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Passing through degenerate points

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Two C^{∞} map-germs are said to be C^{∞} *right-left equivalent* if they coincide under germs of appropriate C^{∞} co-ordinate systems of the source space and the target space.

Let $f_{\alpha}:(\mathbf{R}^{n},0)\rightarrow(\mathbf{R}^{p},0)\quad(\alpha\in \mathbf{R}^{\ell})$ be a family of C^{∞} map-germs. First, we suppose that

Assumption 0. 1 there exists a thin subset Σ of $\mathbf{R}^{\ell}$ such that f_{α} is not C^{∞} right-left equivalent to $f_{\alpha'}$ for any $\alpha\in \mathbf{R}^{\ell}-\Sigma$, $\alpha'\in\Sigma$.

Next, let us assume that

Assumption 0. 2 it seems that f_{α} and $f_{\alpha'}$ are C^{∞} right-left equivalent for any $\alpha, \alpha'\in \mathbf{R}^{\ell}-\Sigma$ and we want to prove it.

If α, α' belong to the same connected component of $\mathbf{R}^{\ell}-\Sigma$, usually we prove it by using Thom-Lenine criterion ([4]) or its refinements (for instance [7], [2]). However, what can we use if α, α' belongs to different connnected components of $\mathbf{R}^{\ell}-\Sigma$? Namely, we want to have a simple systematic method to answer the following problem:

Problem 0. 1 Prove that f_{α} and $f_{\alpha'}$ are C^{∞} right-left equivalent for any α, α' belonging to diffenrent connected components of $\mathbf{R}^{\ell}-\Sigma$?

Until very recently there have been no general methods to answer problem 0.1. The purpose of this paper is to give a brief explanation of a general method to answer this problem , which may be regarded as a partial introduction to the author's paper [12].

In §1, we propose our method. In §2, we explain our strategy which is so constructive that in some cases we can obtain concrete forms of co-ordinate transformations which give right-left equivalence of given two map-germs (see §3). In §3, we give several examples. Assertion 3.2 (2) is an example which answers the more difficult problem than our problem 0.1. The author hopes that §3 shows that our method is genaral and powerful.

1 A criterion

For a given C^{∞} map-germ $f:(\mathbf{R}^{n},0)\rightarrow(\mathbf{R}^{p},0)$, any C^{∞} map-germ $\Phi:(\mathbf{R}^{n}\times \mathbf{R}^{k},(0,0))\rightarrow(\mathbf{R}^{p},0)$ such that $\Phi(x,0)=f(x)$ is called a C^{∞} *deformation-germ* of f. A C^{∞} deformation-germ $\Phi:(\mathbf{R}^{n}\times \mathbf{R}^{k},(0,0))\rightarrow(\mathbf{R}^{p},0)$ of $f:(\mathbf{R}^{n},0)\rightarrow(\mathbf{R}^{p},0)$ is said to be C^{∞} *trivial* if there exist germs of C^∞ diffeomorphisms $h : (\mathbf{R}^n \times \mathbf{R}^k, (0,0)) \to (\mathbf{R}^n \times \mathbf{R}^k, (0,0))$ and $H : (\mathbf{R}^p \times \mathbf{R}^k, (0,0)) \to (\mathbf{R}^p \times \mathbf{R}^k, (0,0))$ such that the following diagram (*) commutes, where $\pi : (\mathbf{R}^n \times \mathbf{R}^k, (0,0)) \to (\mathbf{R}^k, 0)$, $\pi' : (\mathbf{R}^p \times \mathbf{R}^k, (0,0)) \to (\mathbf{R}^k, 0)$, are canonical projections.

$$(*) \qquad \begin{array}{ccccc} (\mathbf{R}^n \times \mathbf{R}^k, (0,0)) & \xrightarrow{(\Phi,\pi)} & (\mathbf{R}^p \times \mathbf{R}^k, (0,0)) & \xrightarrow{\pi'} & (\mathbf{R}^k, 0) \\ {\scriptstyle h}\downarrow & & {\scriptstyle H}\downarrow & & \parallel \\ (\mathbf{R}^n \times \mathbf{R}^k, (0,0)) & \xrightarrow{(f,\pi)} & (\mathbf{R}^p \times \mathbf{R}^k, (0,0)) & \xrightarrow{\pi'} & (\mathbf{R}^k, 0) \end{array}$$

A C^∞ deformation-germ $\Phi : (\mathbf{R}^n \times \mathbf{R}^k, (0,0)) \to (\mathbf{R}^p, 0)$ of $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ is said to be *transversely C^∞ trivial* if it is C^∞ trivial and the germ $(H(\{0\} \times \mathbf{R}^k), 0)$ is transverse to the germ $(\{0\} \times \mathbf{R}^k, 0)$, where H is the germ of C^∞ diffeomorphism of $(\mathbf{R}^p \times \mathbf{R}^k, 0)$ given in the above commutative diagram (*).

Theorem 1. 1 *Let $f, g : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be C^∞ map-germs. Suppose that there exist a germ of C^∞ diffeomorphism $s : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ and a C^∞ map-germ $M : (\mathbf{R}^n, 0) \to (GL(p, \mathbf{R}), M(0))$ such that*

(1-a) $\quad f(x) = M(x)g(s(x)),$

(1-b) $\quad$ *the C^∞ map-germ $F : (\mathbf{R}^n \times \mathbf{R}^p, (0,0)) \to (\mathbf{R}^p, 0)$ given by*

$$F(x, \lambda) = f(x) - M(x)\lambda$$

is a transversely C^∞ trivial deformation germ of f.

Then, f and g are C^∞ right-left equivalent.

Therem 1.1 is proved in [12] by using the strategy of §2 below.

2 The strategy

In this section, we explain the strategy for proving theorem 1.1. Note that this strategy is constructive.

Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be a C^∞ map-germ and $M : (\mathbf{R}^n, 0) \to (GL(p, \mathbf{R}), M(0))$ be a C^∞ map-germ. We treat two kinds of p-dimensional Euclidean space $\mathbf{R}^p$. If we are considering $\mathbf{R}^p$ as the target space, then we denote it by $\mathbf{R}^p_y$. If we are considering $\mathbf{R}^p$ as the parameter space, then we denote it by $\mathbf{R}^p_\lambda$.
We suppose that the C^∞ deformation-germ $F : (\mathbf{R}^n \times \mathbf{R}^p_\lambda, 0) \to (\mathbf{R}^p_y, 0)$ of f given by

$$F(x, \lambda) = f(x) - M(x)\lambda$$

is C^∞ trivial. Then, from the definition of C^∞ triviality, there exist germs of C^∞ diffeomorphisms $h : (\mathbf{R}^n \times \mathbf{R}^p_\lambda, (0,0)) \to (\mathbf{R}^n \times \mathbf{R}^p_\lambda, (0,0))$ and $H : (\mathbf{R}^p_y \times \mathbf{R}^p_\lambda, (0,0)) \to (\mathbf{R}^p_y \times \mathbf{R}^p_\lambda, (0,0))$ such that the diagram (*) commutes, where $\pi_\lambda : (\mathbf{R}^n \times \mathbf{R}^p_\lambda, (0,0)) \to (\mathbf{R}^p_\lambda, 0)$, $\pi'_\lambda : (\mathbf{R}^p_y \times \mathbf{R}^p_\lambda, (0,0)) \to (\mathbf{R}^p_\lambda, 0)$ are canonical projections.

