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Multiple Existence of Entire Solutions for Semilinear Elliptic problems on R^N

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1. Introduction. Our purpose in this talk is to show the multiple existence of entire solutions of the problem

$$(\mathrm{P}) \qquad -\Delta u+u=g(x,u), \quad u\in H^{1}(R^{N})$$

where $N\geq 2$ and $g:R^{N}\times R\rightarrow R$ is a continuous function with superlinear growth and $g(x,0)=0$ on R^{N}.

We fix p such that $p>1$ when $N=2$ and $1<p<(N+2)/(N-2)$ when $N\geq 3$. It is well known that the problem

$$(\mathrm{P}_{0}) \qquad -\Delta u+u=|u|^{p-1}u, \quad u\in H^{1,2}(R^{N})$$

has **a unique positive solution** u up to translation. The positive solution u is characterized as **the ground state soluiton**. That is if we consider a functional I defined by

$$I(u)=\int_{R_{N}}\frac{1}{2}|\nabla u|^{2}dx-\frac{1}{p+1}\int_{R_{N}}|u|^{p+1}dx \qquad \text{for } u\in H^{1}(R^{N}),$$

then $c=I(u)$ is the minimal positive critical level of I. The existence of positive entire solution of problem

$$(\mathrm{P_Q}) \qquad \begin{cases} -\Delta u+u=Q(x)|u|^{p-1}u, & x\in R^{N} \\ u\in H^{1}(R^{N}) \end{cases}$$

has been studied by several authors. Here $Q(x)$ satisfies $Q(x) \to 1$ as $|x| \to \infty$. In case that $Q(x) \geq 1$ in R^N, the existence of a solution of P_Q was established by Lions using the concentrate compactness method. Lions's result was improved by Zhu and Cao. The case that $Q(x) \mid t \mid^{p-1} t$ is replaced by a more general function $g(x,t)$, the existence of positive solutions is proved by the author .

To attack this kind of probblem, one can take the advantage of variational structure of problem P_Q . That is the solutions of problem (P_Q) is characterized as critical points of functional I_Q defined by

$$I_Q(u) = \int_{R_N} \frac{1}{2} \mid \nabla u \mid^2 dx - \frac{1}{p+1} \int_{R_N} Q(x) \mid u \mid^{p+1} dx, \quad u \in H^1(R^N).$$

As in case that $Q(x) \equiv 1$, we can obtain a positive solution as a ground state solution. In this talk , we consider the case that $g \in C^2(R^N, R)$ satisfies the following conditions:

(g1) There exists $0 < \theta < 1/2$ such that

$$\theta g(x,t)t \geq G(x,t) = \int_0^t g(x,s)ds > 0 \qquad \text{for all } x \in R^N \text{ and } t > 0;$$

(g2)

$$\lim_{|x| \to \infty} g(x,t) / \mid t \mid^{p-1} t = 1$$

uniformly on closed bounded subsets of $(0, \infty)$

(g3) there exists $\rho > 0$ such that

$$\mid g(x,t) - \mid t \mid^{p-1} t \mid \leq \rho \mid t \mid^{p-1} t \qquad \text{for all } x \in R^N \text{ and } t \in R;$$

We can now state our main result.

Theorem 1. *Assume that (g1) and (g2) hold. Then there exists a positive number ρ_0 such that if (g3) hold with $0 < \rho < \rho_0$, then problem (P) possesses at least two nontrivial solutions.*

We next impose the following conditions on g.

(g4) $$g(x,t) = -g(x,-t) \qquad \text{for all } x \in R^N \text{ and } t \in R.$$

(g5) there exist positive numbers a, C such that $a < 1$ and

$$g(x,t)/ \mid t \mid^p t \geq 1 + Ce^{-a|x|} \qquad \text{for all } x \in R^N \text{ and } t \neq 0.$$

Theorem 2 . *Assume that $(g1)(g2), (g4) and (g5)$ hold. Then there exists a positive number ρ_0 such that if (g2) hold with $0 < \rho < \rho_0$, then problem (P) possesses at least two pairs of nontrivial solutions*

To get a sign changing solution of (P), we impose the following condition instead of (g5) .

