵ to a limit u (as $\epsilon \to 0$) and then, as the definition of viscosity solutions of (HJ), use the properties which the limit satisfies through the maximum principle.

Our aim here is to collect several representation formulas of (HJ) with (IC) and dynamic programming (DP in short) type formulas of those for the user's convenience. We will also give brief ideas of the verification theorems.

Furthermore, we will mention about some recent progress.

Upto now, 3 books for the viscosity solution theory for first order PDEs have been published; P.-L. Lions [L1], G. Barles [B1], M. Bardi and I. Capuzzo-Dolcetta [BCD].

2. Preliminaries

Let us recall the definitions of the (standard) viscosity solutions (originally by [CL]) and BJ-viscosity soultions (by Barron-Jensen [BJ]). Before that, let us recall the notion of semicontinuous envelopes of functions $h : \mathbf{R}^n \to \mathbf{R}$;

$$h^*(x) = \lim_{\epsilon\to 0}\sup\{h(y) \mid |x-y| < \epsilon\} \text{ and } h_*(x) = \lim_{\epsilon\to 0}\inf\{h(y) \mid |x-y| < \epsilon\},$$

which are, respectively, called the upper semicontinuous (usc in short) and lower semicontinuous (lsc in short) envelopes of h. Notice that h^* and h_* are, respectively, usc and lsc.

Definition: ([CEL]) *A function u is a (standard) viscosity subsolution (resp., supersolution) of* (HJ) *iff, for $\forall\phi \in C^1, \forall(x,t)$ such that $(u^*-\phi)(x,t) = \max(u^*-\phi)$ (resp., $(u_*-\phi)(x,t)$ $= \min(u-\phi)$), the following holds:*

$$\phi_t(x,t) + H(t,x,u^*(x,t),D\phi(x,t)) \le 0$$

$$(\text{resp., } \phi_t(x,t) + H(t,x,u_*(x,t),D\phi(x,t)) \ge 0)\,.$$

Equivalently,

$$q + H(t,x,u^*(x,t),p) \le 0 \text{ for all } (p,q) \in D^+u(x,t)$$

$$\Big(\text{resp., } q + H(t,x,u_*(x,t),p) \ge 0 \text{ for all } (p,q) \in D^-u(x,t)\Big)\,.$$

A function u is a viscosity solution of (HJ) *iff it is both a viscosity sub- and supersolution of* (HJ).

Here, we denote the sets of superdifferentials and subdifferentials, respectively, by

$$D^+u(x,t) := \{(p,q) \mid u(y,s) \le u(x,t) + \langle p, y-x\rangle + q(s-t) + o(|x-y|+|t-s|)\}\,,$$

and

$$D^-u(x,t) := \{(p,q) \mid u(y,s) \ge u(x,t) + \langle p, y-x\rangle + q(s-t) + o(|x-y|+|t-s|)\}\,.$$

Remark. This definition says the following: In order to verify that u is a viscosity subsolution (resp., supersolution),

(♣) instead of checking u to satisfy the HJ equation (because of the lack of differentiability), we choose any smooth "test" function ϕ touching u from above (resp., below) at (x,t) and check that the one-sided inequality holds inserting the derivatives $(\phi_t(x,t), D\phi(x,t))$ into the place (u_t, Du).

In applications, we often have to deal with the case when the expected solutions are discontinuous. However, as the comparison result for these viscosity solutions implies the continuity of those, we need more sensitive definition.

For the case when $p \to H(t,x,u,p)$ is convex (or concave), the following definition was proposed by Barron-Jensen and has been successful. We remark that their definition is not motivated from the vanishing viscosity method with $(\mathrm{HJ})_\epsilon$ as above. However, the definition is naturally derived from the backward DP argument by Soravia [S] as we will see.