$$(**) \qquad \begin{array}{ccccc} (\mathbf{R}^n \times \mathbf{R}^p_\lambda, (0,0)) & \xrightarrow{(F,\pi_\lambda)} & (\mathbf{R}^p_y \times \mathbf{R}^p_\lambda, (0,0)) & \xrightarrow{\pi'_\lambda} & (\mathbf{R}^p_\lambda, 0) \\ {\scriptstyle h}\downarrow & & {\scriptstyle H}\downarrow & & \parallel \\ (\mathbf{R}^n \times \mathbf{R}^p_\lambda, (0,0)) & \xrightarrow{(f,\pi_\lambda)} & (\mathbf{R}^p_y \times \mathbf{R}^p_\lambda, (0,0)) & \xrightarrow{\pi'_\lambda} & (\mathbf{R}^p_\lambda, 0) \end{array}$$

From the commutativity of (**), we may put

$$h(x,\lambda) = (h_1(x,\lambda),\lambda)$$

and

$$H(y,\lambda) = (H_1(y,\lambda),\lambda).$$

Lemma 2. 1 $f(h_1(x, g(s(x)))) = H_1(0, g(s(x)))$.

For the proof of lemma 2.1, see [12].

Lemma 2. 2 *If the map-germ from* $(\mathbf{R}^p_\lambda, 0)$ *to* $(\mathbf{R}^p_y, 0)$ *given by*

$$\lambda \mapsto H_1(0,\lambda) \tag{2.1}$$

is a germ of C^∞ *diffeomorphism, then the map-germ given by*

$$(x,\lambda) \mapsto (h_1(x,\lambda), H_1(0,\lambda))$$

maps the set-germ $(F^{-1}(0),(0,0))$ *onto the germ of graph of* f *at* $(0,0)$.

For the proof of lemma 2.2, see [12].

Lemma 2. 3 *If the map-germ from* $(\mathbf{R}^p_\lambda, 0)$ *to* $(\mathbf{R}^p_y, 0)$ *given by*

$$\lambda \mapsto H_1(0,\lambda)$$

is a germ of C^∞ *diffeomorphism, then the endomorphism-germ of* $(\mathbf{R}^n, 0)$ *given by*

$$x \mapsto h_1(x, g(s(x)))$$

is also a germ of C^∞ *diffeomorphism.*

For the proof of lemma 2.3, see [12].

By lemmata 2.1 and 2.3, we have

Lemma 2. 4 *Under the above situation, if the map-germ from* $(\mathbf{R}^p_\lambda, 0)$ *to* $(\mathbf{R}^p_y, 0)$ *given by*

$$\lambda \mapsto H_1(0,\lambda)$$

is a germ of C^∞ *diffeomorphism, then* f *and* g *are* C^∞ *right-left equivalent.*

Remark 2.0. 1

(1) After composing the parallel translation $(y,\lambda) \mapsto (y - H_1(0,\lambda),\lambda)$ with H, the map-germ (2.1) for the composed map-germ is the constant zero map-germ. Thus, the map-germ (2.1) seems to be meaningless. However, before composing the parallel translation, the map-germ (2.1) is meaningful.

(2) The essential point of the above strategy is to pay attention to the map-germ (2.1). Our method differs from Martinet's one ([5]) in this respect. Thanks to the map-germ (2.1), we can treat C^∞ map-germs which are not necessarily C^∞ stable. In fact, careful observations of the map-germ (2.1) lead us to theorem 1.1.

(3) We are considering only C^∞ right-left equivalence of very special type. Nevertheless, to our surprize, in almost all cases it seems to be enough to consider only such C^∞ right-left equivalence of very special type.

(4) Our method works well even in topological cases ([9], [10],[11], [13]).

3 Examples

In this section, we denote standard co-ordinates in the source space by lower case letters x, y, z and standard co-ordinates in the target space by upper case letters X, Y, Z.

3.1

Let $f_\alpha : (\mathbf{R}^3, 0) \to (\mathbf{R}^3, 0)$ be given by

$$f_\alpha(x, y, z) = (x, xy + yz + z^2, \alpha xy + y^2),$$

where $\alpha \in \mathbf{R}$.

Assertion 3. 1 *f_α is C^∞ right-left equivalent to f_0 if $\alpha \neq -4$.*

Remark 3.1. 1

(1) By using "Transversal" elaborated by N. Kirk ([3]), we see that codimension of $T\mathcal{A}(f_\alpha)$ in $J^4(3,3)$ is 12 for $\alpha \neq -4$ and codimension of $T\mathcal{A}(f_{-4})$ in $J^4(3,3)$ is 17. Thus, f_α and f_{-4} are not C^∞ right-left equivalent for $\alpha \neq -4$.

(2) Since $(0, 0, x^i z)$ is not contained in $T\mathcal{A}(f_0)$ for any $i \in \mathbf{N}$, f_0 is not finitely $\mathcal{A}$-determined by Mather's characterization ([6]). By assertion 3.1, f_α is not finitely $\mathcal{A}$-determined for any $\alpha \in \mathbf{R}$. However, by the geometric characterization ([1], [14]) f_α is finitely $\mathcal{K}$-determined for any $\alpha \in \mathbf{R}$. Nevertheless, existence of degenerate parameter value ($\alpha = -4$) prevents us from using Gaffney-du Plessis criterion ((1.23) of [2]).

(3) As by-products of proving assertion 3.1 by our method, we can easily obtain simple concrete forms of germs of C^∞ diffeomorphisms

$$s_\alpha, t_\alpha : (\mathbf{R}^3, 0) \to (\mathbf{R}^3, 0),$$

which give C^∞ right-left equivalence of f_α and f_0 for $\alpha \neq -4$ (for details, see remark 3.1.2 (1)).

(4) In spite of remark 3.1.1 (2), by using Thom-Levine criterion ([4]) we can obtain concrete forms of C^∞ map-germs

$$\widetilde{s}_\alpha, \widetilde{t}_\alpha : (\mathbf{R}^3, 0) \to (\mathbf{R}^3, 0)$$

such that

(a) $\widetilde{s}_\alpha, \widetilde{t}_\alpha$ are germs of C^∞ diffeomorphisms if $\alpha > -4$,

(b) $f_0 \circ \widetilde{s}_\alpha = \widetilde{t}_\alpha \circ f_\alpha$ if $\alpha > -4$.

Furthermore, since $\widetilde{s_\alpha}$ and $\widetilde{t_\alpha}$ are concrete, we see directly

(c) $\widetilde{s}_\alpha, \widetilde{t}_\alpha$ are germs of C^∞ diffeomorphisms even if $\alpha < -4$,

(d) $f_0 \circ \widetilde{s}_\alpha = \widetilde{t}_\alpha \circ f_\alpha$ even if $\alpha < -4$.

