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A problem on blow-analytic sufficiency of jets

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Let $f : U \to \mathbf{R}$, U open in $\mathbf{R}^n$, be a continuous function. We say that f is *blow-analytic*, if there exists a multi-blowing-up β such that the composition $f \circ \beta$ is analytic. Let $h : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ be a local homeomorphism. We say that h is *blow-analytic*, if the components of both h and h^{-1} are blow-analytic functions. Given $f, g : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$, we say that they are *blow-analytically equivalent*, if there exists such an h with $f = g \circ h$. The notion of blow-analytic equivalence was introduced by T.C. Kuo [9].

Concerning the blow-analytic equivalence, T. Fukui ([2]) gave the following invariant. For an analytic function $f : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$, set

$$A(f) = \{O(f \circ \lambda) \in \mathbf{N} \cup \{\infty\} \mid \lambda : (\mathbf{R}, 0) \to (\mathbf{R}^n, 0)\ C^\omega\}.$$

If two analytic functions $f, g : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ are blow-analytically equivalent, then $A(f) = A(g)$. If the reader wants to know more about blow-analyticity, consult the survey article [3].

Let $J^r(n, 1)$ denote the set of r-jets in analytic function germs. An r-jet $w \in J^r(n, 1)$ is called *blow-analytically sufficient*, if any two analytic function germs f, g with $j^r f(0) = j^r g(0) = w$ are blow-analytically equivalent. We shall identify r-jets with polynomial representatives of degree not exceeding r.

In the Singularity Theory, the notion of sufficiency of jets (or finite determinacy) becomes very important to consider the stability problem or the classification problem. Nevertheless, we don't know any criterion or characterization for blow-analytic sufficiency of jets. Here we discribe a problem on it.

Problem. *For an r-jet $w \in J^r(n, 1)$, are the following equivalent ?*

(1) *w is blow-analytically sufficient.*

(2) *For any analytic function $f : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ with $j^r f(0) = w, A(f) = A(w)$ and f has an isolated singularity at $0 \in \mathbf{R}^n$.*

Condition (1) always implies condition (2). Because $A(f)$ is an invariant for blow-analytic equivalence as stated above, and even C^0 finite determinacy of an analytic function implies an isolated singularity ([1]).

I have the following impressions to this problem.

(I) *In the case $n = 2$, the answer is yes !*

In this case, is the following also equivalent ?

(3) *For any analytic function $f : (\mathbf{R}^2, 0) \to (\mathbf{R}, 0)$ with $j^r f(0) = w,\ A(f) = A(w)$.*

(II) *In the case $n \geq 3$, the answer is no !*

We give a 6-jet $w = x^3 - 3xy^5 + z^3 \in J^6(3,1)$ as a candidate of a negative example to the problem above. This jet was originally given by W. Kucharz [8] as an example such that w is C^0-sufficienct in C^8 functions as a 6-jet, but not C^0-sufficient in C^7 functions as a 7-jet.

Let $f : (\mathbf{R}^3, 0) \to (\mathbf{R}, 0)$ be any analytic function with $j^6 f(0) = w$. Then $A(f) = A(w) = \{3, 4, 5, \cdots, \infty\}$. Furthermore it is easy to see that f has an isolated singularity at $0 \in \mathbf{R}^3$. Therefore this w satisfies condition (2) in the Problem.

In the similar way to [7] (see [4] and [6] also), we can show that f is blow-analytically equivalent to $w = x^3 - 3xy^5 + z^3$ or $v = x^3 + y^7 + z^3$.

If w is a negative example to the Problem, then w is not blow-analytically equivalent to v. How can we show it ? This is a problem.

Remark 1. *Recently, L. Paunescu* [10] *has given a new invariant for blow-analytic equivalence which preserves the order of any analytic arc. Even if we use the invariant, we can't distinguish the above w from v.*

Remark 2. *In the case $n \geq 3$, condition* (3) *does not necessarily imply condition* (2). *Consider a 6-jet $w = x^3 + y^3 \in J^3(3,1)$. For any analytic function $f : (\mathbf{R}^3, 0) \to (\mathbf{R}, 0)$ with $j^3 f(0) = w$, $A(f) = A(w) = \{3, 4, 5, \cdots, \infty\}$. But w does not have an isolated singularity.*

Remark 3. *Fukui gave a formula for $A(f)$ in case f is a non-degenerate function. A formula in the general case is obtained in* [5].

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