Definition: ([BJ]) *A lsc function u is a BJ-viscosity subsolution (resp., supersolution) of* (HJ) *iff, $\forall \phi \in C^1, \forall (x,t)$ such that $(u-\phi)(x,t) = \min(u-\phi)$, the following holds:*

$$\phi_t(x,t) + H(t,x,u(x,t),D\phi(x,t)) \le 0 \text{ (resp., } \ge 0).$$

Equivalently,

$$q + H(t,x,u(x,t),p) \le 0 \text{ (resp., } \ge 0) \text{ for all } (p,q) \in D^-u(x,t).$$

A lsc function u is a BJ-viscosity solution of (HJ) *iff it is both a BJ-viscosity sub- and supersolution of* (HJ).

Remark. Notice that, in this definition, we only have test functions touching from below but the equality holds true by inserting the derivatives of those. Compare this with (♣).

However, surprisingly, for continuous functions, this definition is equivalent to the standard one. See [BJ].

Under appropriate assumptions, we expect to derive the comparison principle:

$$\textbf{(CP)} \qquad \boxed{\textbf{subsolution} \;\le\; \textbf{supersolution}}$$

This implies the uniqueness of (both of the standard and BJ-) viscosity solutions and the continuity of the standard ones;

$$\textbf{(CP)} \Longrightarrow \begin{cases} \textbf{Uniqueness \& continuity} & \textbf{for viscosity solutions} \\ \\ \textbf{Uniqueness} & \textbf{for BJ-viscosity solutions} \end{cases}$$

The reason why these are the correct notions of weak solutions is that the expected solution (the value function) from the optimal control theory and differential game theory is the unique viscosity solution of (HJ).

Here, let us briefly show how the comparison principle is proved.

Idea of the proof of (CP)

For the sake of simplicity, we consider the following HJ equation:

$$u_t + u + H(Du) - f(x) = 0,$$

where given functions are Lipschitz continuous (and bounded if necessary).

Let u be a (bounded, usc) viscosity subsolution and v a (bounded, lsc) viscosity supersolution of the above, such that $u(x,0) \leq v(x,0)$.

We suppose that $A := \sup(u-v) > 0$ is finite and then, will get a contradiction. Suppose for simplicity that the supremum can be attained at a unique point $(\hat{x},\hat{t})$. Notice $\hat{t} > 0$.

Take any C^1 function ϕ in R such that supp $(\phi - 1) \subset [-1,1]$, $\phi'(r) > 0$ for $r \in (0,1)$ (i.e. $\exists \phi^{-1}$ in $(0,1)$) and $\phi(r) > \phi(0) = 0$ for $r \neq 0$.

Then, let $(x_\epsilon, y_\epsilon, t_\epsilon, s_\epsilon)$ be the maximum of the mapping

$$(x,y,t,s) \to u(x,t) - v(y,s) - \frac{1}{2\epsilon}\phi\left(|x-y|^2 + (t-s)^2\right).$$

Note that $(x_\epsilon, y_\epsilon, t_\epsilon, s_\epsilon) \to (\hat{x},\hat{x},\hat{t},\hat{t})$ as $\epsilon \to 0$ because of the uniqueness of maximum points.

Freezing (y_ϵ, s_ϵ), we see that (x_ϵ, t_ϵ) is the maximum of the mapping $(x,t) \to u(x,t) - v(y_\epsilon, s_\epsilon) - \frac{1}{2\epsilon}\phi(|x-y_\epsilon|^2 + (t-s_\epsilon)^2)$. Hence, the definition yields

$$\frac{t_\epsilon - s_\epsilon}{\epsilon}\phi'(r_\epsilon) + u(x_\epsilon, t_\epsilon) + H\left(\frac{(x_\epsilon - y_\epsilon)}{\epsilon}\phi'(r_\epsilon)\right) - f(x_\epsilon) \leq 0,$$

where $r_\epsilon = |x_\epsilon - y_\epsilon|^2 + (t_\epsilon - s_\epsilon)^2$.

Similarly,

$$0 \leq \frac{t_\epsilon - s_\epsilon}{\epsilon}\phi'(r_\epsilon) + v(y_\epsilon, s_\epsilon) + H\left(\frac{(x_\epsilon - y_\epsilon)}{\epsilon}\phi'(r_\epsilon)\right) - f(y_\epsilon).$$

Therefore, for some constant $C > 0$, we have

$$\frac{A}{2} \leq u(x_\epsilon, t_\epsilon) - v(y_\epsilon, s_\epsilon) \leq \|Df\|_\infty |x_\epsilon - y_\epsilon| \leq C\phi^{-1}(C\sqrt{\epsilon}).$$

This is a contradiction for small $\epsilon > 0$.

