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THE TANGENT SEMICONE AND LIMITS OF TANGENT SPACES TO REAL SURFACES

LESLIE WILSON
University of Hawaii

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It has been 40 years since Whitney's seminal paper [W1] on the structure of real algebraic varieties. This paper supplied proofs of a number of results in the folklore, but stopped short of any treatment of local properties. Some years later, Whitney [W2], [W3] returned to study local properties, distinguishing carefully between a number of different tangent cones, and relating them to stratification properties of varieties. In so doing, however, he confined himself largely to the complex analytic category. In the early 1980's, Lê, Teissier, and others [HL], [L], [LT] worked out the structure of spaces of limiting tangent spaces at a singularity, again in the complex analytic setting.

However, there has, to our knowledge, been no attempt to extend these results to real varieties. This is striking because the tangent cone and the space of limiting tangent spaces encode important geometric features of a variety near a singular point. In this paper, we undertake this study for real surfaces.

We begin with surfaces because this is the case of interest for computer graphics. In fact, if we are given a surface $V \subset \mathbf{R}^3$ implicitly by an equation $f=0$ where $f \in \mathbf{R}[x, y, z]$ is a polynomial which vanishes at the origin and which has a singular point there, there are no reasonable methods known for sketching V in a neighborhood of the origin. Even when V can be parameterized, it can still be difficult to determine the fine geometric structure at a singular point. Since the best methods for sketching implicitly defined singular plane curves use a mix of numerical and algebraic techniques, it is reasonbable to seek a finite set of computable, algebraic invariants which would allow one to reliably sketch the main features of a real surface near a singular point. The first such invariants to investigate are clearly the tangent cone and the space of limits of tangent spaces.

Most of the results extend to general semi-algebraic surfaces and higher dimensional semialgebraic sets in arbitrary codimension, as well as to subanalytic sets. There is a limit to these generalizations. Kwiecinski and Trotman have recently shown [KT] that any continuum can be realized as the Nash fiber to a C^∞ Whitney stratified surface with isolated singularity.

It turns out that the extension of the complex analytic results to the reals is not as straightforward as one might hope. In [OW1] and [OW2], we show that the underlying

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definitions require modification. The proofs of the main results, which we supply in [OW3], require rather different techniques than the analogous complex results.

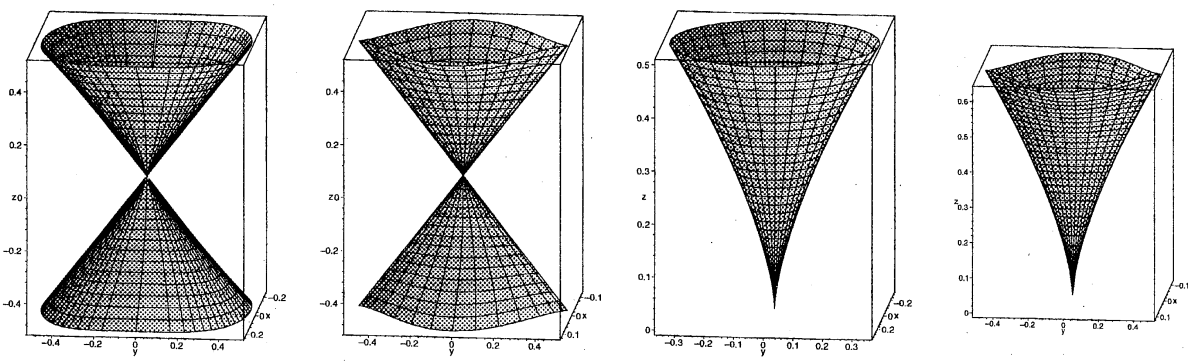
A *semicone* is a subset of $\mathbf{R}^n$ invariant under multiplication by $\mathbf{R}^+$. Given $0 \in V \subset \mathbf{R}^n$, the *tangent semicone* of V is

$$C(V) \equiv \{x : \exists x_i \in V,\, t_i \in \mathbf{R}^+, \text{ such that } x_i \to 0 \text{ and } t_i x_i \to x\}.$$

Suppose that $V \subset \mathbf{C}^n$ is the variety $f = 0$ defined by a polynomial (or holomorphic function) f; let f_i denote the i-th degree component of the Taylor expansion of f at 0, and let r be the smallest number so that f_r is not identically 0. Then $C(V) = \{f_r = 0\}$, and hence is a variety as well as a complex cone (invariant under multiplication by $\mathbf{C}$).