(g5') there exist positive numbers a, C such that $a < 1$ and

$$g(x,t)/ \mid t \mid^p t \geq 1 + C \mid x \mid^N \qquad \text{for all } x \in R^N \text{ and } t \neq 0.$$

Theorem 3. *Assume that $(g1)(g2), (g4)$ and $(g5')$ hold. Then there exists a positive number ρ_0 such that if (g2) hold with $0 < \rho < \rho_0$, then problem (P) possesses at least two pairs of nontrivial solutions. Moreover (P) possesses at least one pair of sign changing solutions.*

2. Preliminaries .

We put $H = H^1(R^N)$ and
$\| z \|^2 = | \nabla z |_2^2 + | z |_2^2$ for $z \in H$.

For each $a \in R$ and each functional $F : H \to R$, we denote by F_a the set $F_a = \{v :\in X : F(v) \leq a\}$. We call a real number d a critical value of a functional F if there exists a sequence $\{v_n\} \subset H$ such that $\lim_{n\to\infty} F(v_n) = d$ and $\lim_{n\to\infty} \| F'(v_n) \| = 0$.

For $z \in H$, $D \subset H$ and $x \in R^N$, we denote by z_x and D_x,

$$z_x(y) = z(y - x) \qquad \text{for } y \in R^N \text{ and } \qquad D_x = \{z_x : z \in D\}.$$

For each $x \in R^N$, the function u_x is a solution of I with $I(u_x) = c$. It is also known that there exist no critical value of I in $(0, 2c)\backslash\{c\}$.

We define a functional J^∞ on $H^1(R^N)$ by

$$J^\infty(v) = \int_{R_N} \frac{1}{2}(|\nabla v|^2 + |v|^2)dx - \int_{R_N} \int_0^{v(x)} g(x,t)dtdxdx$$

for $v \in H^1(R^N)$. We put

$$M = \{v \in H\backslash\{0\} : \| v \|^2 = \int_{R^N} | v |^{p+1}\}$$

Noting that

$$c = I(u) = \min\{I(v) : \| v \|^2 = \int_{R^N} | v |^{p+1} dx\}, \tag{2.1}$$

we have that

$$I(v) \geq c \quad \text{on } M. \tag{2.2}$$

It is also easy to see that

$$M \cap \{\lambda v : v \in H\backslash\{0\}, \lambda \geq 0\} \text{ is a unique point,} \tag{2.3}$$
$$I(v) = \max\{I(\lambda v) : \lambda \geq 0\} \qquad \text{for each } v \in M \tag{2.4}$$

and each critical point of I is contained in M(cf. [12]).

Let $\epsilon_0 > 0$ with $2\epsilon_0 < c$.

The following results is well known.

Lemma 2.1. *For each $\epsilon > 0$ with $\epsilon < c$, there exists $V_\epsilon \subset M$ such that*

$$I_{c+\epsilon} \cap M = V_\epsilon \cup -V_\epsilon, \quad V_\epsilon \cap -V_\epsilon = \phi.$$

Here we put

$$X_{1/2} = \{\mu v \in M, \mu \geq \frac{1}{2}\}$$

Then $M \subset intX_{1/2}$. Let V_0, V_1 be bounded neighborhoods of $V_{\epsilon_0}(\subset M \cap I_{c+\epsilon_0})$ such that

$$V_0 \subset intV_1 \subset X_{1/2} \text{ and } V_1 \subset I^{-1}[\epsilon_0, c+2\epsilon_0]$$

Then we have that

$$\delta_0 = \inf\{\| I(v) \| : v \in V_1 \backslash V_0\} > 0.$$

We next define a functional J. $\alpha(x) : H \to [0,1]$ be a continuous function such that

$$\alpha(x) = \begin{cases} 1 & \text{for } x \in V_1^c \\ 0 & \text{for } x \in V_0 \end{cases}$$

and we put

$$J(v) = \alpha(v)I(v) + (1-\alpha(x))J^\infty(v) \qquad \text{for all } v \in H.$$

Then from the definition, $J \equiv J^\infty$ on V_0 and $J \equiv I$ on V_1^c.

