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HYPERSURFACES HAVING HIGHER ORDER CONTACT WITH SINGULAR SPACES

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ABSTRACT. This note is to study collections of functions whose zero level hypersurfaces are tangent to the regular part of a singular space X along a subspace Y of X. These collections are called the higher primitive ideal of the defining ideal $\mathfrak{g}$ of Y relative to X. We establish the connection between higher iterated primitive ideals and the (relative) symbolic powers of the ideal $\mathfrak{g}/\mathfrak{h}$ and give an effective algorithm to compute both, where $\mathfrak{h}$ is the defining ideal of X.

Let $X \supset Y$ be germs of analytic subspaces of $(\mathbb{C}^n, 0)$, defined by ideals $\mathfrak{h} \subset \mathfrak{g} \subset \mathcal{O} = \mathcal{O}_{\mathbb{C}^n,0}$, respectively. Assume that X and Y are reduced and pure dimensional. The main object of this note is to study a higher order analog of the so-called primitive ideal of $\mathfrak{g}$ relative to X introduced by Jiang–Pellikaan–Siersma (cf. [5], [6], [2]).

In terms of singularity theory, the primitive ideal of $\mathfrak{g}$ relative to X consists of all functions with critical loci (restricted to X and relative to the logarithmic stratification of X) containing Y. Geometrically, this ideal collects all functions whose zero level hypersurfaces pass through Y and are tangent to $X_{\text{reg}} = X \setminus X_{\text{sing}}$ along $Y \cap X_{\text{reg}}$. From geometric point of view, it is reassonable to study the collections of functions whose zero level hypersurfaces pass through Y and have higher order of contact with X along $Y \cap X_{\text{reg}}$. These collections are called higher relative primitive ideals. There are actually two notions of higher primitive ideals in the relative case due to the fact that higher (algebraic) differential operators on a singular space are not just the iterated composites of differential operators of order one. Since this intrinsic difficulty leads to a few delicate points in the theory, we made a choice to work with the iterated version.

In case of $X = \mathbb{C}^n$, the higher primitive ideals coincide with the higher symbolic powers (phased out by one) of the ideal $\mathfrak{g}$ - this is a consequence of the well-known Nagata–Zariski theorem on holomorphic functions. It therefore seems quite natural to ask if the present notion of higher primitive ideals in the relative case $\mathfrak{h} \subset \mathfrak{g}$ bears likewise a close relation to the higher symbolic powers of the ideal $\mathfrak{g}/\mathfrak{h}$ in the residue class ring $\mathcal{O}/\mathfrak{h}$. This paper deals systematically with this question for higher order. We answer this question under a reasonable condition: the critical locus of X does not contain any irreducible components of Y. We also show that the higher (iterated) primitve ideals are effectively computable in terms of syzygies of certain matrices. As a bonus, we answer under certain conditions a question posed by Vasconcelos (cf. [9]) as to how actually compute (relative) symbolic powers in an effective way.

In studying functions with non-isolated singularities on singular spaces, we found that the topoplogy of the Milnor fibre of a function depends also on how its singular locus Y is

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embedded in X. There arises a problem: how to distinguish the different embeddings of Y in X? We include some examples to show that the higher (iterated) primitive ideals can be used to tell how Y sits in X.

Vector fields of higher order contact

Let $\mathcal{O}$ be a commutative ring and let $k \subset \mathcal{O}$ be a subring. The set $\mathrm{End}_k(\mathcal{O})$ of k-module endomorphisms of $\mathcal{O}$ is an $\mathcal{O}$-module in the usual way whereby scalar multiplication $a\rho$ ($a \in \mathcal{O}, \rho \in \mathrm{End}_k(\mathcal{O})$) is to be taken as the composite $\mu_a \circ \rho$, with μ_a denoting the endomorphism multiplication by a.

Given an integer $r \geq 0$, let $\mathrm{Diff}^{(r)}(\mathcal{O}) = \mathrm{Diff}_k^{(r)}(\mathcal{O}) \subset \mathrm{End}_k(\mathcal{O})$ denote the $\mathcal{O}$-submodule consisting of the differential operators of order $\leq r$ of the k-algebra $\mathcal{O}$. The traditional reference for differential operators of algebras is [1] (cf. also [4]). Our main interest will stem from the following notion.