Thus, assertion 3.1 can be proved also by using Thom-Levine criterion and obtaining integral curves of vector fields explicitly (for details, see remark 3.1.2 (2)).

(5) However, there are two merits of our method.

First, in our method, concrete forms of s_α and t_α are merely by-products. We do not need such concrete forms to prove assertion 3.1. For our method, concrete forms of constant terms of vector fields are sufficient. On the other hand, concrete forms of $\widetilde{s}_\alpha$ and $\widetilde{t}_\alpha$ are absolutely necessary if we want to prove assertion 3.1 by the standard method. Without obtaining integral curves of vector fields explicitly (without further makeshift calculations), we can not pass through $\alpha = -4$ when we use the standard method. In other words, our method is simpler and more systematic than the standard method.

Second, even for calculations for concrete forms of germs of C^∞ diffeomorphisms, our method is much easier than the standard method.

Proof of assertion 3.1 We see that

$$f_0(x,y,z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\alpha y & 0 & 1 \end{bmatrix} f_\alpha(x,y,z).$$

Thus, the condition (1-a) of theorem 1.1 is satisfied.

We consider the C^∞ deformation-germ of f_0 given by

$$F_\alpha(x,y,z,\lambda_1,\lambda_2,\lambda_3) = f_0(x,y,z) - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\alpha y & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}.$$

It is clear that

$$-\frac{\partial F_\alpha}{\partial \lambda_2} = \frac{\partial}{\partial Y} \quad \text{and} \quad -\frac{\partial F_\alpha}{\partial \lambda_3} = \frac{\partial}{\partial Z}.$$

Furthermore, we see that

$$
\begin{aligned}
-\frac{\partial F_\alpha}{\partial \lambda_1} = \begin{bmatrix} 1 \\ 0 \\ -\alpha y \end{bmatrix} &= \frac{\partial}{\partial X} + \begin{bmatrix} 0 \\ 0 \\ -\alpha y \end{bmatrix} \\
&= \frac{\partial}{\partial X} - \frac{\alpha}{2}\frac{\partial}{\partial y}(f_0) + \frac{\alpha}{2}\begin{bmatrix} 0 \\ x+z \\ 0 \end{bmatrix} \\
&= \frac{\partial}{\partial X} - \frac{\alpha}{2}\frac{\partial}{\partial y}(F_\alpha) - \frac{\alpha}{2}\begin{bmatrix} 0 \\ 0 \\ -\alpha \end{bmatrix}\lambda_1 + \frac{\alpha}{2}\begin{bmatrix} 0 \\ x \\ 0 \end{bmatrix} + \frac{\alpha}{2}\begin{bmatrix} 0 \\ z \\ 0 \end{bmatrix} \\
&= \frac{\partial}{\partial X} - \frac{\alpha}{2}\frac{\partial}{\partial y}(F_\alpha) + \frac{\alpha^2}{2}\lambda_1\frac{\partial}{\partial Z} + \frac{\alpha}{2}(X+\lambda_1)\frac{\partial}{\partial Y} \\
&\qquad + \frac{\alpha}{4}\frac{\partial}{\partial z}(f_0) - \frac{\alpha}{4}\begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} \\
&= \frac{\partial}{\partial X} - \frac{\alpha}{2}\frac{\partial}{\partial y}(F_\alpha) + \frac{\alpha^2}{2}\lambda_1\frac{\partial}{\partial Z} + \frac{\alpha}{2}(X+\lambda_1)\frac{\partial}{\partial Y} \\
&\qquad + \frac{\alpha}{4}\frac{\partial}{\partial z}(F_\alpha) - \frac{\alpha}{4}\frac{\partial}{\partial x}(f_0) + \frac{\alpha}{4}\frac{\partial}{\partial X} \\
&= \left(-\frac{\alpha}{4}\frac{\partial}{\partial x} - \frac{\alpha}{2}\frac{\partial}{\partial y} + \frac{\alpha}{4}\frac{\partial}{\partial z}\right)(F_\alpha) \\
&\qquad + \frac{(4+\alpha)}{4}\frac{\partial}{\partial X} + \frac{\alpha}{2}(X+\lambda_1)\frac{\partial}{\partial Y} + \frac{\alpha^2}{2}\lambda_1\frac{\partial}{\partial Z}.
\end{aligned}
$$

Put

$$\xi_{1,\alpha} = -\frac{\alpha}{4}\frac{\partial}{\partial x} - \frac{\alpha}{2}\frac{\partial}{\partial y} + \frac{\alpha}{4}\frac{\partial}{\partial z},\ \xi_{2,\alpha} = 0,\ \xi_{3,\alpha} = 0$$

and

$$\eta_{1,\alpha} = -\frac{(4+\alpha)}{4}\frac{\partial}{\partial X} - \frac{\alpha}{2}(X+\lambda_1)\frac{\partial}{\partial Y} - \frac{\alpha^2}{2}\lambda_1\frac{\partial}{\partial Z},\ \eta_{2,\alpha} = -\frac{\partial}{\partial Y},\ \eta_{3,\alpha} = -\frac{\partial}{\partial Z}.$$

Then, we have

$$-\frac{\partial F_\alpha}{\partial \lambda_i} = \xi_{i,\alpha}(F_\alpha) - \eta_{i,\alpha} \circ (F_\alpha, \pi_\lambda) \qquad (i = 1,2,3) \tag{3.1.1}$$

and

$$\eta_{1,\alpha}(0,0) = -\frac{(4+\alpha)}{4}\frac{\partial}{\partial X},\ \eta_{2,\alpha}(0,0) = -\frac{\partial}{\partial Y},\ \eta_{3,\alpha}(0,0) = -\frac{\partial}{\partial Z}. \tag{3.1.2}$$

By (3.1.1), integrating germs of C^∞ vector fields

$$\xi_{i,\alpha} + \frac{\partial}{\partial \lambda_i},\ \eta_{i,\alpha} + \frac{\partial}{\partial \lambda_i} \qquad (i = 1,2,3)$$

yields germs of C^∞ diffeomorphisms

$$
\begin{aligned}
h_\alpha^{-1} &: (\mathbf{R}^3 \times \mathbf{R}^3, (0,0)) \to (\mathbf{R}^3 \times \mathbf{R}^3, (0,0)) \\
H_\alpha^{-1} &: (\mathbf{R}^3 \times \mathbf{R}^3, (0,0)) \to (\mathbf{R}^3 \times \mathbf{R}^3, (0,0))
\end{aligned}
$$

such that the following diagram commutes.

$$\begin{array}{ccccc}
(\mathbf{R}^3 \times \mathbf{R}^3, (0,0)) & \xrightarrow{(F_\alpha, \pi)} & (\mathbf{R}^3 \times \mathbf{R}^3, (0,0)) & \xrightarrow{\pi'} & (\mathbf{R}^3, 0) \\
\uparrow h_\alpha^{-1} & & \uparrow H_\alpha^{-1} & & \| \\
(\mathbf{R}^3 \times \mathbf{R}^3, (0,0)) & \xrightarrow{(f_0, \pi)} & (\mathbf{R}^3 \times \mathbf{R}^3, (0,0)) & \xrightarrow{\pi'} & (\mathbf{R}^3, 0)
\end{array}$$

Thus, F_α is C^∞ trivial.

