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A SURVEY ON SINGULARITIES OF SOLUTIONS FOR FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

This is a survey article on recent results about the singularities of solutions for first order partial differential equations. Firstly we consider the following two kinds of first order partial differential equations:

$$\sum_{i=1}^{n} a_i(x,y)\frac{\partial y}{\partial x_i} - b(x,y) = 0 \tag{Q}$$

$$H(x_1,\dots,x_n,\frac{\partial y}{\partial x_1},\dots,\frac{\partial y}{\partial x_n}) = 0, \tag{H}$$

where $a_i(x,y)$, $b(x,y)$ and $H(x,p)$ are C^∞-functions. Here the equation (Q) is called *a quasilinear first order partial differential equation (briefly, a quasilinear equation)* and (H) is called *a Hamilton-Jacobi equation.* These equations are well studied in several articles ([2–9, 11–14, 19–22, 24–28], etc.). For the study of quasilinear equations, the theory of entropy solutions has provided the right weak setting (see, for example [22]). For Hamilton-Jacobi equations, the theory of viscosity solutions is appropriate one ([5–7]). However, these notions of weak solutions have quite different features. Under the some assumptions, the entropy solutions are discontinuous and the viscosity solutions are continuous.

We refer the following two typical examples of these equations.

Example 1.1. We consider the following equations.

$$\frac{\partial y}{\partial x_1} + 2y\frac{\partial y}{\partial x_2} = 0 \tag{Q$'$}$$

$$\frac{\partial y}{\partial x_1} + (\frac{\partial y}{\partial x_2})^2 = 0. \tag{H$'$}$$

We can explicitly solve these equations by the classical method of characteristics, when the initial condition is $y(0,x_2) = \sin x_2$. The pictures of the graph of geometric (multi-valued) solutions of these equations are given in Figure 1. These pictures are useful to understand the difference between these two equations. We can observe that the geometric solution for (Q′) is a smooth submanifold but for (H′) is not smooth in the (x_1, x_2, y)-space.

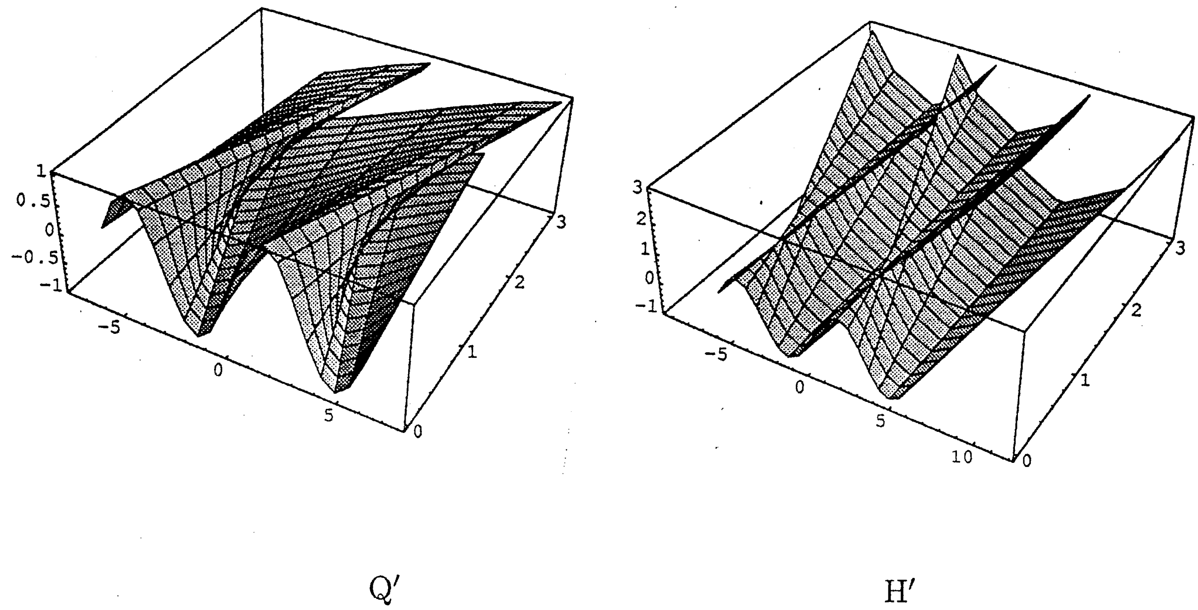

Figure 1

We can easily choose the continuous branch of the multi valued solution for (H'). However, we cannot choose the continuous branch of the multivaleud solutions for (Q').

2. Geometric framwork for time-dependent Hamilton-Jacobi equations

In which we give a brief review of the geometric framework for the study of singularities geometric solutions of the time-evolutional Hamilton-Jacobi equations ([14–17]):

$$(\mathrm{P}') \qquad \begin{cases} \dfrac{\partial y}{\partial t} + H(t, x_1, \ldots, x_n, \dfrac{\partial y}{\partial x_1}, \cdots, \dfrac{\partial y}{\partial x_n}) = 0 \\ y(0, x_1, \cdots, x_n) = \phi(x_1, \cdots, x_n), \end{cases}$$

We describe the theory for the general case here.

Let $J^1(\mathbb{R}^n, \mathbb{R})$ be the 1-jet bundle of functions of n-variables which may be considered as $\mathbb{R}^{2n+1}$ with a natural coordinate system $(x_1, \ldots, x_n, y, p_1, \ldots, p_n)$, where $(x_1, \ldots, x_n)$ is a coordinate system of $\mathbb{R}^n$. We also have a natural projection $\pi : J^1(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}^n \times \mathbb{R}$ given by $\pi(x, y, p) = (x, y)$.

An immersion germ $i : (L_0, u_0) \to J^1(\mathbb{R}^n, \mathbb{R})$ is said to be *a Legendrian immersion germ* (i.e., Legendrian submanifold germ) if $\dim L = n$ and $i^*\theta = 0$, where $\theta = dy - \sum_{i=1}^n p_i \cdot dx_i$. The image of $\pi \circ i$ is called *the wave front set* of i and it is denoted by $W(i)$. We also consider the 1-jet bundle $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and the canonical 1-form Θ on that space. Let $(t, x_1, \ldots, x_n)$ be a canonical coordinate system on $\mathbb{R} \times \mathbb{R}^n$ and $(t, x_1, \ldots, x_n, y, s, p_1, \ldots, p_n)$ the corresponding coordinate system on $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$. Then, the canonical 1-form is given by $\Theta = dy - \sum_{i=1}^n p_i \cdot dx_i - s \cdot dt = \theta - s \cdot dt$.

We define the natural projection $\Pi : J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}) \to (\mathbb{R}\times\mathbb{R}^n)\times\mathbb{R}$ by $\Pi(t,x,y,s,p) = (t,x,y)$. We call the above 1-jet bundle *an unfolded 1-jet bundle.*

A Hamilton-Jacobi equation is defined to be a hypersurface

$$\text{(G-H-J)} \qquad E(H) = \{(t,x,y,s,p)\in J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}) | s + H(t,x,p) = 0\}$$

in $J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$. *A geometric (multi-valued) solution* of $E(H)$ is a Legendrian submanifold L in $J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ lying in $E(H)$. In this case the wave front set $W(i)$ is "the graph" of the geometric solution which is generally a hypersurface with singularities.