Definition 1. Let $\mathfrak{h} \subset \mathcal{O}$ be an ideal. The *rth differential idealizer of* $\mathfrak{h}$ is

$$\mathbb{I}_{\mathfrak{h}}^{(r)} := \left\{ \delta \in \mathrm{Diff}^{(r)}(\mathcal{O}) \mid \delta(\mathfrak{h}) \subset \mathfrak{h} \right\}.$$

If no confusion arises, we will refer to the rth differential idealizer simply as the idealizer (of order r). ¿From the structure of $\mathcal{O}$-module of $\mathrm{End}_k(\mathcal{O})$ as above easily follows that the idealizer of order r of $\mathfrak{h}$ is an $\mathcal{O}$-submodule of $\mathrm{Diff}^{(r)}(\mathcal{O})$. We have $\mathrm{Diff}^{(r)}(\mathcal{O}) = \mathbb{I}_{\mathfrak{h}}^{(0)} = \mathcal{O}$. Since $\mathrm{Diff}^{(r)}(\mathcal{O}) = \mathcal{O} \oplus \mathrm{Der}^{(r)}(\mathcal{O})$, where $\mathrm{Der}^{(r)}(\mathcal{O})$ is the $\mathcal{O}$-module of k-derivations of order r of $\mathcal{O}$, then $\mathbb{I}_{\mathfrak{h}}^{(r)} = \mathcal{O} \oplus \widehat{\mathbb{I}_{\mathfrak{h}}^{(r)}}$, where $\widehat{\mathbb{I}_{\mathfrak{h}}^{(r)}} = \{\delta \in \mathrm{Der}^{(r)}(\mathcal{O}) \mid \delta(\mathfrak{h}) \subset \mathfrak{h}\}$. The latter module is the *essential part* of the differential idealizer. Throughout we set $\mathbb{I}_{\mathfrak{h}} = \mathbb{I}_{\mathfrak{h}}^{(1)}$.

For a given $r \geq 0$, we consider yet another $\mathcal{O}$-module, namely, the $\mathcal{O}$-submodule $(\mathbb{I}_{\mathfrak{h}})^r \subset \mathbb{I}_{\mathfrak{h}}^{(r)}$ generated by the elements of $\mathbb{I}_{\mathfrak{h}}$. In other words, the elements of this submodule are the $\mathcal{O}$-linear combinations of composites of t elements of $\mathbb{I}_{\mathfrak{h}}$ for values $t \leq r$.

Remark 2. Most of the results in this paper are valid for any field k of characteristic zero. To keep it more geometric, we work in the case where $k = \mathbb{C}$ and $\mathcal{O}$ is a polynomial ring $\mathbb{C}[z_1, \ldots, z_n]$ (resp. a power series ring $\mathbb{C}\{z_1, \ldots, z_n\}$). Then one can interpret $Der(\mathcal{O}) = \mathrm{Der}^{(1)}(\mathcal{O})$ as (algebraic) vector fields on the affine space $\mathbb{C}^n$ (resp. germs of analytic vector fields at $0 \in \mathbb{C}^n$), then the essential part of the differential idealizer of order 1 of $\mathfrak{h}$ is to be interpreted as the module of vector fields tangent to the subvariety $V \subset \mathbb{C}^n$ (resp. to the germ $(V, 0)$ of the analytic subspace at 0) defined by the ideal $\mathfrak{h}$. In case V is a hypersurface (a divisor), these vector fields were dubbed logarithmic vector fields along V (cf. [7]). For higher values of r, (the essential part of) the differential idealizer yields a notion of vector fields of higher order of contact along V.