Since

$$\frac{\partial H_\alpha^{-1}}{\partial \lambda_i} = \eta_{i,\alpha} + \frac{\partial}{\partial \lambda_i} \qquad (i = 1, 2, 3),$$

by (3.1.2) we see if $\alpha \neq -4$ then the germ $(H_\alpha^{-1}(\{0\} \times \mathbf{R}^3), (0,0))$ is transverse to the germ $(\{0\} \times \mathbf{R}^3, (0,0))$ and therefore $(H_\alpha(\{0\} \times \mathbf{R}^3), (0,0))$ is transverse to the germ $(\{0\} \times \mathbf{R}^3, (0,0))$. Thus, F_α is transversely C^∞ trivial if $\alpha \neq -4$.

By theorem 1.1, f_α is C^∞ right-left equivalent to f_0 if $\alpha \neq -4$. □

Remark 3.1. 2

(1) Since vector fields

$$\xi_{i,\alpha}, \ \eta_{i,\alpha} \qquad (i = 1, 2, 3)$$

are concrete and simple, we can obtain concrete forms of germs of C^∞ diffeomorphisms h_α^{-1} and H_α^{-1} $(\alpha \neq -4)$ by solving differential equations directly. Since our method is constructive (see §2), we can obtain concrete forms of germs of C^∞ diffeomorphisms s_α and t_α which give C^∞ right-left equivalence of f_0 and f_α for $\alpha \neq -4$ as follows.

Let $\Xi_{i,\alpha} : (\mathbf{R} \times \mathbf{R}^3, (0,0)) \to (\mathbf{R}^3, 0)$ be the germs of local flow for $\xi_{i,\alpha}$ $(i = 1, 2, 3)$. Then, we have

$$\begin{aligned}
& \Xi_{1,\alpha}(\lambda_1; \Xi_{2,\alpha}(\lambda_2; \Xi_{3,\alpha}(\lambda_3; (x, y, z)))) \\
= \ & (x - \frac{\alpha}{4}\lambda_1, y - \frac{\alpha}{2}\lambda_1, z + \frac{\alpha}{4}\lambda_1).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
& h_\alpha((x, y, z), (\lambda_1, \lambda_2, \lambda_3)) \\
= \ & ((x + \frac{\alpha}{4}\lambda_1, y + \frac{\alpha}{2}\lambda_1, z - \frac{\alpha}{4}\lambda_1), (\lambda_1, \lambda_2, \lambda_3))
\end{aligned}$$

and therefore

$$\begin{aligned}
& h_\alpha((x, y, z), f_\alpha(x, y, z)) \\
= \ & ((\frac{(4+\alpha)}{4}x, y + \frac{\alpha}{2}x, z - \frac{\alpha}{4}x), f_\alpha(x, y, z)).
\end{aligned}$$

Put

$$s_\alpha(x, y, z) = (\frac{(4+\alpha)}{4}x, y + \frac{\alpha}{2}x, z - \frac{\alpha}{4}x).$$

Next, let $\Theta_{i,\alpha} : (\mathbf{R} \times \mathbf{R}^3, (0,0)) \to (\mathbf{R}^3, 0)$ be the germ of local flow for $\eta_{i,\alpha}$ $(i = 1,2,3)$. Then, we have

$$\begin{aligned} & \Theta_{1,\alpha}(\lambda_1; \Theta_{2,\alpha}(\lambda_2; \Theta_{3,\alpha}(\lambda_3; (X,Y,Z)))) \\ = \ & \Theta_{1,\alpha}(\lambda_1; \Theta_{2,\alpha}(\lambda_2; (X, Y, Z - \lambda_3))) \\ = \ & \Theta_{1,\alpha}(\lambda_1; (X, Y - \lambda_2, Z - \lambda_3)) \\ = \ & (X - \frac{(4+\alpha)}{4}\lambda_1, Y - \lambda_2 - \frac{\alpha}{2}X\lambda_1 + \frac{\alpha^2}{16}\lambda_1^2, Z - \lambda_3 - \frac{\alpha^2}{4}\lambda_1^2). \end{aligned}$$

Thus, by putting

$$H_{2,\alpha} = Y + \lambda_2 + \frac{\alpha}{2}(X + \frac{(4+\alpha)}{4}\lambda_1)\lambda_1 - \frac{\alpha^2}{16}\lambda_1^2,$$

we have

$$\begin{aligned} & H_\alpha((X,Y,Z),(\lambda_1,\lambda_2,\lambda_3)) \\ = \ & ((X + \frac{(4+\alpha)}{4}\lambda_1, H_{2,\alpha}, Z + \lambda_3 + \frac{\alpha^2}{4}\lambda_1^2), (\lambda_1,\lambda_2,\lambda_3)) \end{aligned}$$

and therefore

$$\begin{aligned} & H_\alpha((0,0,0,0),(\lambda_1,\lambda_2,\lambda_3)) \\ = \ & ((\frac{(4+\alpha)}{4}\lambda_1, \lambda_2 + \frac{(8\alpha+\alpha^2)}{16}\lambda_1^2, \lambda_3 + \frac{\alpha^2}{4}\lambda_1^2), (\lambda_1,\lambda_2,\lambda_3)). \end{aligned}$$

Put

$$t_\alpha(X,Y,Z) = (\frac{(4+\alpha)}{4}X, Y + \frac{(8\alpha+\alpha^2)}{16}X^2, Z + \frac{\alpha^2}{4}X^2).$$

Then, we see

$$f_0 \circ s_\alpha(x,y,z) = t_\alpha \circ f_\alpha(x,y,z)$$

as desired (for details see §2).

(2) We try to show assertion 3.1 by the standard method. We consider the one-parameter family given by

$$\begin{aligned} \tilde{G}_\alpha(x,y,z,t) &= (1-t)f_\alpha + tf_0 \\ &= (x, xy + yz + z^2, (1-t)\alpha xy + y^2). \end{aligned}$$

Then, by the similar calculations as in the proof of assertion 3.1, we see that

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \\ xy \end{bmatrix} = \ & \frac{x}{4}(\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y} - \frac{\partial}{\partial z})(\tilde{G}_\alpha) \\ & -\frac{X}{4}(\frac{\partial}{\partial X} + 2X\frac{\partial}{\partial Y} + 2\alpha X(1-t)\frac{\partial}{\partial Z}) - \alpha\frac{(1-t)}{4}\begin{bmatrix} 0 \\ 0 \\ xy \end{bmatrix}. \end{aligned}$$

Therefore, we have that for $4+\alpha(1-t)\neq 0$

$$\begin{aligned}
-\frac{\partial \tilde{G}_\alpha}{\partial t} &= \begin{bmatrix} 0 \\ 0 \\ \alpha xy \end{bmatrix} \\
&= \frac{\alpha x}{(4+\alpha(1-t))}\left(\frac{\partial}{\partial x}+2\frac{\partial}{\partial y}-\frac{\partial}{\partial z}\right)(\tilde{G}_\alpha) \\
&\quad -\frac{\alpha X}{(4+\alpha(1-t))}\left(\frac{\partial}{\partial X}+2X\frac{\partial}{\partial Y}+2\alpha X(1-t)\frac{\partial}{\partial Z}\right).
\end{aligned}$$

Thus, by Thom-Levine criterion ([4]), f_α and f_0 are C^∞ right-left equivalent if $\alpha > -4$.