In order to study (P) we need the following framework: For any $c\in(\mathbb{R},0)$, we define

$$E(H)_c = \{(c,x,y,-H(c,x,p),p) | (x,y,p)\in J^1(\mathbb{R}^n,\mathbb{R})\}.$$

Then, $E(H)_c$ is a $(2n+1)$-dimensional submanifold of $J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ and $\Theta_c = \Theta | E(H)_c = dz - \sum_{i=1}^n p_i dx_i$ gives a contact structure on $E(H)_c$. We define a mapping $\iota_c : J^1(\mathbb{R}^n,\mathbb{R}) \to E(H)_c$ by $\iota_c(x,y,p) = (c,x,y,-H(c,x,p),p)$. The mapping i_c is a contact diffeomorphism and the following diagram is commutative:

$$\begin{tikzcd} J^1(\mathbb{R}^n,\mathbb{R}) \arrow[r, "\iota_c"] \arrow[d, "\pi"'] & E(H)_c \arrow[d, "\pi_c"] \\ \mathbb{R}^n\times\mathbb{R} \arrow[r, equal] & \mathbb{R}^n\times\mathbb{R}. \end{tikzcd}$$

We say that *a geometric Cauchy problem (with initial condition L') associated with the time parameter(GCPT) is given for an equation $E(H)$* if there is given an n-dimensional submanifold $i : L' \subset E(H)$ with $i^*\Theta = 0$ and $i(L')\subset E(H)_c$ for some $c\in(\mathbb{R},0)$. Since $X_H \notin TE(H)_c$, we have $X_H\notin TL'$, where X_H is the characteristic vector field given by

$$X_H = \frac{\partial}{\partial t} + \sum_{i=1}^n \frac{\partial H}{\partial p_i}\frac{\partial}{\partial x_i} + (\sum_{i=1}^n p_i\frac{\partial H}{\partial p_i} - H)\frac{\partial}{\partial y} - \frac{\partial H}{\partial t}\frac{\partial}{\partial s} - \sum_{i=1}^n \frac{\partial H}{\partial x_i}\frac{\partial}{\partial p_i}.$$

By using the classical characteristic method, we can show that there exists a unique geometric solutions around L'.

We remark that Cauchy problem (P) is a GCPT. The initial submanifold is given by

$$L_{\phi,0} = \left\{(0,x,\phi(x),-H(0,x,\frac{\partial\phi}{\partial x}),\frac{\partial\phi}{\partial x}) | x\in\mathbb{R}^n\right\} \subset E(H)_0.$$

The problem of studying the singularities of the graph of the geometric solution is formulated as follows:

Geometric Problem. *Classify the generic bifurcations of wave fronts of*

$$\pi_t| : L\cap E(H)_t \to \mathbb{R}^n\times\mathbb{R}$$

with respect to the parameter t (i.e., the generic bifurcations of wave fronts of geometric solutions along the time parameter).

Following [16], in order to study the singularities of the geometric solution we identify geometric solutions with one-parameter Legendrian unfoldings. Let R be an $(n+1)$-dimensional smooth manifold, $\mu : (R, u_0) \to (\mathbb{R}, t_0)$ be a submersion germ and $\ell : (R, u_0) \to J^1(\mathbb{R}^n, \mathbb{R})$ be a smooth map germ. We say that the pair (μ, ℓ) is *a Legendrian family* if $\ell_t = \ell|\mu^{-1}(t)$ is a Legendrian immersion germ for any $t \in (\mathbb{R}, t_0)$. Then there exist a unique element $h \in C^\infty_{u_0}(R)$ such that $\ell^*\theta = h \cdot d\mu$, where $C^\infty_{u_0}(R)$ is the ring of smooth function germs at u_0. Define a map germ $\mathcal{L} : (R, u_0) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ by

$$\mathcal{L}(u) = (\mu(u), x \circ \ell(u), y \circ \ell(u), h(u), p \circ \ell(u)).$$

We can easily show that $\mathcal{L}$ is a Legendrian immersion germ. If we fix 1-forms Θ and θ, the Legendrian immersion germ $\mathcal{L}$ is uniquely determined by the Legendrian family (μ, ℓ). We call $\mathcal{L}$ *a Legendrian unfolding associated with the Legendrian family* (μ, ℓ).

We have to study how various branches of the multi-valued graph $W_t = (\{t\} \times \mathbb{R}^n \times \mathbb{R}) \cap W(i)$ intersecting at a point bifurcate in time for an arbitrary Hamiltonian $H(t, x, p)$ in [17]. We classify the bifurcations of the branches of the graph by classifying the bifurcations of singularities of multi-Legendrian unfoldings which are expressed in terms of multi-germs.

Let $\mathcal{L}_i : (R, u_0) \to (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z_i)$ $(i = 1, \dots, r)$ be Legendrian unfoldings with $\Pi(z_i) = 0$ where $z_1, \dots, z_r$ are distinct. We call $(\mathcal{L}_1, \dots, \mathcal{L}_r)$ *a multi-Legendrian unfolding*. Let $(\mathcal{L}_1, \dots, \mathcal{L}_r)$ and $(\mathcal{L}'_1, \dots, \mathcal{L}'_r)$ be multi-Legendrian unfoldings. We say that these are *$P_{(r)}$-Legendrian equivalent* if there exist contact diffeomorphism germs

$$K_i : (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z_i) \to (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z'_i) \quad (i = 1, \dots, r)$$

of the form $K_i(t, x, y, s, p) = (\phi_1(t), \phi_2(t, x, y), \phi_3(t, x, y), \phi^i_4(t, x, y, s, p), \phi^i_5(t, x, y, s, p))$ and a diffeomorphism germ $\Psi : (R, u_0) \to (R, u'_0)$ such that $K_i \circ \mathcal{L}_i = \mathcal{L}'_i \circ \Psi$ for any $i = 1, \dots, r$. It is clear that if two multi-Legendrian unfoldings are $P_{(r)}$-Legendrian equivalent, then there exists a diffeomorphism germ $\Phi : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0)$ of the form $\Phi(t, x, y) = (\phi_1(t), \phi_2(t, x, y), \phi_3(t, x, y))$ such that $\Phi(\cup_{i=1}^r W(\mathcal{L}_i)) = \cup_{i=1}^r W(\mathcal{L}_i)$. Thus the above equivalence describes how bifurcations of wavefronts (i.e. graphs of solutions) interact. We can define the notion of stability with respect to the $P_{(r)}$-Legendrian equivalence in the same way as for the ordinary Legendrian stability (see [1,29]). Motivated by Arnol'd-Zakalyukin's theory ([1, 29]), we can construct multi-generating families of multi-Legendrian unfoldings and give a classification of $P_{(r)}$-Legendrian stable Legendrian unfoldings by using the classification of multi-families of function germs in Zakalyukin [29]. We get a list of classifications for $n = 1, 2, 3$ in [17]. However, we only present the list of classifications for $n = 1$. For the case $n = 2, 3$, see [17].

