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SINGULARITIES OF VISCOSITY SOLUTIONS OF HAMILTON–JACOBI EQUATIONS

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Abstract

A viscosity solution of a Hamilton–Jacobi equation is the asymptotics of the solution with the same initial condition of the original Hamilton–Jacobi equation regularized by vanishing viscosity. Even if the initial condition is smooth, the viscosity solution can have singularities. In the case of a convex smooth Hamiltonian and a generic smooth initial condition we describe a full classification of these singularities and their perestroikas (= bifurcations, metamorphoses) in spaces of physically interesting dimensions 1, 2, and 3.

Singularities and Perestroikas of Shock Waves

We consider the Cauchy problem for a viscosity solution u of a Hamilton–Jacobi equation:

$$u_t + H(t,x,u_x) = 0, \quad u(0,x) = v(x), \quad 0 \le t \le T < +\infty, \quad x \in \mathbb{R}^n, \tag{1}$$

where the initial condition v is a bounded smooth function and the Hamiltonian H is a smooth function being the Legendre transform of a convex (along the velocities y) smooth Lagrangian L:

$$H(t,x,p) = \max_y \left(py - L(t,x,y)\right), \quad p \in \mathbb{R}^n, \quad y \in \mathbb{R}^n, \quad \left\| \frac{\partial^2 L}{\partial y^2} \right\| > 0. \tag{2}$$

The last condition means that the Jacobian matrix is positive definite at any point. In particular, the Hamiltonian is convex along the momenta p: $\|\partial^2 H/\partial p^2\| > 0$. In the present paper "smooth" means "infinitely smooth".

DEFINITION ([8]). A continuous function u is a viscosity solution of the Hamilton–Jacobi equation (1) if and only if

- any smooth function $\varphi(t,x)$ such that $u - \varphi$ has a local minimum at (t_0, x_0) satisfies the inequality $\varphi_t(t_0,x_0) + H(t_0, x_0, \varphi_x(t_0,x_0)) \ge 0$, and
- any smooth function $\varphi(t,x)$ such that $u - \varphi$ has a local maximum at (t_0, x_0) satisfies the inequality $\varphi_t(t_0,x_0) + H(t_0, x_0, \varphi_x(t_0,x_0)) \le 0$.

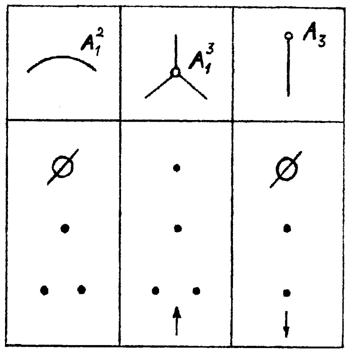

Figure 1: Singularities of big shock waves and perestroikas of momentary shock waves if $n = 1$

It is well known that the viscosity solution u can become singular even if the initial condition v and the Hamiltonian H are smooth. At each time t these singularities form the momentary shock waves in n-dimensional x-space:

$$MSW_t = \{x \in \mathbb{R}^n \mid u \text{ is not smooth at } (t, x)\}$$

depending on time t. If the initial condition is generic then the momentary shock waves are hypersurfaces with singularities which experience *perestroikas* (= bifurcations, metamorphoses) with time. The momentary shock waves are the sections wave with isochrones $t =$ const of the big shock wave

$$BSW = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n \mid u \text{ is not smooth at } (t, x)\}$$

lying in $(n + 1)$-dimensional space-time. If the initial condition is generic then the big shock wave is a hypersurface with singularities.

Problem. To describe singularities and perestroikas of the momentary shock waves in physically interesting cases $n = 2$ and 3 provided the initial condition v is generic.

REMARK. As usually, *generic* initial conditions form an open dense subset in the space of all smooth functions with respect to the so-called fine Whitney topology (see, for example, [3]). In other words, the following theorems are true only for an open dense subset of initial conditions.