Higher primitive ideals

Definition 3. (i) Let $\mathfrak{h} \subset \mathfrak{g} \subset \mathcal{O}$ be ideals, with $\mathfrak{h}$ radical and let $r \geq 0$ be an integer. The *rth primitive ideal* of $\mathfrak{g}$ (relative to X) is

$$\oint_{\mathfrak{h}}^{(r)} \mathfrak{g} := \{f \in \mathfrak{g} \mid \mathbb{I}_{\mathfrak{h}}^{(r)}(f) \subset \mathfrak{g}\}.$$

(ii) The *rth iterated primitive ideal* of $\mathfrak{g}$ relative to X is

$$\int_{\mathfrak{h}}^{(r)} \mathfrak{g} := \{f \in \mathfrak{g} \mid (\mathbb{I}_{\mathfrak{h}})^r(f) \subset \mathfrak{g}\}.$$

Clearly, $\oint_{\mathfrak{h}}^{(r)} \mathfrak{g} \subset \int_{\mathfrak{h}}^{(r)} \mathfrak{g}$ and, trivially, $\oint_{\mathfrak{h}}^{(0)} \mathfrak{g} = \int_{\mathfrak{h}}^{(0)} \mathfrak{g} = \mathfrak{g}$. If $r = 1$, we omit the superscript, setting $\oint_{\mathfrak{h}}^{(1)} \mathfrak{g} = \int_{\mathfrak{h}}^{(1)} \mathfrak{g} = \int_{\mathfrak{h}} g$. Note that for this value of r the present definition coincides with that introduced by Pellikaan (cf. [5], [6], also [2]). Fixing the ideal $\mathfrak{h}$, the question as to when one has equalities $\oint_{\mathfrak{h}}^{(r)} \mathfrak{g} = \int_{\mathfrak{h}}^{(r)} \mathfrak{g}$ throughout for all ideals $\mathfrak{g} \supset \mathfrak{h}$ and for all values of r is rather subtle. We suspect it may be very close to implying thet $\mathcal{O}/\mathfrak{h}$ be a regular ring. The question is also related to the well known conjecture of Nakai on differential operators.

As it turns, the first notion is rather unwielding in several aspects, theoretical as well as computational. The clear advantage of the second notion is that it is recursively defined (see Proposition 7).

We now establish the connections between the iterated primitive ideal and the symbolic powers of a given ideal. Recall that for any commutative noetherian ring A and any ideal $I \subset A$, we have the notion of the sth symbolic power $I^{(s)}$ of I, where s is a nonegative integer. Namely, $I^{(s)}$ is the intersection of the primary components of I^s relative to the minimal primes of I. Equivalently, $I^{(s)} = \{a \in A \mid \exists x \in A \setminus \cup_{P \in \mathrm{Min}(A/I)} P,$ with $xa \in I^s\}$.

Proposition 4. *Let $Y \subset X \subset \mathbb{C}^n$ defined by ideals $\mathfrak{g} \supset \mathfrak{h}$, with X reduced. Then:*

(i) *For every $r \geq 0$, the support of the $\mathcal{O}$-module $\int_{\mathfrak{h}}^{(r)} \mathfrak{g} / \int_{\mathfrak{h}}^{(r+1)} \mathfrak{g}$ is contained in Y.*

(ii) *If $\mathfrak{g}$ has no embedded associated primes then $Ass\,(\mathcal{O}/\mathfrak{g}) = Ass\,(\mathcal{O}/\int_{\mathfrak{h}}^{(r)} \mathfrak{g})$ for every $r \geq 0$.*

(iii) *If $\mathfrak{g}$ has no embedded associated primes then, for every $r \geq 0$, $\mathfrak{g}^{(r+1)} \subset \widetilde{\mathfrak{g}^{(r+1)}} \subset \int_{\mathfrak{h}}^{(r)} \mathfrak{g}$, where $\widetilde{\mathfrak{g}^{(r+1)}}$ denotes the inverse image in $\mathcal{O}$ of $(\mathfrak{g}/\mathfrak{h})^{(r+1)} \subset \mathcal{O}/\mathfrak{h}$.*

(iv) *$(\int_{\mathfrak{h}}^{(r)} \mathfrak{g})/\mathfrak{h} = \int_{0}^{(r)} (\mathfrak{g}/\mathfrak{h})$, where $0 = \mathfrak{h}/\mathfrak{h}$.*

(v) *The sequence $\{\int_{\mathfrak{h}}^{(r)} \mathfrak{g}\}_{r \geq 0}$ is a descending multiplicative filtration.*

Proof. (i) For every $r \geq 0$ the stalks of the module at the points outside Y are trivial.