Theorem 2.1 [1]. *Suppose that $n = 1$. Then a generic multi-Legendrian unfolding is $P_{(r)}$-Legendrian equivalent to one of the multi-Legendrian unfoldings in the following list :*

$r = 1$;

${}^0A_1$: $(t, u, 0, 0, 0)$;

${}^0A_2$: $(t, 3u^2, 2u^3, 0, u)$;

${}^1A_3\ :\ (t, 4u^3+2ut, 3u^4+u^2t, -u^2, u).$
$r=2\ ;$
${}^0({}^0A_1\,{}^0A_1)\ :\ ((t,u,-u,0,-1),(t,u,u,0,1))\ ;$
${}^1({}^0A_1\,{}^0A_1)\ :\ ((t,u,t\pm u^2,1,\pm 2u),(t,u,0,0,0))\ ;$
${}^1A_2\,{}^0A_1\ :\ ((t,3u^2-t,2u^3,u,u),(t,u,-u,0,-1)).$
$r=3\ ;$
${}^0A_1\,{}^0A_1\,{}^0A_1\ :\ ((t,u,t-u,1,-1),(t,u,0,0,0),(t,u,u,0,1)).$

When we consider the geometric solution, we can get rid of the germ ${}^1({}^0A_1\,{}^0A_1)$ from the above list because the geometric solution is a one-to-one immersions into the unfolded 1-jet space. For the purpose, we need a kind of non-degeneracy condition on the Hamiltonian function. We say that a Hamiltonian function $H(t,x,p)$ is *non-degenerate at* (t_0,x_0,p_0) if it $\frac{\partial^2 H}{\partial p_i \partial p_j}(t_0,x_0,p_0) \neq 0$ for some $1 \leq i,j \leq n$. This condition is weaker than the condition that $H(t,x,p)$ is convex (or concave) with respect to $(p_1,\dots,p_n)$-variables at (t_0,x_0,p_0) for $n \geq 2$. The following theorem is a realization theorem for generic singularities for a given Hamilton-Jacobi equation.

Theorem 2.2 ([17,18]). *Let $H(t,x,p)$ be a non-degenerate Hamiltonian function germ at (t_0,x_0,p_0) and $\mathcal{L} : (R,u_0) \to (J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),(t_0,x_0,y_0,s_0,p_0))$ be a $P_{(1)}$-Legendrian stable Legendrian unfolding associated with (μ,ℓ). Then there exists a Legendrian unfolding $\mathcal{L}'$ which is a geometric solution of the Hamilton-Jacobi equation $s+H(t,x,p)=0$ such that $\mathcal{L}$ and $\mathcal{L}'$ are $P_{(1)}$-Legendrian equivalent.*

We remark that ${}^1A_3$ singularity (even for general n) describes how the singularity appears from a smooth solution. These are $P_{(1)}$-Legendrian stable Legendrian unfoldings, so that these can be realized as geometric solutions at the non-degenerated point for a given Hamilton-Jacobi equation. We can asserts the detailed statement for the case that the Hamiltonian function depends only on $(p_1,\dots,p_n)$-variables. In this case the Cauchy problem is given by

$$(\mathrm{P}')\qquad \begin{cases} \dfrac{\partial y}{\partial t}+H(\dfrac{\partial y}{\partial x_1},\cdots,\dfrac{\partial y}{\partial x_n})=0 \\ y(0,x_1,\cdots,x_n)=\phi(x_1,\cdots,x_n), \end{cases}$$

where H and ϕ are C^∞-functiions. Then we have the following proposition.

Proposition 2.3. *Let $s+H(p)=0$ be a Hamilton-Jacobi equation. If a singularity of geometric solution for the Cauchy problem (P′) appears at a point (t_0,x_0,p_0), then H is non-degenrated at (t_0,x_0,p_0).*

In this case the characteristic equation is given by

$$(\mathrm{C}')\qquad \begin{cases} \dfrac{dx_i}{dt}=\dfrac{\partial H}{\partial p_i}(p)\ (i=1,\dots,n), \\ \dfrac{dp_i}{dt}=0\ (i=1,\dots,n), \\ \dfrac{dy}{dt}=-H(p)+\displaystyle\sum_{i=1}^n p_i\cdot\dfrac{\partial H}{\partial p_i}(p),\ x,p\in\mathbb{R}^n, \\ x(0)=u,\ p(0)=\dfrac{\partial\phi}{\partial u}(u),\ y(0)=\phi(u),\ u\in\mathbb{R}^n. \end{cases}$$

We can explicity solve the charcteristic equation as follows:

$$
(\mathrm{S}') \qquad \begin{cases} x_i(t,u) = u_i + t\dfrac{\partial H}{\partial p_i}(\dfrac{\partial \phi}{\partial u}(u)) \ (i=1,\dots,n), \\ p_i(t,u) = \dfrac{\partial \phi}{\partial u_i}(u) \ (i=1,\dots,n), \\ y(t,u) = t\{-H(\dfrac{\partial \phi}{\partial u}(u)) + \displaystyle\sum_{i=1}^{n} \dfrac{\partial \phi}{\partial u_i}(u)\cdot \dfrac{\partial H}{\partial p_i}(\dfrac{\partial \phi}{\partial u}(u))\} + \phi(u). \end{cases}
$$

3. Viscosity solutions

The viscosity solutions for nonlinear equations of first order have been introduced by Crandall and Lions [7]. Such solutions need not be differentiable everywhere, as the only regularity required in the definition is that of continuity. The function $y_{\mathfrak{v}} \in C(\mathcal{O})$ is a viscosity solution of

$$
(\text{H-J}) \qquad \frac{\partial y}{\partial t} + H(t,x,\frac{\partial y}{\partial x_1},\dots,\frac{\partial y}{\partial x_n}) = 0
$$

in the open domain $\mathcal{O} \subset \mathbb{R}^+ \times \mathbb{R}^n$ provided

$$
\frac{\partial \psi}{\partial t}(t,x) + H(t,x,\frac{\partial \psi}{\partial x_1}(t,x),\dots,\frac{\partial \psi}{\partial x_n}(t,x)) \le 0, (resp. \ge 0)
$$

for any $\psi \in C^1(\mathcal{O})$ for which $y_{\mathfrak{v}} - \psi$ attains a local maximum (resp. local minimum) at the point $(t,x) \in \mathcal{O}$. The function $y_{\mathfrak{v}} \in C([0,\infty)\times\mathbb{R}^n)$ is a viscosity solution of the Cauchy problem (P) if and only if it is a viscosity solution of (H-J) in the domain $(0,\infty)\times\mathbb{R}^n$ and satisfies the initial condition $\lim_{t\to 0+} y_{\mathfrak{v}}(t,x) = \phi(x)$. The above inequality will be referred as the *viscosity criterion* at the point (t,x). We next state the viscosity criterion in a form which is more useful for the construction of the solution. To this end, assume that $\mathcal{O} \subset (0,\infty)\times\mathbb{R}^n$ is open and that there is a smooth hypersurface Γ of $\mathbb{R}^+\times\mathbb{R}^n$, which divides $\mathcal{O}$ into two open sets $\mathcal{O}^+$ and $\mathcal{O}^-$, $\mathcal{O} = \Gamma \cup \mathcal{O}^+ \cup \mathcal{O}^-$. Then we have the following theorem.