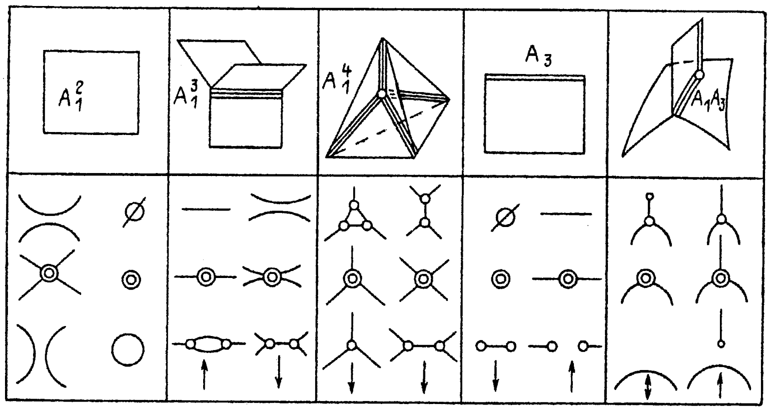

Figure 2: Singularities of big shock waves and perestroikas of momentary shock waves if $n = 2$

In the one-dimensional case $n = 1$ the solution of Problem is well known. Momentary shock waves are sets of isolated points and there exist two perestroikas shown in Fig. 1 by the arrows: two sticking points and one arising point. The big shock wave, besides smooth points A_1^2, can consist of singularities A_1^3 and A_3 shown in Fig. 1 as well.

In the many-dimensional cases $n = 2$ and 3 Problem is solved in [5] (see also [1] and Chapter 2, § 3 in [2]), provided $H(t, x, p) = p^2/2$ and $L(t, x, y) = y^2/2$. In the present paper it is solved for general convex Hamiltonians (2).

THEOREM 1. *If $n = 2$ or 3 then at a fixed instant there exist only 3 (A_1^2, A_1^3, A_3) or 5 (A_1^2, A_1^3, A_1^4, A_3, A_1A_3) smoothly different germs of momentary shock waves provided the initial condition is generic.*

If $n = 1$ or 2 and the initial condition is generic then the big shock wave can have only the same germs (A_1^2, A_1^3, A_3 for $n = 1$ and A_1^2, A_1^3, A_1^4, A_3, A_1A_3 for $n = 2$) which are shown in Fig. 1 or Fig. 2 respectively.

THEOREM 2. *If $n = 2$ or 3 then there exist respectively 9 or 26 topologically different perestroikas of momentary shock waves provided the initial condition is generic. These perestroikas are shown by the arrows in Fig. 2 or Fig. 3 respectively.*