As pointed out in the above argument, we have avoided to deal with the following cases:
(1) $A = \infty$,
(2) $\hat{x} = \infty$ or $\hat{t} = T$,
(3) $(\hat{x},\hat{t})$ is not a unique maximum point.
To overcome these (also for more general (HJ)), we refer to [B1] and [BCD].

For a proof of uniqueness of BJ-viscosity solutions, we refer to [B2].

3. Representation formulas

(i) Value functions

In this subsection, we consider the HJ equations represented with controls in the following manner:

(HJB) $$u_t + \max_{a \in A}\{-\langle g(x,a), Du\rangle - f(x,a)\} = 0$$

and

$$\text{(HJI)} \qquad \hat{u}_t + \min_{b\in B}\max_{a\in A}\{-\langle g(x,a,b), D\hat{u}\rangle - f(x,a,b)\} = 0,$$

which are called the Bellman and Isaacs equations, respectively. Here, A, $B \subset \mathbf{R}^k$ (for some k) are control sets. Notice that (HJB) has a convex Hamiltonian.

In this note we only treat the case when f and g are independent of t for simplicity.

We will suppose the following regularity in (HJB) and (HJI), respectively:

$$\begin{cases} \text{the mapping } (x,a) \to \langle g(x,a), p\rangle + f(x,a) \text{ is uniformuly continuous for } \forall p \in \mathbf{R}^n, \\ \sup_{a\in A} (\|g(\cdot,a)\|_{W^{1,\infty}} + \|f(\cdot,a)\|_{L^\infty}) < \infty, \end{cases}$$

and

$$\begin{cases} \text{the mapping } (x,a,b) \to \langle g(x,a,b), p\rangle + f(x,a,b) \text{ is uniformuly continuous for } \forall p \in \mathbf{R}^n, \\ \sup_{(a,b)\in A\times B} (\|g(\cdot,a,b)\|_{W^{1,\infty}} + \|f(\cdot,a,b)\|_{L^\infty}) < \infty. \end{cases}$$

Let us recall the value functions associated with (HJB) and (HJI) under the initial condition (IC). To this end, we need several notations from the control and game theory:

$$\mathcal{A} := \{\alpha : [0,\infty) \to A \text{ measurable}\} \cdots \text{the set of controls for } A$$

$$\mathcal{B} := \{\beta : [0,\infty) \to B \text{ measurable}\} \cdots \text{the set of controls for } B$$

$$\Delta := \{\delta : \mathcal{A} \to \mathcal{B} \text{ non-anticipating starategy}\} \cdots \text{the set of strategies from } A \text{ to } B$$

The terminology "non-anticipating" means that, for $\forall t > 0$, α, $\hat{\alpha} \in \mathcal{A}$, if $\alpha = \hat{\alpha}$ a.e. in $(0,t)$, then $\delta[\alpha] = \delta[\hat{\alpha}]$ a.e. in $(0,t)$.

The value functions of (HJB) and (HJI), respectively, with (IC) are given by

$$u(x,t) = \inf_{\alpha\in\mathcal{A}} \left(\int_0^t f(X(s;x,\alpha),\alpha(s))ds + \psi(X(t;x,\alpha)) \right),$$

and

$$\hat{u}(x,t) = \sup_{\delta\in\Delta}\inf_{\alpha\in\mathcal{A}} \left(\int_0^t f(Y(s;x,\alpha,\delta[\alpha]),\alpha(s),\delta[\alpha](s))ds + \psi(Y(t;x,\alpha,\delta[\alpha])) \right).$$