Here we note that

$$\lim_{\rho \to 0} | I(v) - J^\infty(v) | = \lim_{\rho \to 0} \| \nabla I(v) - \nabla J^\infty(v) \| = 0 \text{ uniformly on } V_1. \quad (2.5)$$

Then there exists $\rho_0 > 0$ such that if $\rho \leq \rho_0$,

$$| I(v) - J(v) | < c/2 \quad \text{on } V_1$$

and

$$\| \nabla J^\infty(v) - \nabla I(v) \| < \delta_0/2 \quad \text{on } V_1.$$

Therefore we have that

$$\| \nabla J(v) \| > \delta_0/2 \qquad \text{for all } v \in V_1 \backslash V_0.$$

This implies that if $\rho \leq \rho_0$,

$$\| \nabla J(v) \| < \delta_0/2 \text{ and } 2c > J(v) > 0 \text{ implies that } v \in V_0$$

and therefore $J(v) = J^\infty(v)$. This implies that if we find a critical point v of J with $2c > J(v) > 0$, then v is a critical point of J^∞ in V_0.

3. Homology groups . Our purpose in this section is to calculate homology groups $H_*(I_{c+\epsilon}, I_{c-\epsilon})$ for $0<\epsilon<c+2\epsilon_0$. To calculate the homology groups $H_*(I_{c+\epsilon}, I_{c-\epsilon})$, we will find subsets K and U of V_0 satisfying

(a) $$K \subset \mathrm{int} U;$$

(b) $$\pm K_0 = \{\pm u_x : x \in R^N\} \subset K$$

for some $r>0$, where ∂K denotes the boundary of K in H;

(c) there exists $\epsilon_1 > 0$ such that $I_{c/2}$ is a strong deformation retract of $I_{c+\epsilon}\backslash K$ for $0<\epsilon<\epsilon_1$.

For U and K satisfying (a), (b) and (c) , we have the following lemma.

Lemma 3.1. *Suppose that U and K satisfies (a), (b) and (c). Then for each $0<\epsilon<\epsilon_1$,*

$$H_*(I_{c+\epsilon}, I_{c-\epsilon}) = H_*(U \cap I_{c+\epsilon}, (U\backslash K) \cap I_{c+\epsilon})$$

We will define subsets U and K of V_0 satisfying (a), (b) and (c).

Lemma 3.3. *For each $0<\epsilon<c+2\epsilon_0$,*

$$I^M_{c+\epsilon} \cong \{u\} \cup \{-u\}$$

where I^M is the restriction of I on M.

We put $\widetilde{U} = I^M_{c+2\epsilon_0}$ and $\widetilde{K} = I^M_{c+\epsilon_0}$. Then it follows that

We next define U and K. We fix positive numbers r_1^-, r_2^- with $r_1^- > r_2^-$. We assume that r_1^- is so small that

$$c/2 < I(v+\lambda v) \qquad \text{for all } v \in \widetilde{U} \text{ and } \lambda \in R \text{ with } |\lambda| \le r_1^-. \tag{3.1}$$

By (3.4) and Lemma 3.2, there exists $\widetilde{\epsilon} > 0$ such that

$$I(v+\lambda v) < I(v) - \widetilde{\epsilon}. \qquad \text{for } v \in \widetilde{U} \text{ and } r_2^- \le |\lambda| \le r_1^- \tag{3.2}$$

Then by choosing r_2^+ small enough , we have that $\sup\{I(v) : v \in \widetilde{U}\} < c+\widetilde{\epsilon}/2$. Then by (3.2) that

$$I(v + \lambda v) < c \qquad \text{for all } v \in \widetilde{U} \text{ and } r_2^- \leq \mid \lambda \mid \leq r_1^- . \tag{3.3}$$

It also follows from Lemma 3.2 that

$$\text{mapping } t \to I(v + tv) \text{ is decreasing on } [0,1] \text{ for } v \in \widetilde{U}. \tag{3.4}$$

Now we set

$$U = \{v + \lambda v : v \in \widetilde{U}, \mid \lambda \mid \leq r_1^-\}, \quad K = \{v + \lambda v : v \in \widetilde{K}, \mid \lambda \mid \leq r_2^-\}.$$