(ii) It follows from the definition that, for any two ideals $\mathfrak{g}_1, \mathfrak{g}_2$ containing $\mathfrak{h}$, $\int_{\mathfrak{h}}^{(r)} (\mathfrak{g}_1 \cap \mathfrak{g}_2) = \int_{\mathfrak{h}}^{(r)} \mathfrak{g}_1 \cap \int_{\mathfrak{h}}^{(r)} \mathfrak{g}_2$. One may stick to the case where $\mathfrak{g}$ is a primary ideal. Assuming $Ass\,(\mathcal{O}/\mathfrak{g}) = \{P\}$, one can easily prove that $\mathfrak{g}/\int_{\mathfrak{h}}^{(r)} \mathfrak{g}$ is P-primary.

(iii) The first inclusion follows trivially from the definition of the symbolic powers. One can use (i) and (ii) to prove the second inclusion by induction on r.

(iv)This follows immediately from the definitions and from the recursive character of both sides of the required equality.

(v) This follows from the definition. □

We will next state a result in terms of geometric conditions. A similar result holds for $\mathcal{O}$ a polynomial ring over a feild k of characteristic zero.

Recall that for a reduced analytic subspace X of $\mathbb{C}^n$ defined by a radical ideal (sheaf) $\mathfrak{h}$, the critical locus X_{sing} is well defined by the Jacobian ideal $\mathfrak{J}(\mathfrak{h})$ of $\mathfrak{h}$. In fact, $\mathfrak{J}(\mathfrak{h})$ is the ideal $(\mathfrak{J}(\mathfrak{h}) + \mathfrak{h})/\mathfrak{h}$, where $\mathfrak{J}(\mathfrak{h}) \subset \mathcal{O}$ is the ideal generated by the $c \times c$ minors of the jacobian matrix of $\{f_1, \dots, f_m\}$, a generator set of $\mathfrak{h}$, where c stands for the codimension of X. Note that this is a Fitting ideal, hence it is independent of the choice of generators of $\mathfrak{h}$.

Proposition 5. *Let $Y \subset X$ be reduced subspaces of $\mathbb{C}^n$ defined by radical ideals $\mathfrak{h} \subset \mathfrak{g} \subset \mathcal{O}$, respectively. If X_{sing} does not contain any irreducible components of Y then $\int_{\mathfrak{h}}^{(r)} \mathfrak{g} = \widetilde{\mathfrak{g}^{(r+1)}}$ for every $r \geq 0$.*

Proof. The inclusion $\widetilde{\mathfrak{g}^{(r+1)}} \subset \int_{\mathfrak{h}}^{(r)} \mathfrak{g}$ has been noted in Proposition 4, (iii). To prove the reverse inclusion it is sufficient to prove $(\int_{\mathfrak{h}}^{(r)} \mathfrak{g})/\mathfrak{h} = \widetilde{\mathfrak{g}^{(r+1)}}/\mathfrak{h}$, the latter is $(\mathfrak{g}/\mathfrak{h})^{(r+1)}$ by definition.