Theorem 3.1. *Let $y_{\mathfrak{v}} \in C(\mathcal{O})$ and $y_{\mathfrak{v}} = y_{\mathfrak{v}}^+$ in $\mathcal{O}^+ \cup \Gamma, y_{\mathfrak{v}} = y_{\mathfrak{v}}^-$ in $\mathcal{O}^- \cup \Gamma$ where $y_{\mathfrak{v}}^\pm \in C^1(\mathcal{O}^\pm \cup \Gamma)$. Then $y_{\mathfrak{v}}$ is a viscosity solution of (H-J) in $\mathcal{O}$ if and only if the following conditions hold:*

a) $y_{\mathfrak{v}}^+$ and $y_{\mathfrak{v}}^-$ are classical solutions of (H-J) in $\mathcal{O}^+$ and $\mathcal{O}^-$ respectively,

b) If the vector $\vec{\eta} = (H(t,x,\frac{\partial y^+}{\partial x}) - H(t,x,\frac{\partial y^+}{\partial x}), -(\frac{\partial y^+}{\partial x_1} - \frac{\partial y^-}{\partial x_1},\dots,\frac{\partial y^+}{\partial x_n} - \frac{\partial y^-}{\partial x_n}))$ points into $\mathcal{O}^+$, then

$$
H\left(t,x,(1-\lambda)\frac{\partial y_{\mathfrak{v}}^+}{\partial x} + \lambda\frac{\partial y_{\mathfrak{v}}^-}{\partial x}\right) - (1-\lambda)H(t,x,\frac{\partial y_{\mathfrak{v}}^+}{\partial x}) - \lambda H(t,x,\frac{\partial y_{\mathfrak{v}}^-}{\partial x}) \le 0
$$

(resp. ≥ 0), where $\lambda \in [0,1]$.

In particular, the graph of H lies respectively below or above the line segment joining the points $\left(\frac{\partial y_{\mathfrak{v}}^+}{\partial x}, H(t,x,\frac{\partial y_{\mathfrak{v}}^+}{\partial x})\right)$ *and* $\left(\frac{\partial y_{\mathfrak{v}}^-}{\partial x}, H(t,x,\frac{\partial y_{\mathfrak{v}}^-}{\partial x})\right)$.

The proof of Theorem 3.1 is given in ([20, 21]) as a direct application of Theorem 1.3 in [5]. The condition b) will be referred in the sequel as the *viscosity criterion.* The hypersurface Γ in the neighbourhood of which $y_{\mathfrak{v}}$ has the properties specified in the above theorem is the *shock surface.* If the Hamiltonian is uniformly convex (or concave), we can automatically construct viscosity solutions from our normal forms, so that we can easily draw the pictures of shock surfaces for lower dimensional cases. In [4] Bogaevskii has shown that the potential solution of the Burgers system with vanishing viscosity is given by the minimum function of a certain family of smooth functions. It corresponds to the viscosity solution of the Hamilton-Jacobi equation when the Hamiltonian is given by $H(p_1,\dots p_n) = \frac{1}{2}p_1^2 + \dots + \frac{1}{2}p_n^2$. He has drawn the pictures of shocks for this case. Our pictures are same as his pictures, so we do not present these in here (see [4]).

On the other hand, Bogaevskii used Florin-Hopf-Cole method ([10, 12]) to detect the solution for the Hamilton-Jacobi equation correspoding to the Burgers system. However, his method works for geral Hamilton-Jacobi eqauations which are convex with respect to $(p_1,\dots,p_n)$-variables. In this case we apply Bardi-Evans' result[2] to our situations in stead of Florin-Hopf-Cole method. The geometric solution for (P′) is given by

$$\text{(S)} \qquad L_{\phi,t} = \{(t, x(t,u), y(t,u), -H(p(t,u)), p(t,u)) | u \in \mathbb{R}^n\},$$

where

$$\begin{cases} x(t,u) = u + t\dfrac{\partial H}{\partial p}(\dfrac{\partial \phi}{\partial x}(u))), \\ p(t,u) = \dfrac{\partial \phi}{\partial x}(u) \\ y(t,u) = t\{-H(\dfrac{\partial \phi}{\partial x}(u)) + < \dfrac{\partial \phi}{\partial x}(u), \dfrac{\partial H}{\partial p}(\dfrac{\partial \phi}{\partial x}(u)) >\} + \phi(u). \end{cases}$$

We consider a family of functions $F(t,x,p,q) = \phi(q) + < p, (x-q) > -H(p)t$, where $(t,x,p,q) \in \mathbb{R} \times \mathbb{R}^n \times (\mathbb{R}^n \times \mathbb{R}^n)$ and $<,>$ is the canonical inner product on $\mathbb{R}^n$. We have

$$\Sigma(F) = \{(t, q + \frac{\partial H}{\partial p}(\frac{\partial \phi}{\partial q}(q))t, \frac{\partial \phi}{\partial q}(q), q) | (t,q) \in \mathbb{R} \times \mathbb{R}^n\},$$

where $\Sigma(F)$ is the set defined to be $\frac{\partial F}{\partial p_i} = 0$ and $\frac{\partial F}{\partial q_i} = 0$. We now define a map $\Phi_F : \Sigma(F) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ by $\Phi_F(t,x,p,q) = (t, x, F(t,x,p,q), \frac{\partial F}{\partial t}, \frac{\partial F}{\partial x})$. It follows that

$$\begin{aligned} &\Phi_F(t, q + \frac{\partial H}{\partial p}(\frac{\partial \phi}{\partial q}(q))t, \frac{\partial \phi}{\partial q}(q), q) = \\ &\qquad (t, q + \frac{\partial H}{\partial p}(\frac{\partial \phi}{\partial q}(q))t, -H(\frac{\partial \phi}{\partial q}(q))t + < \frac{\partial \phi}{\partial q}(q), \frac{\partial H}{\partial p}(\frac{\partial \phi}{\partial q}(q)) > + \phi(q), \\ &\qquad\qquad -H(\frac{\partial \phi}{\partial q}(q)), \frac{\partial \phi}{\partial q}(q)). \end{aligned}$$

This shows that the image of the map Φ_F is equal to $L_{\phi,t}$, namely, F is a global *generating family* of $L_{\phi,t}$.

We refer the following result of Bardi-Evans [2].

Theorem 3.2. *Assume that the Hamiltonian $H(p_1,\dots,p_n)$ is convex, then*

$$y(t,x) \equiv \inf_q \sup_p \{\phi(q) + < p, (x-q) > - H(p)t\}$$

is the unique viscosity solution of (P).

Then we have the following theorem as a corollary of the above theorem.

Theorem 3.3. *Assume that H is uniformly convex and ϕ has the minimum. Let $L_{\phi,t}$ be the geometric solution* (S) *of the Cauchy problem* (*rm P*$'$). *Then*

$$y(t,x) \equiv \min_y \{y | (t,x,y) \in \Pi(L_{\phi,t})\}$$

is the unique viscosity solution of (P$'$).