Then it is obvious that U and K satisfies (a) and (b). Moreover we have

Lemma 3.4. *There exists $\epsilon_1 > 0$ such that for each $0 < \epsilon < \epsilon_1$, $I_{c/2}$ is a strong deformation retract of $I_{c+\epsilon} \backslash K$*

For each $v \in \widetilde{U}$. we put

$$U_v = \{v + \lambda v :, \mid \lambda \mid \leq r_1^-\}, \quad K_v = \begin{cases} \{v + \lambda v : \mid \lambda \mid \leq r_2^-\} & \text{if } v \in \widetilde{K} \\ \{\phi\} & \text{if } v \notin \widetilde{K}. \end{cases}$$

Then

Lemma 3.6. *Let $0 < \epsilon < \epsilon_0$. Then for each $v \in \widetilde{U}$,*

$$(U_v \backslash K_v) \cap I_{c+\epsilon} \cong v + \{-r_1^- v, r_1^- v\} \cong S^0 \cong \{-1, 1\}. \tag{3.5}$$

Lemma 3.7. *For $0 < \epsilon < \min\{\epsilon_1, \epsilon_0\}$,*

$$H_*(U \cap I_{c+\epsilon}, (U \backslash K) \cap I_{c+\epsilon}) = H_*(S^0 \times D^1, S^0 \times S^0) \oplus H_*(S^0 \times D^1, S^0 \times S^0).$$

Proof. Let $0 < \epsilon < \min\{\epsilon_1, \epsilon_0\}$. By Lemma 3.5 and the definition , we have that

$$U \cap I_{c+\epsilon} \cong U \cong \widetilde{U} \times D^1 \cong \{u\} \times D^1 \cup \{-u\} \times D^1$$

On the other hand, by Lemma 3.6, we have that

$$(U \backslash K) \cap I_{c+\epsilon} \cong \widetilde{U} \times S^0 \cong \{u\} \times S^0 \cup \{-u\} \times S^0$$

Then the assertion follows. ∎

By Lemma 2.1 and Lemma 3.7, we have

Proposition 3.8. *For each* $0 < \epsilon < c$

$$H_n(I_{c+\epsilon}, I_{c-\epsilon}) = \begin{cases} 2 & \text{for } n = 1 \\ 0 & \text{otherwise} . \end{cases}$$

4. Proofs of Theorem 1. In this section, we calculate the homology groups for J and prove Theorem 1. From (2.1?), . we have that there exists $\rho_2 > 0$ such that for $0 < \rho < \rho_1$ sufficiently small, that

$$H_*(I_{c+\epsilon}, I_{c/2}) \cong H_*(J_{c+\epsilon}, J_{c/2}) \qquad \text{for } 0 < 2\epsilon < c. \tag{4.1}$$

We will prove Theorem 1 by contradiction. That is we assume that J possesses no critical point different from 0.

Here we state a direct consequence from Lions's concentrate compactness lemma.

Now assume that $\rho < \rho_0$ and we define a manifold $\mathcal{M}$ by

$$\mathcal{M} = \{v \in H \backslash \{0\} : \| v \|^2 = \int_{R^N} \int_0^{v(x)} g(x,t) dt dx\}$$

It is easy to check that for each $v \in H\backslash\{0\}$, the set $\{\lambda v : \lambda \geq 0\}$ intersect to $\mathcal{M}$ at exactly one point. For each $x \in R$, we define a positive number $\alpha_{+,x}$ and a negative number $\alpha_{-,x}$ by

$$\alpha_{+,x} u_x \in \mathcal{M} \text{ and } \alpha_{-,x} u_x \in \mathcal{M}.$$

From condition (g3) , we have that

$$\lim_{|x| \to \infty} \alpha_{\pm,x} = \pm 1. \tag{4.2}$$

For $r > 0$, we put

$$K_{\pm,r} = \{\alpha_{\pm,x} u_x : x \in R^N, | x | \geq r\}.$$

Then

$$\lim_{r \to \infty} \sup\{J(v) : v \in K_{\pm,r}\} = c. \tag{4.3}$$

Lemma 4.2. *For each $\epsilon > 0$ with $2\epsilon < c$, there exists $r_\epsilon > 0$ and*

$$J^{\mathcal{M}}_{c+\epsilon} \cong K_{+,r_\epsilon} \cup K_{-,r_\epsilon} \cong S^{N-1} \cup S^{N-1}.$$

Now we put $\widetilde{\mathcal{K}} = J^{\mathcal{M}}_{c+\epsilon}$ and $\widetilde{\mathcal{U}} = J^{\mathcal{M}}_{c+2\epsilon}$.