If V is semialgebraic in $\mathbf{R}^n$, then $C(V)$ is also semialgebraic. However, if V is a variety in $\mathbf{R}^n$, $C(V)$ is not necessarily a variety, as the following examples show.

Let V_i denote the real surface $f_i = 0$, where $f_1 = x^2 + y^4 - z^4$, $f_2 = x^2 + (y^2 - z^2)^3$, $f_3 = x^2 + y^2 - z^3$ and $f_4 = x^2 + (y^2 - z^3)^3$. Then $C(V_1) = C(V_2) = \{x = 0, |z| \geq |y|\}$ and $C(V_3) = C(V_4) = \{x = y = 0, z \geq 0\}$. These four varieties are shown in order below.



So which semialgebraic semicones are tangent semicones of algebraic varieties? This was completely answered in the recent paper by M. Ferrarotti (Torino), E. Fortuna (Pisa) and the author([FFW]; a partial result appeared in [FF]):

Theorem. *If C is any semialgebraic semicone of dimension less than n in $\mathbf{R}^n$, then there exists a real algebraic variety $V \subset \mathbf{R}^n$ such that $C = C(V)$.*

The proof is partially constructive; e.g. if C in $\mathbf{R}^3$ is defined by $\{z = 0,\ x \geq 0,\ y \geq 0\}$, then the proof produces V given by $(z^2 - x^3)^2 - y^7 = 0$.

The rest of this paper concerns another important invariant of a variety V: the Nash fiber. If V has dimension r, the *Nash fiber* of V (at 0) is $\mathcal{N}(V)$, the set of r-planes T with the property that there exists some sequence $\mathbf{x}_n$ of smooth points of V converging to 0 such that T is the limit of tangent spaces to V at the points $\mathbf{x}_n$ (this is called the Nash fiber because it is a fiber of the "Nash blow-up"). One easily shows that if $T \in \mathcal{N}(V)$ is a limit of tangent spaces at a sequence of smooth points $\mathbf{x}_n$ then, by passing to a subsequence if necessary, we can assume that the sequence $\{\mathbf{x}_n\}$ approaches the origin tangent to some ray ℓ. Necessarily, $\ell \subset C(V)$. We let $\mathcal{N}_\ell(V) \subset \mathcal{N}(V)$ denote the set of planes that can be obtained as limits of tangent spaces along sequences tending to the origin tangent to ℓ. Whitney shows that if V is algebraic, then $T \in \mathcal{N}_\ell(V)$ implies $\ell \subset T$.

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If $V \subset \mathbf{C}^n$ is a complex analytic variety, $0 \in V$, Whitney showed that any limit of tangent spaces to the tangent cone of a variety is, in fact, a limiting tangent space to the variety (that is, $\mathcal{N}(C(V)) \subset \mathcal{N}(V)$), but not conversely. The limits of tangent spaces to V which are not limits of tangent spaces to the tangent cone reveal features of the local geometry of a variety which are not captured by the tangent cone and, hence, are the parts of the Nash fiber of particular interest. There is a very satisfactory theory describing the structure of the Nash fiber due to Lê, Teissier, and others. The nicest formulation is in terms of the conormal fiber, which coincides with the Nash fiber for hypersurfaces.

In the case of surfaces, the Nash fiber is determined by the tangent cone together with a finite set of distinguished lines in the tangent cone. The following result, due to Lê and Henry [HL] in the case of an isolated singularity and to Lê [L,LT] in general, characterizes the "extra" limiting tangent spaces to a surface in $\mathbf{C}^3$.