It is easy to see that taking primitive ideals commutes with localization (cf. [8]). Similarly, it is a known fact that taking symbolic powers commutes with localization. Therefore, at each point $P \in X_{reg}$, one has $\left((\int_{\mathfrak{h}}^{(r)} \mathfrak{g})/\mathfrak{h}\right)_P = (\mathfrak{g}/\mathfrak{h})_P^{(r+1)}$ for every $r \geq 0$. Hence outside X_{sing}, we have $(\int_{\mathfrak{h}}^{(r)} \mathfrak{g})/\mathfrak{h} = \widetilde{\mathfrak{g}^{(r+1)}}/\mathfrak{h}$ everywhere. By Rückert Nullstellensatz, a power of $\mathfrak{J} := \mathfrak{J}(\mathfrak{h})$ annihilates $\int_{\mathfrak{h}}^{(r)} \mathfrak{g}/\widetilde{\mathfrak{g}^{(r+1)}}$ on $\mathcal{O}$ as both ideals contain $\mathfrak{h}$ by definition. In other words, we get $\int_{\mathfrak{h}}^{(r)} \mathfrak{g} \subset \widetilde{\mathfrak{g}^{(r+1)}} : J^\infty$. However, under the present hypothesis on $\mathfrak{g}$, $\widetilde{\mathfrak{g}^{(r+1)}}$ and $\mathfrak{g}$ have the same associated primes. Thus, the condition that X_{sing} does not contain any irreducible components of Y implies that $\widetilde{\mathfrak{g}^{(r+1)}} : J^\infty = \widetilde{\mathfrak{g}^{(r+1)}}$. □

The computation of the primitive ideal

In this section we state two results without proofs to show how to effectively compute the rth iterated primitive ideal. The basic computational result refers to the first primitive ideal. We make the convention of writing the elements of the $\mathcal{O}$-submodule $\mathbb{I}_{\mathfrak{h}} \subset Der(\mathcal{O}) = \sum_{i=1}^n \mathcal{O} dz_i$ as column vectors with coordinates in $\mathcal{O}$.

Proposition 6. *Let $\mathfrak{h} \subset \mathfrak{g} \subset \mathcal{O}$ be ideals, with $\mathfrak{h}$ radical. Let $\mathfrak{g} = (g) = (g_1, \dots, g_m)$ be generators of $\mathfrak{g}$, let $\Theta(g)$ denote the jacobian matrix of g and let $\Delta(\mathfrak{h})$ denote the matrix whose columns are a set of generators of the $\mathcal{O}$-submodule $\mathbb{I}_{\mathfrak{h}} \subset Der(\mathcal{O})$. Then*

$$\int_{\mathfrak{h}} \mathfrak{g} = I_1(g \cdot \Psi),$$

where Ψ denotes the lifting to $\mathcal{O}$ of the syzygy matrix of the transpose of $\Theta(g) \cdot \Delta(\mathfrak{h})$ modulo the ideal $\mathfrak{g}$.

Proposition 7. *Let $\mathfrak{h} \subset \mathfrak{g} \subset \mathcal{O}$ be ideals, with $\mathfrak{h}$ radical. Then $\int_{\mathfrak{h}}^{(r)} \mathfrak{g} = \{f \in \mathfrak{g} \,|\, \mathbb{I}_{\mathfrak{h}}(f) \subset \int_{\mathfrak{h}}^{(r-1)} \mathfrak{g}\}$. And consequently $\int_{\mathfrak{h}}^{(r)} \mathfrak{g} = \int_{\mathfrak{h}} \left(\int_{\mathfrak{h}}^{(r-1)} \mathfrak{g}\right)$.*

Remark 8. By shifting emphasis, the result of Proposition 5 can be viewed as a means of computing the relative symbolic power $(\mathfrak{g}/\mathfrak{h})^{(r+1)} \subset \mathcal{O}/\mathfrak{h}$. It is quite effective by Proposition 7. This partially answers a question posed by Vasconcelos (cf. [9]) as to whether one can compute symbolic powers of an ideal in a residue ring of a polynomial ring in a relatively fast effective way (that avoids, say, taking the unmixed part of the ordinary powers by means of computing with Ext modules and resolutions).