However, for general (non-convex) Hamiltonian, situations are quite different.

4. Non convex Hamiltonians in one space variable

In this section we stick to the Cauchy problem of Hamilton-Jacobi equation in one space variable as follows:

$$\text{(P)} \qquad \begin{cases} \dfrac{\partial y}{\partial t} + H(\dfrac{\partial y}{\partial x}) = 0 \\ y(0,x) = \phi(x), \end{cases}$$

where H and ϕ are C^∞-functions. Since $H(p)$ is not assumed to be uniformly convex (or concave), we cannot use Theorem 3.3, so that the situations should be quite complicated even for the one space variables case.

In this case the geometric solution is given by

$$L_{\phi,t} = \{(t, x(t,u), y(t,u), -H(p(t,u)), p(t,u)) | u \in \mathbb{R}\},$$

where

$$\begin{cases} x(t,u) = u + tH'(\phi'(u)), \\ p(t,u) = \phi'(u) \\ y(t,u) = t\{-H(\phi'(u)) + \phi'(u)H'(\phi'(u))\} + \phi(u). \end{cases}$$

Before the first critical time that characteristics cross in the (t,x)-plane, W_t is the graph of the viscosity solution y_v. After the characteristics cross, W_t becomes singular. Theorem 2.1 describes the generic singularities of W_t. The first singularity appears in the form of ${}^1A_3$. See Figure 2a, where we show the shape of the appearing singularity. By Proposition 2.3, these appear at the convex or the concave points of the Hamiltonian function. Away from the singularity, the viscosity solution is given by W_t. In ([17], [18]) we have constructed the unique viscosity solution past the first critical time by selecting a single-valued branch of W_t. Assume that the singularity of type ${}^1A_3$ appears at the point (t_0, x_0, p_0). After the critical time t_0, the wave front W_t is three-valued on an interval $(x_1(t), x_2(t))$; see Figure 2b. Let y_i, $i = 1,2,3$ be the three branches of W_t, where y_1 is defined on a neighborhood of $x_1(t)$ and y_2 on a neighborhood of $x_2(t)$. Then y_1, y_2 intersect at one point $\chi(t) \in (x_1(t), x_2(t))$, for $t > t_0$. We define the viscosity solution past t_0 by selecting a continuous single-valued branch of W_t as follows:

Theorem 4.1. *There exists an $\varepsilon > 0$ such that the function $y_{\mathfrak{v}}(t,x), (t,x) \in (t_0, t_0+\varepsilon) \times (x_1(t), x_2(t))$, defined by*

$$(*) \qquad y_{\mathfrak{v}}(t,x) = \begin{cases} y_1(t,x), \; x \leq \chi(t) \\ y_3(t,x), \; x \geq \chi(t), \end{cases}$$

is the viscosity solution of (P) in a neighborhood of x_0 past the time t_0.

In view of Proposition 2.3 the viscosity criterion (see Section 3) is satisfied across $\chi(t)$ while $y_{\mathfrak{v}}$ defined by (*) is a classical solution away from $\chi(t)$. Hence, by the uniqueness of the viscosity solution, (*) gives the viscosity solution of (P) past t_0.

By this construction, we have extended the viscosity solution beyond the first critical time t_0.

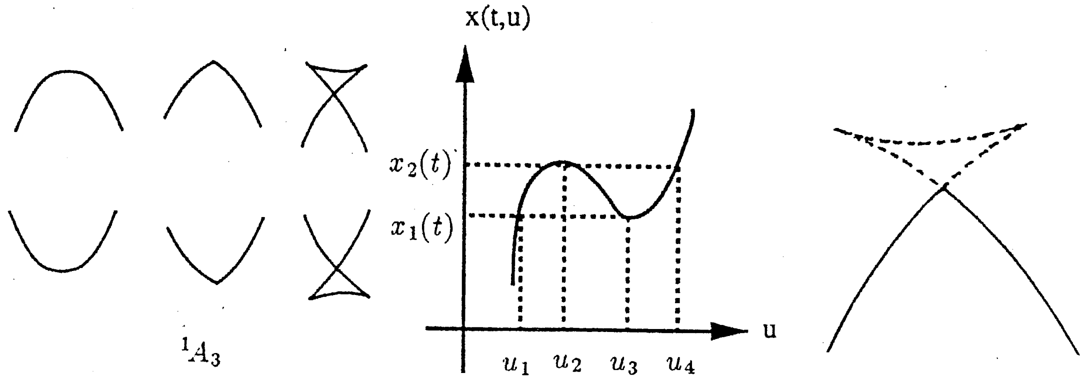

Figure 2a Figure 2b Figure 2c

According to Theorem 2.5 the shock is generated in a convex or concave domains of $H(p)$, so the viscosity criterion is automatically satisfied. The graph of the viscosity solution past the first critical time is depicted by a full line in Figure 2c, where we assume that H is convex in the neighborhood of the appearing singularity $^1A_3$. The shock corresponds to the intersection of the two branches and it is called a *genuine shock.* The genuine shock is defined as the intersection of two incoming characteristics (or waves) and its speed is given by the Rankine-Hugoniot condition

$$\chi'(t) = \frac{H(y^+_{\mathfrak{v},x}(t,\chi(t))) - H(y^-_{\mathfrak{v},x}(t,\chi(t)))}{y^+_{\mathfrak{v},x}(t,\chi(t)) - y^-_{\mathfrak{v},x}(t,\chi(t))},$$

where $y^\pm_{\mathfrak{v},x} = \frac{\partial y^\pm_{\mathfrak{v}}}{\partial x}$ and $\chi'(t) = \frac{d\chi}{dt}(t)$. Therefore in order to follow the evolution of the shock we have to study the following questions:

a) How different branches of the multi-valued graph of W_t intersecting at one point bifurcate in time.

b) If the two branches initially defining the shock continue to cross, whether the viscosity criterion is satisfied across the intersection.

If the viscosity criterion is satisfied at the time $t_\alpha = t_0 + \varepsilon$, we can choose the correct branch of the graphs of the geometric solutions as viscosity solutions.

We will now investigate how the viscosity criterion can be violated across the intersection of two branches. Assume that a generated shock is defined by two intersecting branches y^- and y^+. We denote by y^- (resp. y^+) the branch representing the viscosity solution for $x < \chi(t)$ (resp. $x > \chi(t)$). If the two branches remain intersected they evolve according to ${}^0({}^0A_1\,{}^0A_1)$. We denote by $\chi(t)$ the intersection of the two branches. In the case when $H(p)$ has only one inflection point Kossioris [20] studied this problem and constructed the viscosity solutions. We consider the general situation here. It is clear that for generic Hamiltonian function $H(p)$, H has only Morse type critical points and no tritangent lines. So we assume that the Hamiltonian has the above properties. By Theorem 2.1, we have the following theorem.