Theorem (Lê, Henry). *Suppose that V is an algebraic surface, $0 \in V \subset \mathbf{C}^3$. There exists a finite (possibly empty) set of lines $l_1, \ldots, l_r \subset C(V)$, $0 \in l_i$ for all $1 \leq i \leq r$, (called* **exceptional lines***) such that $\mathcal{N}(V) = \mathcal{N}(C(V)) \cup (\bigcup_{i=1}^{r} G(l_i))$ where $G(l_i)$ denotes the set of all planes in $\mathbf{C}^3$ containing l_i.*

The data {tangent cone, exceptional lines} are called the *aureole* of V at 0 and are explicitly computable because there are a number of useful equivalent characterizations of exceptional lines.

It is natural to ask what parts of the complex theory go over to real surfaces. However, as we point out in [OW1] and [OW2], tangent cones and Nash fibers have been comparatively little studied over the reals, and important modifications of the central notions are necessary to even state, much less prove, analogues of the complex results. The rest of this paper is a description of joint work with D. O'Shea (Mt. Holyoke), to appear in [OW3].

¿From now on we assume that V is a 2-dimensional semialgebraic subset of $\mathbf{R}^3$, containing 0. For each ray $\ell \in C(V)$, $\mathcal{N}_\ell(V)$ is a closed, semialgebraic subset of G_ℓ, the set of all planes containing ℓ. By an *interval* in G_ℓ we mean a non-empty connected subset of G_ℓ which is neither a point nor all of G_ℓ. Every nontrivial semialgebraic subset of G_ℓ is a disjoint, finite union of points and/or intervals.

Given any ℓ and any closed semialgebraic $A \subset G_\ell$, we can construct a V with $\ell \subset C(V)$ and $\mathcal{N}_\ell(V) = A$. However, if we restrict to ℓ's which are not tangent to the singular set of V, then $\mathcal{N}_\ell(V)$ is not nearly so arbitrary, and there is again a strong relationship between $\mathcal{N}_\ell(V)$ and $C(V)$.

Structure Theorem. *Let $V \subset \mathbf{R}^3$, $0 \in V$, be a semialgebraic surface which contains smooth points arbitrarily close to 0. Let $C(V)$ and $\mathcal{N}_\ell(V)$ be as above, and let $S(V)$ denote the singular locus of V. There exist finitely many rays $\ell_1, \ldots, \ell_r$, $\ell_{r+1}, \ldots, \ell_s$ such that*
1) $\mathcal{N}(V) = \mathcal{N}(C(V)) \cup (\cup_{i=1}^{s} \mathcal{N}_{\ell_i}(V))$;
2) $\ell_{r+1}, \ldots, \ell_s$ are exactly the rays tangent to $S(V)$ at 0;
3) for $i = 1, \ldots, r$, each $\mathcal{N}_{\ell_i}(V)$ is connected, closed and one-dimensional ($\mathcal{N}_{\ell_i}(V)$ is either all of G_{ℓ_i} or is a closed interval);
4) for any ray $\ell \in C(V) - C(S(V))$ other than $\ell_1, \ldots, \ell_r$, $\mathcal{N}_\ell(V) = \mathcal{N}_\ell(C(V))$ consists of a single point.

Furthermore, we can closely relate $\mathcal{N}_{\ell_i}(V)$, $i = 1, \ldots, r$, to $C(V)$; for example (and assuming in each case that ℓ is not tangent to $S(V)$):

• If ℓ is isolated in $C(V)$ in some deleted neighborhood of 0, then $\mathcal{N}_\ell(V) = G_\ell$ (see for example V_3; compare to V_4, in which $\ell =$ positive z-axis is isolated, but $\mathcal{N}_\ell$ is a point—of course, ℓ is tangent to $S(V_4)$).

• If ℓ is contained in the boundary of $C(V)$, then $\mathcal{N}_\ell(V) = G_\ell$ (see for example V_1, where the boundary is $x = 0$, $z = \pm y$; compare to V_2).