Let $\alpha\neq 0,-4$ and $\tilde{\Xi}_\alpha, \tilde{\Theta}_\alpha : U\times \mathbf{R}^3 \to \mathbf{R}^3$ be the flow of

(3.1.3) $$\frac{\alpha x}{(4+\alpha(1-t))}\left(\frac{\partial}{\partial x}+2\frac{\partial}{\partial y}-\frac{\partial}{\partial z}\right),$$

(3.1.4) $$\frac{\alpha X}{(4+\alpha(1-t))}\left(\frac{\partial}{\partial X}+2X\frac{\partial}{\partial Y}+2\alpha X(1-t)\frac{\partial}{\partial Z}\right)$$

respectively, where U is the connected component of $\mathbf{R}-\{\frac{(4+\alpha)}{\alpha}\}$ which contains the origin. Then, by solving (3.1.3) and (3.1.4) explicitly, we have

$$\begin{aligned}
\tilde{\Xi}_\alpha(t;(x,y,z)) &= \left(\frac{(4+\alpha)}{(4+\alpha(1-t))}x, y+\frac{2\alpha t}{(4+\alpha(1-t))}x, z-\frac{\alpha t}{(4+\alpha(1-t))}x\right) \\
\tilde{\Theta}_\alpha(t;(X,Y,Z)) &= \left(\frac{(4+\alpha)}{(4+\alpha(1-t))}X, Y+\frac{(8\alpha t+2\alpha^2 t-\alpha^2t^2)}{(4+\alpha(1-t))^2}X^2,\right. \\
&\qquad \left. Z+\frac{(8\alpha^2 t+2\alpha^3 t-4\alpha^2t^2-2\alpha^3t^2)}{(4+\alpha(1-t))^2}X^2\right).
\end{aligned}$$

Note that these integral curves can be extended for any $t\in \mathbf{R}-\{\frac{(4+\alpha)}{\alpha}\}$.

Taking $t=1$, we put

$$\begin{aligned}
\tilde{s}_\alpha(x,y,z) &= \left(\frac{(4+\alpha)}{4}x, y+\frac{\alpha}{2}x, z-\frac{\alpha}{4}x\right) \\
\tilde{t}_\alpha(X,Y,Z) &= \left(\frac{(4+\alpha)}{4}X, Y+\frac{(8\alpha+\alpha^2)}{16}X^2, Z+\frac{\alpha^2}{4}X^2\right).
\end{aligned}$$

Note that $\tilde{s_\alpha}$ and $\tilde{t_\alpha}$ are germs of C^∞ diffeomorphisms for any $\alpha\neq -4$. By substituting, we have directly

$$f_0\circ \tilde{s}_\alpha(x,y,z) = \tilde{t}_\alpha\circ f_\alpha(x,y,z).$$

for any $\alpha\neq -4$. Thus, we see that f_α is C^∞ right-left equivalent to f_0 by obtaining integral curves of vector fields explicitly. □

(3) Surprisingly,

$$s_\alpha = \tilde{s}_\alpha \quad\text{and}\quad t_\alpha = \tilde{t}_\alpha.$$

This fact might indicate the nature of our method.

3.2

Let $f_{(\alpha,\beta)} : (\mathbf{R}^3, 0) \to (\mathbf{R}^3, 0)$ be given by

$$f_{(\alpha,\beta)}(x,y,z) = (x, xy + \alpha yz + z^2, \beta xy + y^2),$$

where $\alpha, \beta \in \mathbf{R}$.

Assertion 3. 2

(1) *$f_{(\alpha,\beta)}$ is C^∞ right-left equivalent to $f_{(0,0)}$ if $4 + \alpha^2\beta \neq 0$.*

(2) *$f_{(\alpha,-\frac{4}{\alpha^2})}$ is C^∞ right-left equivalent to $f_{(1,-4)}$ for any $\alpha \neq 0$.*

Remark 3.2. 1

(1) By using "Transversal" elaborated by N. Kirk ([3]), we see that codimension of $T\mathcal{A}(f_{(\alpha,\beta)})$ in $J^4(3,3)$ is 12 for $\alpha^2\beta + 4 \neq 0$ and codimension of $T\mathcal{A}(f_{(\alpha,-\frac{4}{\alpha^2})})$ in $J^4(3,3)$ is 17 for $\alpha \neq 0$. Thus, $f_{(\alpha,\beta)}$ and $f_{(1,-4)}$ are not C^∞ right-left equivalent for $\alpha^2\beta + 4 \neq 0$.

(2) By remark 3.2.1 (2) and assertion 3.2 (1), we see that $f_{(\alpha,\beta)}$ is not finitely $\mathcal{A}$-determined for any $\alpha, \beta \in \mathbf{R}$. However, by the geometric characterization ([1], [14]) $f_{(\alpha,\beta)}$ is finitely $\mathcal{K}$-determined for any $\alpha, \beta \in \mathbf{R}$. Nevertheless, non-connectedness of the parameter space prevents us from using Gaffney-du Plessis criterion ((1.23) of [2]) for either assertion.

(3) For any α, β with $\alpha \neq 0$, $f_{(\alpha,\beta)}$ is not $\mathcal{C}$-equivalent to $f_{(0,0)}$. Thus, for our method we need to seek suitable co-ordinate transformations of the source space in advance.

(4) For either assertion, as by-products of the proof by using our method, we can easily obtain simple concrete forms of germs of C^∞ diffeomorphisms which give C^∞ right-left equivalence.

Proof of assertion 3.2 (1) We see that

$$f_{(0,0)}(x,y,z) = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{\alpha^2\beta}{4}y & 1 & \frac{\alpha^2}{4} \\ -\beta y & 0 & 1 \end{bmatrix} f_{(\alpha,\beta)}(x, y, z - \frac{1}{2}\alpha y).$$

Thus, the condition (1-a) of theorem 1.1 is satisfied.
We consider the C^∞ deformation-germ of $f_{(0,0)}$ given by

$$F_{(\alpha,\beta)}(x,y,z,\lambda_1,\lambda_2,\lambda_3) = f_{(0,0)}(x,y,z) - \begin{bmatrix} 1 & 0 & 0 \\ -\frac{\alpha^2\beta}{4}y & 1 & \frac{\alpha^2}{4} \\ -\beta y & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}.$$

It is clear that

$$-\frac{\partial F_{(\alpha,\beta)}}{\partial \lambda_2} = \frac{\partial}{\partial Y} \quad \text{and} \quad -\frac{\partial F_{(\alpha,\beta)}}{\partial \lambda_3} = \frac{\alpha^2}{4}\frac{\partial}{\partial Y} + \frac{\partial}{\partial Z}.$$