Applications

A couple of examples may indicate the applications of higher relative primitive ideals. Let $X \supset Y$ be germs of reduced analytic subspaces in $(\mathbb{C}^n, 0)$, defined by radical ideals $\mathfrak{h} \subset \mathfrak{g} \subset \mathcal{O}$ as in the earlier sections. The higher iterated primitive ideals roughly tell us how Y is embedded in X. The case when X is smooth was considered by Pellikaan [5]. In case X is singular and Y is a smooth curve, the (first) primitive ideal was computed in [2]. It turns out that, for fixed Y, which is taken to be the first axis in the local coordinates of $\mathbb{C}^{n+1}$, i.e., $\mathfrak{g} = (y_1, \dots, y_n)$, the (first) primitive ideal of $\mathfrak{g}$ stays the same for each X with an isolated complete intersection singularity and containing Y (see also the examples below). Also when X is a two dimensional isolated singular hypersurface, the length λ of the torsion submodule T of the conormal module of $\mathfrak{g}/\mathfrak{h}$ was used as a relative invariant of Y with respect to X [3]. It was proved that $T = (\int_{\mathfrak{h}}^{(1)} \mathfrak{g})/\mathfrak{g}^2 + \mathfrak{h}$. It is quite conceivable that λ remains unchanged for different smooth curves on the same surface X with an isolated singularity. However, for higher iterated primitive ideals the story is quite different.

Example 1. Let $h_k = xy + (1/k+1)z^{k+1}$ be the well known A_k singularity and let $\mathfrak{g} = (y, z)$ be line contained therein. Then

$$\int_{h_k}^{(r)} \mathfrak{g} = \begin{cases} (y, z^{r+1}) & \text{if } r \leq k \\ y \int_{h_k}^{(r-k-1)} \mathfrak{g} + (h_k) & \text{otherwise} \end{cases}$$

Example 2. In the first example the line in consideration had $\lambda = 1$. If we take a line with $\lambda = l$ on an A_{2l-1} surface to be the x-axis, then X_l is defined by $h_l = x^l y + z^2 + yz$ (see [3]). In this case we have

$$\int_{h_l}^{(2k)} \mathfrak{g} = y^k(y, z) + (h_l), \qquad \int_{h_l}^{(2k+1)} \mathfrak{g} = y^k(y, z^2) + (h_l).$$

Example 3. Consider the D_5 surface $D_{5,2}$, with defining equation $h_1 = x^2y + y^4 + z^2$ with respect to the line defined by the ideal $\mathfrak{g} = (y, z)$. Then

$$\int_{h_1}^{(1)} \mathfrak{g} = (y, z^2), \qquad \int_{h_1}^{(2)} \mathfrak{g} = y\mathfrak{g} + (h_1), \qquad \int_{h_1}^{(r+1)} \mathfrak{g} = (y^r) + (h_1), \quad (r > 2).$$

Example 4. Consider a D_5 surface once more, only now with respect to a different line being the x-axis. The equation of the surface is $D_{5,2}^* : h_2 = x^2y + xz^2 + y^2$. Here we obtain:

$$\int_{h_2}^{(1)} \mathfrak{g} = (y, z^2), \qquad \int_{h_2}^{(2)} \mathfrak{g} = (yz, xy + z^2) + (h_2), \qquad \int_{h_2}^{(3)} \mathfrak{g} = (yz^2, xy + z^2) + (h_2),$$

and, for $k \geq 1$,

$$\int_{h_2}^{(4k)} \mathfrak{g} = (xy + z^2)^k \mathfrak{g} + (h_2), \qquad \int_{h_2}^{(4k+1)} \mathfrak{g} = (xy + z^2)^k \int_{h_2}^{(1)} \mathfrak{g} + (h_2),$$

$$\int_{h_2}^{(4k+2)} \mathfrak{g} = (xy + z^2)^k \int_{h_2}^{(2)} \mathfrak{g} + (h_2), \qquad \int_{h_2}^{(4k+3)} \mathfrak{g} = (xy + z^2)^k \int_{h_2}^{(3)} \mathfrak{g} + (h_2).$$

We note that the invariant $\lambda = 2$ for both $D_{5,2}$ and $D_{5,2}^*$.

For higher primitive ideals one can introduce the corresponding torsion number

$$\lambda_r := \dim_{\mathbb{C}} \frac{\int_{\mathfrak{h}}^{(r)} \mathfrak{g}}{\mathfrak{g}^{r+1} + \mathfrak{h}}, \qquad r = 0, 1, \dots .$$

Clearly, $\lambda_1 = \lambda$.

A calculation shows that $\lambda_2 = 4$ for $D_{5,2}$ and 5 for $D_{5,2}^*$. Therefore, these higher invariants distinguish between the two singularities.

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