Here, for $\alpha \in \mathcal{A}$ and $\delta \in \Delta$, $X(\cdot;x,\alpha)$ and $Y(\cdot;x,\alpha,\delta[\alpha])$, respectively, are the unique solutions of

$$\begin{cases} \dfrac{dX}{dt}(t) = g(X(t),\alpha(t)) \quad \text{for } t > 0, \\ X(0) = x, \end{cases}$$

and

$$\begin{cases} \dfrac{dY}{dt}(t) = g(Y(t),\alpha(t),\delta[\alpha](t)) \quad \text{for } t > 0, \\ Y(0) = x. \end{cases}$$

From the definitions, we easily obtain the so-called DP principle, which is a key tool to verify that these are viscosity solutions: for $0 < r < t$,

$$u(x,t) = \inf_{\alpha\in\mathcal{A}} \left(\int_0^r f(X(s;x,\alpha),\alpha(s))ds + u(X(r;x,\alpha),t-r) \right),$$

$$\hat{u}(x,t) = \sup_{\delta\in\Delta} \inf_{\alpha\in\mathcal{A}} \left(\int_0^r f(Y(s;x,\alpha,\delta[\alpha]),\alpha(s),\delta[\alpha](s))ds + \hat{u}(Y(r;x,\alpha,\delta[\alpha]),t-r) \right).$$

The proof of these DP principle can be done only from the definitions.

For a proof of the verification theorms, we refer to [ES] provided that these are (semi) continuous. However, in general, we need a little more careful observation since we have to consider semicontinuous envelopes. See [BCD] for example.

It is worth mentioning the fact: For a function $u : \mathbf{R}^n \to \mathbf{R}$,

$$\boxed{u \textbf{ is a viscosity solution if and only if the DP principle holds.}}$$

This is shown even for second order PDEs (with the associated stochastic control) by Lions [L2]. Also the same equivalence holds for the BJ-viscosity solutions. See [AKN] for this.

To show that the value function for (HJB) is a BJ-viscosity solution, we refer to [S]. If we allow ψ to be only lsc, the value function could be lsc in general. Still it is possible to show that such a value function is a BJ-viscosity solution.

(ii) Hopf-Lax formulas

On the other hand, in the literature, we know the Lax or Hopf formula for solutions of the following simplified HJ equations:

$$\text{(SHJ)} \qquad \begin{cases} u_t + H(Du) = 0 & \text{in } \mathbf{R}^n \times (0,T) \\ u(0,x) = \psi(x) & \text{for } x \in \mathbf{R}^n, \end{cases}$$

in case when H is convex or g is convex, respectively.

Before giving the formulas, we recall the Legendre transformation of convex functions $h : \mathbf{R}^n \to \mathbf{R}$:

$$h^\star(p) := \sup_{q\in\mathbf{R}^n} \{\langle p,q\rangle - h(q)\}$$

It is well-known that $(h^\star)^\star = h$ if h is convex and coercive; $\lim_{|p|\to\infty} h(p)/|p| = \infty$. Thus, we will suppose this coercivity for H whenever we discuss the Lax formula.

<u>(a) Lax formula (the case when H is convex.)</u>

$$u(x,t) := \inf_{y\in\mathbf{R}^n} \left\{ tH^\star\left(\frac{x-y}{t}\right) + \psi(y) \right\}$$

$$\left(= \inf_{y\in\mathbf{R}^n} \sup_{z\in\mathbf{R}^n} \{\langle z, x-y\rangle - tH(z) + \psi(y)\} \right)$$

The associated DP type equality is as follows: for $0 < r < t$,

$$u(x,t) = \inf_{y\in\mathbf{R}^n} \left\{ (t-r)H^\star\left(\frac{x-y}{t-r}\right) + u(y,r) \right\}$$

$$\left(= \inf_{y\in\mathbf{R}^n} \sup_{z\in\mathbf{R}^n} \{ \langle z, x-y\rangle - (t-r)H(z) + u(y,r) \} \right)$$

The next verification result is in Evans' text [E1]. To compare the argument with Proposition 2 below, we give the full-proof.

Proposition 1: *If u is continuous, then it is a viscosity solution of* (SHJ).