Now we set

$$\mathcal{U} = \{v + \lambda v : v \in \widetilde{\mathcal{U}}, \mid \lambda \mid \leq r_1^-\}, \quad \mathcal{K} = \{v + w : v \in \widetilde{\mathcal{U}}, w \mid \lambda \mid \leq r_2^-\}.$$

Then by a parallel argument as in the proof of Lemma 2.5, we can see that there exists $\overline{\epsilon}_1 > 0$ such that $J_{c/2}$ is a strong deformation retract of $J_{c_0+c+\epsilon}\backslash\mathcal{K}$ for each $0 < \epsilon < \overline{\epsilon}_1$. That is we have

$$H_*(J_{c+\epsilon}, J_{c/2}) = H_*(\mathcal{U} \cap J_{c_0+c+\epsilon}, (\mathcal{U}\backslash\mathcal{K}) \cap J_{c_0+c+\epsilon}) \tag{4.4}$$

for each $0 < \epsilon < \overline{\epsilon}_1$.

We also have

Lemma 4.3. *For each $0 < \epsilon < \overline{\epsilon}_0$,*

$$\mathcal{U} \cap J_{c_0+c+\epsilon} \cong \mathcal{U} \cong K_0.$$

The proof of Lemma 4.5 is the same as that of Lemma 2.5. Then we omit the proof. As in section 2, we put

$$\mathcal{U}_v = \{v + \lambda v :\mid \lambda \mid \leq r_1^-\}, \quad \mathcal{K}_v = \begin{cases} \{v + \lambda v :\mid \lambda \mid \leq r_2^-\} & \text{if } v \in \widetilde{\mathcal{K}} \\ \{\phi\} & \text{if } v \notin \widetilde{\mathcal{K}}. \end{cases}$$

for each $v \in \widetilde{U}$. Then by the same argument as in section 2, we have

Lemma 4.4. *Let $0 < \epsilon < \overline{\epsilon}_0$. Then for each $v \in \widetilde{\mathcal{U}}$,*

$$(\mathcal{U}_v\backslash\mathcal{K}_v) \cap I_{c+\epsilon} \cong v + \{-r_1^- v, r_1^- v\} \cong S^0. \tag{4.5}$$

Then by using Lemma 4.5 and Lemma 4.6, we obtain

Lemma 4.7. *For each* $0 < \epsilon < \min\{\bar{\epsilon}_0, \bar{\epsilon}_1\}$,

$$\begin{aligned} &H_*(\mathcal{U} \cap J_{c+\epsilon}, (\mathcal{U} \backslash \mathcal{K}) \cap J_{c+\epsilon}) \\ &\qquad = H_*(S^{N-1} \times D^1, S^{N-1} \times S^0) \oplus H_*(S^{N-1} \times D^1, S^{N-1} \times S^0). \end{aligned}$$

Thus we obtain by (4.1) and Lemma 4.7 that

Proposition 4.8.

$$H_n(J_{c+\epsilon}, J_{c/2}) = \begin{cases} 2 & \text{for } n = 1 \text{ or } n = N \\ 0 & \text{otherwise .} \end{cases}$$

We can now finish the proof of Theorem.

Proof of Theorem 1. By (4.5) and (4.0) , we have that if $\rho \leq \rho_0$, then for each $0 < \epsilon < c$,

$$H_*(J_{c+\epsilon}, J_{c/2}) \cong H_*(I_{c+\epsilon}, I_{c/2}) \cong H_*(I_{c+\epsilon}, I_{c-\epsilon}). \tag{4.6}$$

But we can see from Proposition 3.8 and Proposition 4.8 that the equality does not holds. This is a contradiction. Thus we obtain that there exists at least two solutions of (P). ∎