Theorem 4.2. *For a generic initial function ϕ, if the viscosity criterion is violated at t_α, then the only following 8 cases may occur:*
(1) The normal form is ${}^0({}^0A_1\,{}^0A_1)$ and $\overline{P^+P^-}$ is tangent to the graph of $H(p)$ at only one of the points P^+, P^- and the line is not tangent to the graph at other points between these points.
(2) The normal form is ${}^0({}^0A_1\,{}^0A_1)$ and $\overline{P^+P^-}$ is not tangent to the graph of $H(p)$ at each point P^+, P^- and there exists only one another point between these points at where the above line is tangent to the graph.
(3) The normal form is ${}^1A_2\,{}^0A_1$ and $\overline{P^+P^-}$ is tangent to the graph of $H(p)$ at only one of the points P^+, P^- and the line is not tangent to the graph at other points between these points.
We denote $\overline{P^+P^-}$ the line through P^+, P^- in the $(p, H(p))$-plane, where

$$P^+ = (y_x^+(t_\alpha, \chi(t_\alpha)), H(y_x^+(t_\alpha, \chi(t_\alpha))),$$

$P^- = (y_x^-(t_\alpha, \chi(t_\alpha)), H(y_x^-(t_\alpha, \chi(t_\alpha)))$.

We can show that the case 3) cannot ocur if the viscosity criterion is satisfied before the perestroika time t_α. We can solve local Riemann problems and construct viscosity solutions for each case in the above theorem. However, we only consider the cases (1) in this note. For the deatiled consideration , please refere [16]

Case (1). We assume that the graph of the viscosity solution at the time $t \le t_\alpha$ is depicted as in Figure 3a.

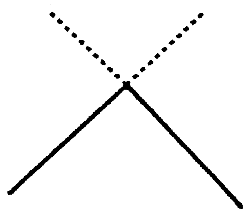

Figure 3a

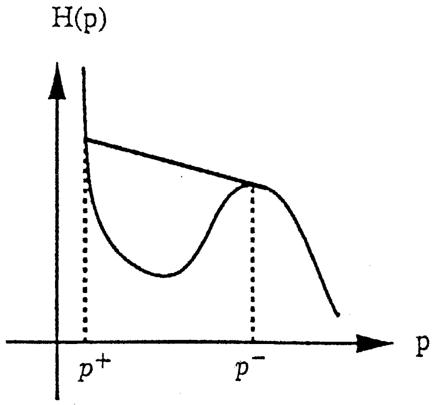

Figure 3b

Without the loss of generality, we may assume that $\overline{P^+P^-}$ is tangent to the graph of $H(p)$ at the point $(y_x^-(t_\alpha, \chi(t_\alpha)), H(y_x^-(t_\alpha, \chi(t_\alpha)))$ and

$$H''((y_x^-(t_\alpha, \chi(t_\alpha))) < 0$$

(see Figure 3b). As we already mentioned that the genuine shocks satisfies the Rankine-Hugoniot condition. So we should construct new characteristics which satisfies both of the Rankine-Hugoniot condition and the viscosity criterion. In this case we have

$$H'(y_x^-(t_\alpha,\chi(t_\alpha))) = \frac{H(y_x^+(t_\alpha,\chi(t_\alpha))) - H(y_x^-(t_\alpha,\chi(t_\alpha)))}{y_x^+(t_\alpha,\chi(t_\alpha)) - y_x^-(t_\alpha,\chi(t_\alpha))} = \chi'(t_\alpha).$$

We now distinguish two cases as follows:

a) If

$$H'(y_x^-(t,\chi(t))) \geq \frac{H(y_x^+(t,\chi(t))) - H(y_x^-(t,\chi(t)))}{y_x^+(t,\chi(t)) - y_x^-(t,\chi(t))}$$

for $t_\alpha \leq t < t_\alpha + \varepsilon$ for sufficiently small $\varepsilon > 0$, then we can easily show that the viscosity criterion is satisfied for $t < t_\alpha + \varepsilon$. So we can choose single valued continuous branches of the geometric solution as the viscosity solution.

b) If

$$H'(y_x^-(t,\chi(t))) < \frac{H(y_x^+(t,\chi(t))) - H(y_x^-(t,\chi(t)))}{y_x^+(t,\chi(t)) - y_x^-(t,\chi(t))}$$

for $t_\alpha \leq t < t_\alpha + \varepsilon$ for sufficiently small $\varepsilon > 0$, then we can easily show that the viscosity criterion is violated for $t_\alpha < t < t_\alpha + \varepsilon$, so that a new way to build the solution is required (cf., Figure 4).

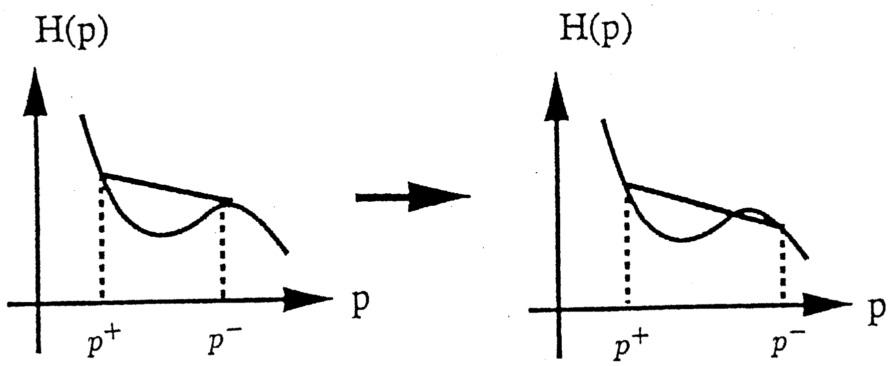

FIGURE 4

In this case we can use the techniques in [20] to construct the contact discontinuity shock curve and then obtain new characteristics. Lets consider the relation $H'(q) = \frac{H(p)-H(q)}{p-q}$ around (q_0, p_0) with $q_0 \neq p_0$, $H'(q_0) = \frac{H(p_0)-H(q_0)}{p_0-q_0}$ and $H''(q_0) \neq 0$. By the implicit function theorem, there exists a smooth function ψ around p_0 such that the above relation is equivalent to $q = \psi(p)$. We will first construct the contact discontinuity as the solution of the following initial value problem.

$$\begin{cases} \chi_c'(t) = H'(\psi(y_x^+(t,\chi_c(t)))), \\ \chi_c(t_\alpha) = \chi(t_\alpha). \end{cases}$$

The characteristic which is started at a point $(\tau,\chi_c(\tau))$ should be satisfied the following:

$$\begin{cases} x'(t)=H'(p(t)),\\ p'(t)=0\\ y'(t)=-H(p(t))+p(t)H'(p(t)),\end{cases}$$

with the initial condition

$$x(\tau)=\chi_c(\tau),\ y(\tau)=y^+(\tau,\chi_c(\tau)) \text{ and } p(\tau)=\psi(y_x^+(\tau,\chi_c(\tau))).$$

So the solution is exactly given as follows:

$$\begin{cases} \tilde{x}(t)=&\chi_c(\tau)+(t-\tau)H'(\psi(y_x^+(\tau,\chi_c(\tau)))),\\ \tilde{p}(t)=&\psi(y_x^+(\tau,\chi_c(\tau)))\\ \tilde{y}(t)=&y^+(\tau,\chi_c(\tau))\\ &+(t-\tau)\{-H(\psi(y_x^+(\tau,\chi_c(\tau))))+\psi(y_x^+(\tau,\chi_c(\tau)))H'(\psi(y_x^+(\tau,\chi_c(\tau))))\}.\end{cases}$$