Proof. subsolution: For any $\phi \in C^1$, suppose (x,t) is the maximum of $u-\phi$. Adding the constant $-(u-\phi)(x,t)$, we may suppose $(u-\phi)(x,t)=0$. Thus, for small $\epsilon > 0$, we have

$$\phi(x,t) = u(x,t) = \inf_{y\in\mathbf{R}^n} \left(\epsilon H^\star\left(\frac{x-y}{\epsilon}\right) + u(y,t-\epsilon) \right) \le \inf_{y\in\mathbf{R}^n} \left(\epsilon H^\star\left(\frac{x-y}{\epsilon}\right) + \phi(y,t-\epsilon) \right).$$

For any $p \in \mathbf{R}^n$, taking $y = x-\epsilon p$ in the above and then, dividing it by $\epsilon > 0$, we have

$$\frac{\phi(x,t)-\phi(x-\epsilon p, t-\epsilon)}{\epsilon} - H^\star(p) \le 0.$$

Sending $\epsilon \to 0$ and then, taking the supremum over all $q \in \mathbf{R}^n$, by the fact that $(H^\star)^\star = H$, we get

$$\phi_t(x,t) + H(D\phi(x,t)) \le 0.$$

supersolution: This part of proof must be more difficult than the other since we need to show the opposite inequality and we are taking the infimum.

The proof is by contradiction. Suppose that (x,t) is the minimum of $u-\phi$ for a C^1 function ϕ but the following inequality holds for some $\theta > 0$:

$$-\theta \ge \phi_t(x,t) + H(D\phi(x,t)).$$

We may assume that $(u-\phi)(x,t) = 0$ as before.

For any $p \in \mathbf{R}^n$, we have

$$-\theta \ge \phi_t(x,t) + \langle p, D\phi(x,t)\rangle - H^\star(p).$$

Then, by continuity, there is small $s > 0$ such that

$$-\frac{\theta}{2} \ge \phi_t(x-rp, t-r) + \langle p, D\phi(x-rp,t-r)\rangle - H^\star(p) \text{ for } r \in [0,s].$$

Hence, integrating the above over $[0,s]$, we have

$$\frac{\theta s}{2} \le sH^\star(p) + \phi(x-sp,t-s) - \phi(x,t) \le sH^\star(p) + u(x-sp,t-s) - u(x,t).$$

Setting $y = x-sp$ and then, taking the infimum over all $y \in \mathbf{R}^n$, we get the right hand side of the above being zero. This is a contradiction. *QED*

Next, we show (assuming that the Lax formula is lsc) that it is a BJ-viscosity solution using the DP type equality in a backward way.

Proposition 2: *If u is lsc, then it is a BJ-viscosity solution of* (HJB).

Proof. The supersolution part of the proof is done in the above. For the other part, we barrow the backward DP type argument.

For any $\phi \in C^1$, let (x,t) be the minimum of $u-\phi$. We may assume $(u-\phi)(x,t)=0$.

For any $p \in \mathbf{R}^n$ and $\epsilon > 0$, we have

$$\phi(x+\epsilon p, t+\epsilon) \le u(x+\epsilon p, t+\epsilon) = \inf_{z \in \mathbf{R}^n} \left(\epsilon H^\star \left(\frac{x+\epsilon p - z}{\epsilon} \right) + u(z,t) \right)$$

$$\le \epsilon H^\star(p) + u(x,t) = \epsilon H^\star(p) + \phi(x,t).$$

Hence,

$$\frac{\phi(x+\epsilon p, t+\epsilon) - \phi(x,t)}{\epsilon} - H^\star(p) \le 0.$$

Sending $\epsilon \to 0$ and then, taking the supremum over $p \in \mathbf{R}^n$, we get the desired inequality. *QED*

Let us try to write down the solution with the value function to observe the relation between the value function and the Lax formula.