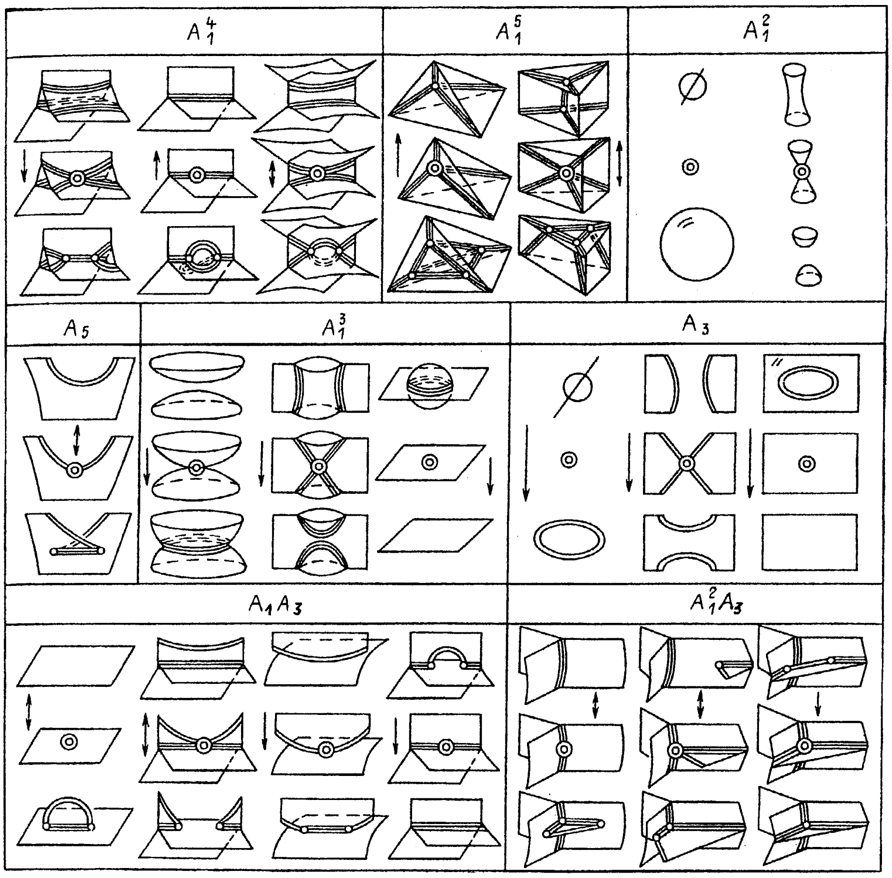

Figure 3: Perestroikas of momentary shock waves if $n = 3$

Viscosity Solutions as Minimum Functions

The above description of singularities and perestroikas of momentary shock waves is based on Theorem 3 giving us an explicit formula for the viscosity solution u of our Cauchy problem.

THEOREM 3. *Let*

$$\mathcal{F}(t,x,y) = v(\gamma(0)) + \int_0^t L(\tau,\gamma(\tau),\dot\gamma(\tau))\,d\tau \tag{3}$$

where $\gamma : [0,t] \to \mathbb{R}^n$ *is the solution of the Euler–Lagrange equation with the following initial conditions:*

$$\frac{d}{d\tau}\frac{\partial}{\partial\dot\gamma}L(\tau,\gamma,\dot\gamma) = \frac{\partial}{\partial\gamma}L(\tau,\gamma,\dot\gamma), \quad \gamma(t) = x, \quad \dot\gamma(t) = y.$$

Then

$$u(t,x) = \min_y \mathcal{F}(t,x,y), \quad y \in \mathbb{R}^n$$

is the viscosity solution of the Cauchy problem (1), (2).

Theorem 3 follows from Lemma 1 proved, for example, in [9] for more general Hamiltonians.

LEMMA. *If* $\mathcal{A}_{t,x} = \{\gamma : [0,T] \to \mathbb{R}^n \mid \gamma$ *is piecesmooth continuous,* $\gamma(t) = x\}$ *then*

$$u(t,x) = \min_{\gamma\in\mathcal{A}_{t,x}} \left\{ v(\gamma(0)) + \int_0^t L(\tau,\gamma(\tau),\dot\gamma(\tau))\,d\tau \right\}$$

is the viscosity solution of the Cauchy problem (1), (2).

Singularities and Perestroikas of Minimum Functions

Let

$$f(\lambda) = \min_y F(\lambda,y)$$

where F is a smooth function of λ and $y \in \mathbb{R}^n$. In spite of the smoothness of F the function f can have singularities. Theorem 4 describes these singularities.

THEOREM 4 ([7]). *If* F *is generic and* $\dim\lambda \le 4$ *then* f *can have only the following germs with respect to diffeomorphisms of* λ *and adding smooth functions of* λ*:*

A_1^m	$f(\lambda) = \min\{\lambda_1,\dots,\lambda_{m-1},0\}$	$\dim\lambda \ge m-1,\ m \ge 1;$
$A_1^m A_3$	$f(\lambda) = \min\{\lambda_1,\dots,\lambda_m, \min_s(s^4+\lambda_{m+1}s^2+\lambda_{m+2}s)\}$	$\dim\lambda \ge m+2,\ m \ge 0;$
A_5	$f(\lambda) = \min_s(s^6+\lambda_1 s^4+\lambda_2 s^3+\lambda_3 s^2+\lambda_4 s)$	$\dim\lambda \ge 4.$

The germ $A_1 = A_1^1$ is smooth. The other ones are singular. The singularities of each of them form a hypersurface in λ-space. In the case A_1^2 this hypersurface ($\lambda_1 = 0$) is smooth, has a boundary in the case $A_3 = A_1^0 A_3$ ($\lambda_1 \le 0$, $\lambda_2 = 0$), and has more complex singularities in the other cases.