By definition of the contact discontinuity, we have

$$\chi_c''(t)=H''(\psi(\phi(u_+(t))\frac{\partial\psi}{\partial p}(\phi'(u_+(t))\phi''(u_+(t))u_+'(t),$$

where $\chi_c(t)=u_+(t)+tH'(\phi(u_+(t))$. Since $\frac{\partial\psi}{\partial p}=\frac{H'(p)-H'(q)}{H''(q)(p-q)}$, we have

$$\chi_c''(t)=\frac{H'(\phi'(u_+(t))-H'(\psi(\phi'(u_+(t))))}{\phi'(u_+(t))-\psi(\phi'(u_+(t)))}\phi''(u_+(t))u_+'(t).$$

We also have

$$\chi'(t)=u_+'(t)\{1+tH''(\phi'(u_+(t))\phi''(u_+(t))\}+H'(\phi'(u_+(t))).$$

It follows that

$$\chi_c''(t)=-\frac{(H'(\phi'(u_+(t)))-H'(\psi(\phi'(u_+(t)))))^2}{\phi'(u_+(t))-\psi(\phi'(u_+(t)))}\frac{\phi''(u_+(t))}{1+tH''(\phi'(u_+(t))\phi''(u_+(t))}.$$

Since

$$\frac{\partial x}{\partial u}(t,u_+(t))=1+tH''(\phi'(u_+(t)))\phi''(u_+(t)),$$

we may assume that $1+tH''(\phi'(u_+(t)))\phi''(u_+(t))>0$. So $\chi_c(t)$ is convex if and only if $\phi''(u_+(t))>0$. We suppose that $\phi''(u_+(t))\le 0$ and denote $\chi_c(t)=u_+(t)+tH'(\phi(u_+(t))=u_-(t)+tH'(\phi(u_-(t)))$, where $u_-(t)$ (resp. $u_+(t)$) is the point corresponding to the characteristic from the right (resp. left) side of $(t,\chi_c(t))$. We distinguish two cases as follows:

b-1) If $\phi''(u_-(t)) > 0$, then ϕ' is monotone. Since $u'_-(t) < 0$, $\phi'_-(u_(t))$ moves to the left direction, so that the viscosity criterion is satisfied across χ.
b-2) If $\phi''(u_-(t)) < 0$ and the viscosity criterion is violated across χ for $t > t_\alpha$, then $1 + tH''(\phi'(u_-(t)))\phi''(u_-(t)) > 0$ near t_α. Differentiate the equality $\chi_c(t) = u_-(t) + tH'(\phi(u_-(t))$ with respect to t, then we have

$$\chi'(t) - H'(\phi'(u_-(t))) = \{1 + tH''(\phi'(u_-(t)))\phi''(u_-(t))\}u'_-(t).$$

Since

$$\chi'(t) = \frac{H(\phi'(u_+(t))) - H(\phi'(u_-(t)))}{\phi'(u_+(t)) - \phi'(u_-(t))} > H'(\phi'(u_-(t))),$$

we have $u'_-(t) > 0$, so that $u_-(t)$ is increase, which is a contradiction.

Hence, if the viscosity criterion is violated for $t > t_\alpha$, the contact discontinuity curve χ is convex and the viscosity solution can be constructed. We draw the picture which is illustrating the situations as follows :

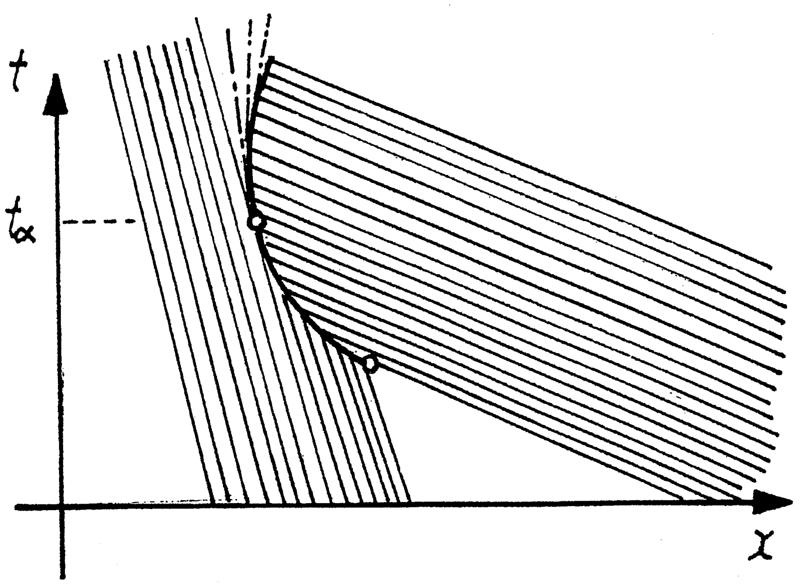

FIGURE 5

Then we can draw the picture of the graph of the viscosity solution for $t > t_\alpha$ and the shock curve around t_α.

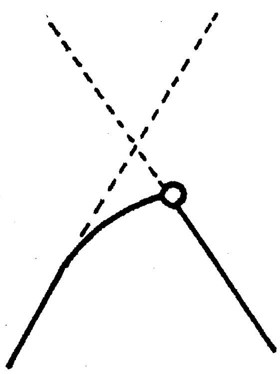

FIGURE 6

5. Big Ray tracing:the Benamou's project

Consider the following Helmholtz equation

$$\Delta u(x,z) + k^2\eta^2(z)u(x,z) = 0,$$

where $\eta(z)$ is a piecewise smooth continuous function. This equation appears in the theorey of underwateracoustics and seismology. The orresponding eikonal euation is

$$\left(\frac{\partial u}{\partial x}(x,z)\right)^2 + \left(\frac{\partial u}{\partial z}(x,z)\right)^2 - \eta^2(z) = 0.$$

Here, we consider the point source case. The source point is $(z_0, 0) \in \mathbb{R}^2$. The classical ray tracing is the integaration of the ray eaquation (i.e., characteristic equation) for the Hamiltonian function

$$H(x,z,p,q) = \frac{1}{2}\left\{p^2 + q^2 - \eta^2(z)\right\}$$

which is an ordinary differential equation:

$$\frac{dx}{d\tau} = p, \ \frac{dz}{d\tau} = q, \ \frac{dp}{d\tau} = 0, \ \frac{dq}{d\tau} = \eta(z)\eta'(z)$$

with the initial data

$$x(0) = 0, \ z(0) = z_0, \ p(0) = \eta(z_0)\cos\theta, \ q(0) = \eta(z_0)\sin\theta.$$

Therefore, we have the solution of the ray equation of the form

$$x(\tau,\theta) = \eta(z_0)\cos\theta\tau, \ z(\tau,\theta) = z(\tau,\theta), \ p(\tau,\theta) = \eta(z_0)\cos\theta, \ q(\tau,\theta) = q(\tau,\theta).$$

By allowing θ to vary and computing a (necessarily finite) number of corresponding ray, we want to cover the region as besta as possible (in order to compute the taravel time etc.) In the classical results, an interpolation process has to be used. However, for hetrogeneous media (i.e., $\eta(z) \neq$ constant), this process may be difficult by the following reason:

(a) zones where few rays enter appear (low density zone) (cf., Fig??)