Furthermore, we see that

$$-\frac{\partial F_{(\alpha,\beta)}}{\partial\lambda_1}=\begin{bmatrix}1\\ -\frac{\alpha^2\beta}{4}y\\ -\beta y\end{bmatrix} = \frac{\partial}{\partial X}-\frac{\alpha^2\beta}{4}\frac{\partial}{\partial x}(f_{(0,0)})+\frac{\alpha^2\beta}{4}\frac{\partial}{\partial X}+\begin{bmatrix}0\\0\\-\beta y\end{bmatrix}$$

$$= -\frac{\alpha^2\beta}{4}\frac{\partial}{\partial x}(F_{(\alpha,\beta)})+\frac{(4+\alpha^2\beta)}{4}\frac{\partial}{\partial X}-\frac{\beta}{2}\frac{\partial}{\partial y}(f_{(0,0)})+\frac{\beta}{2}\begin{bmatrix}0\\x\\0\end{bmatrix}$$

$$= -\frac{\alpha^2\beta}{4}\frac{\partial}{\partial x}(F_{(\alpha,\beta)})+\frac{(4+\alpha^2\beta)}{4}\frac{\partial}{\partial X}$$

$$-\frac{\beta}{2}\frac{\partial}{\partial y}(F_{(\alpha,\beta)})-\frac{\beta}{2}\begin{bmatrix}0\\-\frac{\alpha^2\beta}{4}\\-\beta\end{bmatrix}\lambda_1+\frac{\beta}{2}(X+\lambda_1)\frac{\partial}{\partial Y}$$

$$= -\frac{\alpha^2\beta}{4}\frac{\partial}{\partial x}(F_{(\alpha,\beta)})-\frac{\beta}{2}\frac{\partial}{\partial y}(F_{(\alpha,\beta)})+\frac{(4+\alpha^2\beta)}{4}\frac{\partial}{\partial X}$$

$$+\frac{1}{8}(\alpha^2\beta^2\lambda_1+4\beta(X+\lambda_1))\frac{\partial}{\partial Y}+\frac{\beta^2}{2}\lambda_1\frac{\partial}{\partial Z}.$$

Put

$$\xi_{1,(\alpha,\beta)}=-\frac{\alpha^2\beta}{4}\frac{\partial}{\partial x}-\frac{\beta}{2}\frac{\partial}{\partial y},\ \xi_{2,(\alpha,\beta)}=0,\ \xi_{3,(\alpha,\beta)}=0$$

and

$$\eta_{1,(\alpha,\beta)}=-\frac{(4+\alpha^2\beta)}{4}\frac{\partial}{\partial X}-\frac{1}{8}(\alpha^2\beta^2\lambda_1+4\beta(X+\lambda_1))\frac{\partial}{\partial Y}-\frac{\beta^2}{2}\lambda_1\frac{\partial}{\partial Z},$$

$$\eta_{2,(\alpha,\beta)}=-\frac{\partial}{\partial Y},\ \eta_{3,(\alpha,\beta)}=-\frac{\alpha^2}{4}\frac{\partial}{\partial Y}-\frac{\partial}{\partial Z}.$$

Then, we have

$$-\frac{\partial F_{(\alpha,\beta)}}{\partial\lambda_i}=\xi_{i,(\alpha,\beta)}(F_{(\alpha,\beta)})-\eta_{i,(\alpha,\beta)}\circ(F_{(\alpha,\beta)},\pi_\lambda)\qquad(i=1,2,3)\tag{3.2.1}$$

and

$$\eta_{1,(\alpha,\beta)}(0,0)=-\frac{(4+\alpha^2\beta)}{4}\frac{\partial}{\partial X},\tag{3.2.2}$$

$$\eta_{2,(\alpha,\beta)}(0,0)=-\frac{\partial}{\partial Y},\ \eta_{3,(\alpha,\beta)}(0,0)=-\frac{\alpha^2}{4}\frac{\partial}{\partial Y}-\frac{\partial}{\partial Z}.\tag{3.2.3}$$

By (3.2.2), integrating germs of C^∞ vector fields

$$\xi_{i,(\alpha,\beta)}+\frac{\partial}{\partial\lambda_i},\ \eta_{i,(\alpha,\beta)}+\frac{\partial}{\partial\lambda_i}\qquad(i=1,2,3)$$

yields germs of C^∞ diffeomorphisms

$$h^{-1}_{(\alpha,\beta)}\ :\ (\mathbf{R}^3\times\mathbf{R}^3,(0,0))\to(\mathbf{R}^3\times\mathbf{R}^3,(0,0))$$
$$H^{-1}_{(\alpha,\beta)}\ :\ (\mathbf{R}^3\times\mathbf{R}^3,(0,0))\to(\mathbf{R}^3\times\mathbf{R}^3,(0,0))$$

such that the following diagram commutes.

$$\begin{array}{ccccc}
(\mathbf{R}^3 \times \mathbf{R}^3, (0,0)) & \xrightarrow{(F_{(\alpha,\beta)},\pi)} & (\mathbf{R}^3 \times \mathbf{R}^3, (0,0)) & \xrightarrow{\pi'} & (\mathbf{R}^3, 0) \\
\uparrow h^{-1}_{(\alpha,\beta)} & & \uparrow H^{-1}_{(\alpha,\beta)} & & \| \\
(\mathbf{R}^3 \times \mathbf{R}^3, (0,0)) & \xrightarrow{(f_{(0,0)},\pi)} & (\mathbf{R}^3 \times \mathbf{R}^3, (0,0)) & \xrightarrow{\pi'} & (\mathbf{R}^3, 0)
\end{array}$$

Thus, $F_{\alpha,\beta}$ is C^∞ trivial.

Since

$$\frac{\partial H^{-1}_{(\alpha,\beta)}}{\partial \lambda_i} = \eta_{i,(\alpha,\beta)} + \frac{\partial}{\partial \lambda_i} \qquad (i = 1, 2, 3),$$

by (3.2.2) and (3.2.3) we see if $4 + \alpha^2\beta \neq 0$ then the germ $(H^{-1}_{(\alpha,\beta)}(\{0\} \times \mathbf{R}^3), (0,0))$ is transverse to the germ $(\{0\} \times \mathbf{R}^3, (0,0))$ and therefore $(H_{(\alpha,\beta)}(\{0\} \times \mathbf{R}^3), (0,0))$ is transverse to the germ $(\{0\} \times \mathbf{R}^3, (0,0))$. Thus, $F_{\alpha,\beta}$ is transversely C^∞ trivial if $4+\alpha^2\beta \neq 0$.