• Suppose that $\ell \subset C(V)$, $x \in \ell$, $x \neq 0$, and T is a plane through x transverse to ℓ. Suppose that in a neighborhood of x, $T \cap C(V)$ consists of two analytic arcs emanating from x (in the above case that ℓ is contained in the boundary of $C(V)$, $T \cap C(V)$ consists of exactly one arc). Let ρ_1 and ρ_2 denote the tangent rays at x to these arcs. Let G_{ρ_1,ρ_2} denote the set of all planes in G_ℓ spanned by ℓ and rays lying in the angle ($\leq \pi$) between ρ_1 and $-\rho_2$, the opposite ray to ρ_2 (so G_{ρ_1,ρ_2} consists of one plane if $\rho_1 = -\rho_2$, is all of G_ℓ if $\rho_1 = \rho_2$, and otherwise is a closed interval in G_ℓ). Then $\mathcal{N}_\ell(V) \supset G_{\rho_1,\rho_2}$.

The following example shows that this can be a proper subset. Let V be the variety $z^3 = yx^6$. Then $C(V)$ is the xy-plane. Let ℓ be the positive x-axis. Then ρ_1 and ρ_2 are the positive and negative y-axes, hence G_{ρ_1,ρ_2} consists of the xy-plane. However, $\mathcal{N}_\ell(V)$ is the "half-pencil" consisting of all planes spanned by ℓ and rays in the first quadrent of the yz-plane.

Idea of the proof of (3) of the Structure Theorem.

Suppose that ℓ is not tangent to $S(V)$, and that $\mathcal{N}_\ell(V) \neq G_l$. For $x \in \ell$, $x \neq 0$, let $l_x^\perp$ denote the plane through x perpendicular to ℓ, let L be a line in $l_x^\perp$ through x, and let π be the orthogonal projection of $\mathbf{R}^3$ such that $\pi(L) = x$. Let R be a rectangle in $l_x^\perp$ centered at x with two sides parallel to L. Let R^* denote the interior of the convex closure of $R \cup \{0\}$. Let $U = L^\perp \cap R^*$.

We can pick x, L and R so that: $V^0 = V \cap R^*$ is nonsingular, $\pi|V^0$ is a submersion, $V^0 = V_1 \cup \cdots \cup V_s$, where $V_1, \ldots, V_s$ are the connected components of V^0, and each $\pi|V_i : V_i \to U$ is a diffeomorphism, hence V_i is a graph of a real smooth function f_i defined on U. We can assume that $f_1 \leq f_2 \leq \cdots \leq f_s$ on all of U.

Consider $C(V) \cap R$. It's arcs emanating from x have tangent rays at x; none of these rays are vertical (parallel to L) so they all lie on one side or the other of L. We'll call the one group of rays $\lambda_1, \ldots, \lambda_a$ and the other group $\rho_1, \ldots, \rho_b$ (λ for "left" and ρ "right"). To each V_i is associated one left ray $\lambda(V_i)$ and one right ray $\rho(V_i)$, and the angle $\lambda(V_i)$ makes with the horizontal is nondecreasing in i, similarly for $\rho(V_i)$.

Each V_i contributes a subset $\mathcal{N}(V_i) \subset \mathcal{N}_{\ell_i}(V)$. Using the Intermediate Value Theorem, and the genericity of Thom's condition a_f, we show that $\mathcal{N}(V_i)$ is connected and contains $\mathcal{N}_{\lambda(V_i)\rho(V_i)}$. Of course $\mathcal{N}_\ell(V) = \cup \mathcal{N}(V_i)$. It only remains to observe that the intervals $\mathcal{N}_{\lambda(V_i)\rho(V_i)}$ and $\mathcal{N}_{\lambda(V_j)\rho(V_j)}$ overlap for all i and j.

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University of Hawaii
Honolulu, Hawaii, USA
E-mail address: les@math.hawaii.edu