By using $(H^\star)^\star = H$, (SHJ) can be written in the following way:

$$(SHJ') \qquad u_t + \sup_{a \in \mathbf{R}^n} \{\langle a, Du\rangle - H^\star(a)\} = 0.$$

Then, the value function can be given by

$$u(x,t) = \inf \left\{ \int_0^t H^\star(X(s;x,\alpha))ds + \psi(\alpha(s)) \,|\alpha : [0,\infty) \to \mathbf{R}^n \text{ measurable} \right\}$$

$$= \inf\{ \int_0^1 H^\star(\dot{z}(s))dt + \psi(y) \mid z \in C^1([0,1];\mathbf{R}^n) \text{ such that } z(0)=x \text{ and } z(1)=y \}.$$

Since the uniqueness of viscosity solutions holds, the Lax formula coincide with the above. From the second expression, this means that we only need to take the straight line $z(\cdot)$ between x and y among C^1 functions.

(b) Hopf formula (the case when ψ is convex.)

$$u(x,t) := \sup_{y \in \mathbf{R}^n} \{-\psi^\star(y) - tH(y) + \langle y, x\rangle\}$$

$$\left(= \sup_{y \in \mathbf{R}^n} \inf_{z \in \mathbf{R}^n} \{\psi(y) + \langle y, x-z\rangle - tH(y)\} \right)$$

The associated DP type equality is as follows: for $0 < r < t$,

$$u(x,t) = \sup_{y\in\mathbf{R}^n} \{-u^*(y,r) - (t-r)H(y) + \langle y,x\rangle\}$$
$$= \sup_{y\in\mathbf{R}^n} \inf_{z\in\mathbf{R}^n} \{\langle y, x-z\rangle + u(z,r) - (t-r)H(y)\}$$

As mentioned in the talk by H. Ishii in this conference, by a clever choice of test functions, we can directly show that the Hopf formula is a viscosity solution.

4. Recent topics

(i) Modified HJ equations

Recently, a series of works by Barron-Jensen-Liu treats the following modified HJ equations ([BJL]):

$$\text{(MHJ)} \qquad u_t + H(u, Du) = 0.$$

Let us recall from [BJL] the reason why, in this case, that the "convexity" is not a sufficient assumption in view of the associated characteristic curves, which is the set of ODEs (assume given functions are C^2 say).

$$\begin{cases} \dot{\xi}(t) = H_p(\eta(t), p(t)), & \xi(0) = x, \\ \dot{\eta}(t) = -H(\eta(t), p(t)) + \langle p(t), H_p(\eta(t), p(t))\rangle, & \eta(0) = \psi(x), \\ \dot{p}(t) = -H_u(\eta(t), p(t))p(t), & p(0) = D\psi(x). \end{cases}$$

The reason why the DP type equality for Lax or Hopf formulas are easier to treat than the standard DP Principle for the value functions is that we need only to take straight lines for the DP type equality. To apply this argument to (MHJ), we must first suppose $H_p(\eta(t), p(t))$ is constant since $\dot{\xi}$ is constant. Assuming this, we differentiate H_p w.r.t. t to get

$$0 = H_{up}\dot{\eta} + \langle H_{pp}, \dot{p}\rangle = H_{up}(-H + \langle p, H_p\rangle) - H_u\langle H_{pp}, p\rangle.$$

If H does not depend on u, then this holds true but, if not, in order to make the above hold, the homogenuity degree 1 is the correct choice. (i.e. $H(u, \lambda p) = \lambda H(u,p)$ for all $\lambda > 0$.) Indeed, this implies $\langle p, H_p\rangle = H$ and $\langle p, H_{pp}p\rangle = 0$. Since $H_{pp} \geq 0$ from its convexity, we have $H_{pp} = 0$.

In this direction, we also refer to [ABI].

(ii) Degenerate case

Let us consider the degenerate case:

$$\text{(DHJ)} \qquad h(x)u_t + H(Du) = 0$$

In order to treat the case when $h \geq 0$ might vanish, there have appeared several works. See [IR], [K] and references therein.

In fact, in this case, under the standard setting, the uniqueness fails. To overcome this difficulty, we need to adapt stronger definitions than the standard ones.

(iii) Level set approach

To treat semicontinuous viscosity solutions, Y. Giga and M.-H. Sato propose a level set approach for HJ equations. This topics will be discussed in their abstract in this Note.

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