By theorem 1.1, $f_{(\alpha,\beta)}$ is C^∞ right-left equivalent to $f_{(0,0)}$ if $4 + \alpha^2\beta \neq 0$. □

Remark 3.2. 2

(1) Since vector fields

$$\xi_{i,(\alpha,\beta)},\ \eta_{i,(\alpha,\beta)} \qquad (i = 1, 2, 3)$$

are concrete and simple, we can obtain concrete forms of germs of C^∞ diffeomorphisms $h^{-1}_{(\alpha,\beta)}$ and $H^{-1}_{(\alpha,\beta)}$ $(\alpha^2\beta+4 \neq 0)$ by solving differential equations directly. Since our method is constructive (see §2), we can obtain concrete forms of germs of C^∞ diffeomorphisms $s_{(\alpha,\beta)}$ and $t_{(\alpha,\beta)}$ which give C^∞ right-left equivalence of $f_{(0,0)}(x, y, z)$ and $f_{(\alpha,\beta)}(x, y, z - \frac{\alpha}{2}y)$ for α, β with $\alpha^2\beta + 4 \neq 0$ as follows.

Let $\Xi_{i,(\alpha,\beta)} : (\mathbf{R} \times \mathbf{R}^3, (0,0)) \to (\mathbf{R}^3, 0)$ be the germs of local flow for $\xi_{i,(\alpha,\beta)}$ $(i = 1, 2, 3)$. Then, we have

$$\begin{aligned}
&\Xi_{1,(\alpha,\beta)}(\lambda_1; \Xi_{2,(\alpha,\beta)}(\lambda_2; \Xi_{3,(\alpha,\beta)}(\lambda_3; (x, y, z)))) \\
= \ & (x - \frac{\alpha^2\beta}{4}\lambda_1, y - \frac{\beta}{2}\lambda_1, z).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&h_{(\alpha,\beta)}((x, y, z), (\lambda_1, \lambda_2, \lambda_3)) \\
= \ & ((x + \frac{\alpha^2\beta}{4}\lambda_1, y + \frac{\beta}{2}\lambda_1, z), (\lambda_1, \lambda_2, \lambda_3))
\end{aligned}$$

and therefore

$$\begin{aligned}
&h_{(\alpha,\beta)}((x, y, z), f_{(\alpha,\beta)}(x, y, z - \frac{\alpha}{2}y)) \\
= \ & ((\frac{(4 + \alpha^2\beta)}{4}x, y + \frac{\beta}{2}x, z), f_{(\alpha,\beta)}(x, y, z - \frac{\alpha}{2}y)).
\end{aligned}$$

Put

$$s_{(\alpha,\beta)}(x, y, z) = (\frac{(4 + \alpha^2\beta)}{4}x, y + \frac{\beta}{2}x, z).$$

Next, let $\Theta_{i,(\alpha,\beta)} : (\mathbf{R}\times\mathbf{R}^3,(0,0)) \to (\mathbf{R}^3,0)$ be the germ of local flow for $\eta_{i,(\alpha,\beta)}$ $(i=1,2,3)$. Then, we have

$$\begin{aligned}
&\Theta_{1,(\alpha,\beta)}(\lambda_1;\Theta_{2,(\alpha,\beta)}(\lambda_2;\Theta_{3,(\alpha,\beta)}(\lambda_3;(X,Y,Z))))\\
= \ &\Theta_{1,(\alpha,\beta)}(\lambda_1;\Theta_{2,(\alpha,\beta)}(\lambda_2;(X,Y-\frac{\alpha^2}{4}\lambda_3,Z-\lambda_3)))\\
= \ &\Theta_{1,(\alpha,\beta)}(\lambda_1;(X,Y-\frac{\alpha^2}{4}\lambda_3-\lambda_2,Z-\lambda_3))\\
= \ &(X-\frac{(4+\alpha^2\beta)}{4}\lambda_1, Y-\frac{\alpha^2}{4}\lambda_3-\lambda_2-\frac{\beta}{2}X\lambda_1, Z-\lambda_3-\frac{\beta^2}{4}\lambda_1^2).
\end{aligned}$$

Thus, by putting

$$H_{2,(\alpha,\beta)} = Y+\frac{\alpha^2}{4}\lambda_3+\lambda_2+\frac{\beta}{2}(X+\frac{(4+\alpha^2\beta)}{4}\lambda_1)\lambda_1,$$

we have

$$\begin{aligned}
&H_{(\alpha,\beta)}((X,Y,Z),(\lambda_1,\lambda_2,\lambda_3))\\
= \ &((X+\frac{(4+\alpha^2\beta)}{4}\lambda_1, H_{2,(\alpha,\beta)}, Z+\lambda_3+\frac{\beta^2}{4}\lambda_1^2),(\lambda_1,\lambda_2,\lambda_3))
\end{aligned}$$

and therefore

$$\begin{aligned}
&H_{(\alpha,\beta)}((0,0,0,0),(\lambda_1,\lambda_2,\lambda_3))\\
= \ &((\frac{(4+\alpha^2\beta)}{4}\lambda_1, \lambda_2+\frac{\alpha^2}{4}\lambda_3+\frac{(4\beta+\alpha^2\beta^2)}{8}\lambda_1^2, \lambda_3+\frac{\beta^2}{4}\lambda_1^2),(\lambda_1,\lambda_2,\lambda_3)).
\end{aligned}$$

Put

$$t_{(\alpha,\beta)}(X,Y,Z) = (\frac{(4+\alpha^2\beta)}{4}X, Y+\frac{\alpha^2}{4}Z+\frac{(4\beta+\alpha^2\beta^2)}{8}X^2, Z+\frac{\beta^2}{4}X^2).$$

Then, we see

$$f_{(0,0)}\circ s_{(\alpha,\beta)}(x,y,z) = t_{(\alpha,\beta)}\circ f_{(\alpha,\beta)}(x,y,z-\frac{\alpha}{2}y)$$

as desired (for details see §2).

Proof of assertion 2.3 (2) We see that

$$f_{(1,-4)}(x,y,z) = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{\alpha^2}(\alpha^2-1)y & 1 & \frac{1}{4}(\alpha^2-1) \\ -\frac{4}{\alpha^2}(\alpha^2-1)y & 0 & 1 \end{bmatrix} f_{(\alpha,-\frac{4}{\alpha^2})}(x,y,z-\frac{1}{2}(\alpha-1)y).$$

Thus, the condition (1-a) of theorem 1.1 is satisfied.

We consider the C^∞ deformation-germ of $f_{(1,-4)}$ given by

$$F_\alpha(x,y,z,\lambda_1,\lambda_2,\lambda_3) = f_{(1,-4)}(x,y,z) - \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{\alpha^2}(\alpha^2-1)y & 1 & \frac{1}{4}(\alpha^2-1) \\ -\frac{4}{\alpha^2}(\alpha^2-1)y & 0 & 1 \end{bmatrix}\begin{bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{bmatrix}.$$

It is clear that

$$-\frac{\partial F_\alpha}{\partial \lambda_2} = \frac{\partial}{\partial Y} \qquad \text{and} \qquad -\frac{\partial F_\alpha}{\partial \lambda_3} = \frac{1}{4}(\alpha^2-1)\frac{\partial}{\partial Y} + \frac{\partial}{\partial Z}.$$