According to Theorem 3,

$$u(t,x) = \min_y \mathcal{F}(t,x,y).$$

So, if $\mathcal{F}$ is generic, then Theorem 4 gives us the full list of possible singularities of the momentary shock waves at a fixed instant t with respect to diffeomorphisms of x-space in physically interesting cases $\dim x \le 3$. If $n = 1$ then the momentary shock waves do not have singularities and consists of isolated points A_1^2. If $n = 2$ then, besides smooth points A_1^2, the momentary shock waves can have singularities A_1^3 and A_3 shown in Fig. 1. If $n = 3$ then, besides smooth points A_1^2, the momentary shock waves can have singularities A_1^3, A_1^4, A_3, and A_1A_3 shown in Fig. 2.

In order to describe perestroikas of the momentary shock waves with time let us consider the big shock wave lying in $(n+1)$-dimensional space-time. The momentary shock waves are the sections of the big shock wave with isochrones $t = \text{const}$. If $\mathcal{F}$ is generic, Theorem 4 describes all possible singularities of the big shock wave with respect to diffeomorphisms of space-time in physically interesting cases $\dim \lambda \leq 4$. But such diffeomorphisms can mix time and space coordinates changing the time function $\tau : (t, x) \mapsto t$. So, perestroikas of the momentary shock waves are just perestroikas of the sections of singularities from Theorem 4 with isochrones $\tau(\lambda) = \text{const}$ where τ is a generic smooth function on λ-space without critical points.

THEOREM 5 ([5]). *If $\mathcal{F}$ is generic and $n = 1$, 2, or 3 then there exist respectively 6, 18 or 42 topologically different perestroikas of momentary shock waves. These perestroikas are shown in Fig. 1, Fig. 2, or Fig. 3 respectively.*

In fact, Theorem 5 describes all germs of the function τ with respect to homeomorphisms preserving the shock wave normalized in Theorem 4 already. For example, if $n = 1$ our 6 perestroikas are described by the following time functions:

$$A_1^2 : \ \tau(\lambda_1, \lambda_2) = \lambda_1 \pm \lambda_2^2; \quad A_1^3 : \ \tau(\lambda_1, \lambda_2) = \pm(\lambda_1 + \lambda_2); \quad A_3 : \ \tau(\lambda_1, \lambda_2) = \pm\lambda_1.$$

Singularities and Perestroikas of Shock Waves

Theorem 5 does not solve our Problem yet because the function $\mathcal{F}$ has the special form (3) and is not generic. For example, if $n = 1$ only two perestroikas from the six ones of Theorem 5 are realized as perestroikas of the momentary shock waves of the viscosity solutions of the Cauchy problem (1), (2). S. N. Gurbatov and A. I. Saichev have observed that this phenomenon is present in the many-dimensional case as well.

THEOREM 6 ([5]). *If $n \leq 3$ and the initial condition is generic then in a neighborhood of the point of a perestroika in space-time the momentary shock waves of the viscosity solutions of the Cauchy problem (1), (2) are contractible (= homotopy-equivalent to a point) after the perestroika.*

The full list of perestroikas of the momentary shock waves of the viscosity solutions of the Cauchy problem (1), (2) consists of all perestroikas of Theorem 5 which satisfy this topological restriction.

Theorems 5 and 6 prove Theorem 2 decreasing the number of possible perestroikas from 6 to 2 for $n = 1$, from 18 to 9 for $n = 2$, and from 42 to 26 for $n = 3$.

In [5] Theorem 6 is proved implicitly only in the case $n \leq 3$. The proof is based on comparing the figures of all perestroikas from Theorem 5 and that ones which are realized as perestroikas of momentary shock waves of viscosity solutions. The realizability is checked with the help of some algebraic (but not topological!) criterion. A general topological proof of Theorem 6 is not known.

Another topological restriction which forbids the same perestroikas for $n \leq 3$ was proposed in [4]. It turns out that the homotopy types of complements to the momentary shock waves of a viscosity solution at the instant of a perestroika and directly after it coincide. This restriction is proved in [4] topologically for all n and for all initial conditions with the exception of a set of infinite codimension.

As a matter of fact, in order to be rigorous it is necessary to show that the function $\mathcal{F}$ in the form (3) with a generic initial condition v and a fixed Lagrangian L can be considered as generic. It is done in [6].

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