Furthermore, we see that

$$\begin{aligned} -\frac{\partial F_\alpha}{\partial \lambda_1} = \begin{bmatrix} 1 \\ \frac{1}{\alpha^2}(\alpha^2-1)y \\ -\frac{4}{\alpha^2}(\alpha^2-1)y \end{bmatrix} &= \frac{\partial}{\partial X} + \frac{1}{\alpha^2}(\alpha^2-1)\frac{\partial}{\partial x}(f_{(1,-4)}) - \frac{1}{\alpha^2}(\alpha^2-1)\frac{\partial}{\partial X} \\ &= \frac{1}{\alpha^2}(\alpha^2-1)\frac{\partial}{\partial x}(F_\alpha) + \frac{1}{\alpha^2}\frac{\partial}{\partial X}. \end{aligned}$$

Thus, by putting

$$\xi_{1,\alpha} = \frac{1}{\alpha^2}(\alpha^2-1)\frac{\partial}{\partial x},\ \xi_{2,\alpha} = 0,\ \xi_{3,\alpha} = 0$$

and

$$\eta_{1,\alpha} = -\frac{1}{\alpha^2}\frac{\partial}{\partial X},\ \eta_{2,\alpha} = -\frac{\partial}{\partial Y},\ \eta_{3,\alpha} = -\frac{1}{4}(\alpha^2-1)\frac{\partial}{\partial Y} - \frac{\partial}{\partial Z},$$

we again obtain the equalities

$$-\frac{\partial F_\alpha}{\partial \lambda_i} = \xi_{i,\alpha}(F_\alpha) - \eta_{i,\alpha} \circ (F_\alpha, \pi_\lambda) \qquad (i = 1,2,3) \tag{3.2.3}$$

and

$$\eta_{1,\alpha}(0,0) = -\frac{1}{\alpha^2}\frac{\partial}{\partial X},\ \eta_{2,\alpha}(0,0) = -\frac{\partial}{\partial Y},\ \eta_{3,\alpha}(0,0) = -\frac{1}{4}(\alpha^2-1)\frac{\partial}{\partial Y} - \frac{\partial}{\partial Z}. \tag{3.2.4}$$

Thus, again by integrating

$$\xi_{i,\alpha} + \frac{\partial}{\partial \lambda_i},\ \eta_{i,\alpha} + \frac{\partial}{\partial \lambda_i} \qquad (i = 1,2,3),$$

we see that F_α is C^∞ trivial.

By (3.2.4), F_α is transversely C^∞ trivial if $\alpha \neq 0$.

Therefore, by theorem 1.1, $f_{(\alpha,-\frac{4}{\alpha^2})}$ is C^∞ right-left equivalent to $f_{(1,-4)}$ if $\alpha \neq 0$. □

Remark 3.2. 3

(1) Since vector fields

$$\xi_{i,\alpha},\ \eta_{i,\alpha} \qquad (i = 1,2,3)$$

are concrete and simple, we can obtain concrete forms of germs of C^∞ diffeomorphisms h_α^{-1} and H_α^{-1} $(\alpha \neq 0)$ by solving differential equations directly. Since our method is constructive (see §2), we can obtain concrete forms of germs of C^∞ diffeomorphisms s_α and t_α which give C^∞ right-left equivalence of $f_{(1,-4)}(x,y,z)$ and $f_{(\alpha,-\frac{4}{\alpha^2})}(x,y,z-\frac{1}{2}(\alpha-1)y)$ for $\alpha \neq 0$ as follows.

Let $\Xi_{i,\alpha} : (\mathbf{R} \times \mathbf{R}^3, (0,0)) \to (\mathbf{R}^3, 0)$ be the germ of local flow for $\xi_{i,\alpha}$ $(i = 1,2,3)$. Then, we have

$$\begin{aligned} &\Xi_{1,\alpha}(\lambda_1; \Xi_{2,\alpha}(\lambda_2; \Xi_{3,\alpha)}(\lambda_3; (x,y,z)))) \\ = \ &(x + \frac{1}{\alpha^2}(\alpha^2-1)\lambda_1, y, z). \end{aligned}$$

Thus, we have

$$\begin{aligned} & h_\alpha((x,y,z),(\lambda_1,\lambda_2,\lambda_3)) \\ = \ & ((x-\frac{1}{\alpha^2}(\alpha^2-1)\lambda_1,y,z),(\lambda_1,\lambda_2,\lambda_3)) \end{aligned}$$

and therefore

$$\begin{aligned} & h_\alpha((x,y,z),f_{(\alpha,-\frac{4}{\alpha^2})}(x,y,z-\frac{1}{2}(\alpha-1)y)) \\ = \ & ((\frac{1}{\alpha^2}x,y,z),f_{(\alpha,-\frac{4}{\alpha^2})}(x,y,z-\frac{1}{2}(\alpha-1)y). \end{aligned}$$

Put

$$s_\alpha(x,y,z)=(\frac{1}{\alpha^2}x,y,z).$$

Next, let $\Theta_{i,\alpha} : (\mathbf{R}\times\mathbf{R}^3,(0,0)) \to (\mathbf{R}^3,0)$ be the germs of local flow for $\eta_{i,\alpha}$ $(i=1,2,3)$. Then, we have

$$\begin{aligned} & \Theta_{1,\alpha}(\lambda_1;\Theta_{2,\alpha}(\lambda_2;\Theta_{3,\alpha}(\lambda_3;(X,Y,Z)))) \\ = \ & \Theta_{1,\alpha}(\lambda_1;\Theta_{2,\alpha}(\lambda_2;(X,Y-\frac{1}{4}(\alpha^2-1)\lambda_3,Z-\lambda_3))) \\ = \ & \Theta_{1,\alpha}(\lambda_1;(X,Y-\frac{1}{4}(\alpha^2-1)\lambda_3-\lambda_2,Z-\lambda_3)) \\ = \ & (X-\frac{1}{\alpha^2}\lambda_1,Y-\frac{1}{4}(\alpha^2-1)\lambda_3-\lambda_2,Z-\lambda_3). \end{aligned}$$

Thus, we have

$$\begin{aligned} & H_\alpha((X,Y,Z),(\lambda_1,\lambda_2,\lambda_3)) \\ = \ & ((X+\frac{1}{\alpha^2}\lambda_1,Y+\frac{1}{4}(\alpha^2-1)\lambda_3+\lambda_2,Z+\lambda_3),(\lambda_1,\lambda_2,\lambda_3)) \end{aligned}$$

and therefore

$$\begin{aligned} & H_\alpha((0,0,0,0),(\lambda_1,\lambda_2,\lambda_3)) \\ = \ & ((\frac{1}{\alpha^2}\lambda_1,\lambda_2+\frac{1}{4}(\alpha^2-1)\lambda_3,\lambda_3),(\lambda_1,\lambda_2,\lambda_3)) \end{aligned}$$

Put

$$t_{(\alpha,\beta)}(X,Y,Z)=(\frac{1}{\alpha^2}X,Y+\frac{1}{4}(\alpha^2-1)Z,Z).$$

Then, we see

$$f_{(1,-4)}\circ s_\alpha(x,y,z)=t_\alpha\circ f_{(\alpha,-\frac{4}{\alpha^2})}(x,y,z-\frac{1}{2}(\alpha-1)y)$$

as desired (for details see §2).

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