(b) zones with complex multivalued travel tie fields appear (different rays cross) (cf., Fif??).

An alternative method for the ray tracing propsing by Benamou[3] is to solve the eikonal equation directly by finite difference or finite element schees (i.e., the eikonal solver). These scheme, however, only compute a single valued viscosity soloutions.

The algorithm given by Benamou is as follows:

(1) Shoot a given number of rays, say M, in regularly spaced directions. We denote these by $(R_i)_{i=1,\dots,M}$ and call this step the *ray sicretization.*

(2) Define around eah ray R_i a local domain Ω_i, also called a *big ray.*

(3) Compute the viscosity solution of the eikonal equation on each Ω_i.

The difficulty lies in step (2). Ω_i hae to satisfy two conflicting properties:

a) They have to be big enough to cover the domain.

b) They have to be small enough so that they do not contains several rays whih intersect.

In [3] Benamou preseted an example as follows: He considered the case when the graph of teh velocity index $\eta(z)$ is depicted in Fig. 7. He used a third-order Runge-Kutta algorithm to integarate the ray equations. We first shoot 200 rays (Fi. 8), and the 100 rays (Fig. 9). Here, we only put the pictures of Big rays and Travel times given by Benamou[3] in the remaining pages.

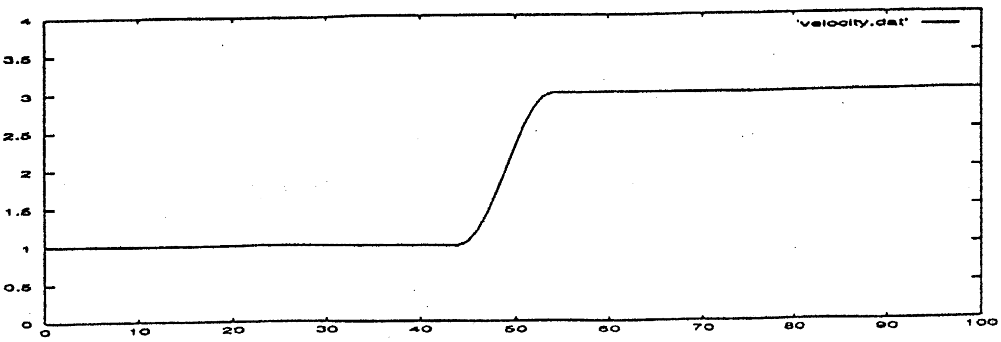

FIG. 7 Velocity profile; the horizontal axis is z.

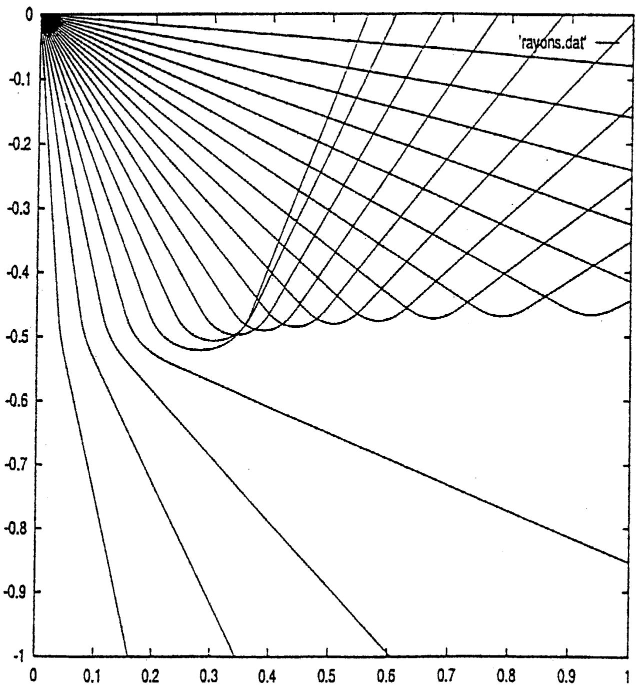

FIG. 8 Twenty rays shot with regularly spaced initial directions.

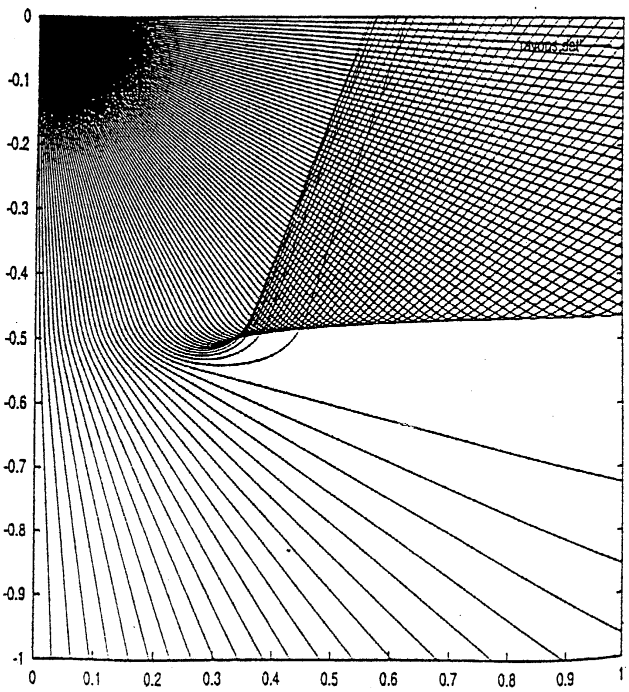

FIG. 9 One hundred rays shot with regularly spaced initial directions.

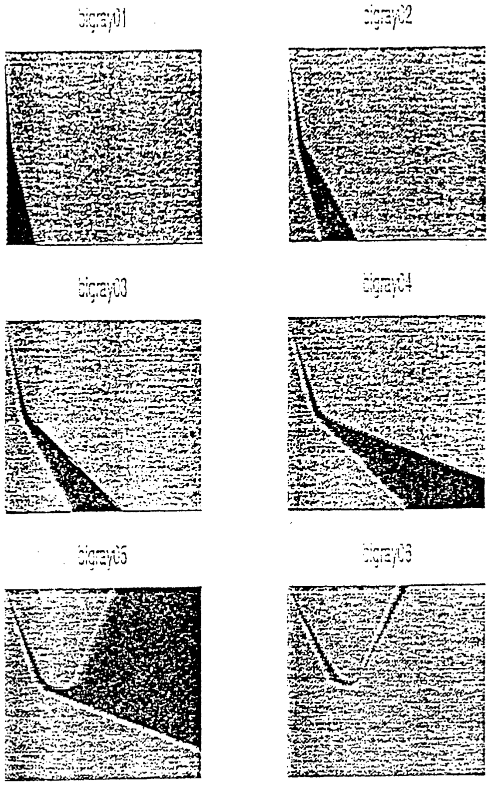

FIG. 10 Big rays: in white the two rays around which the big ray envelope is generated (in black).

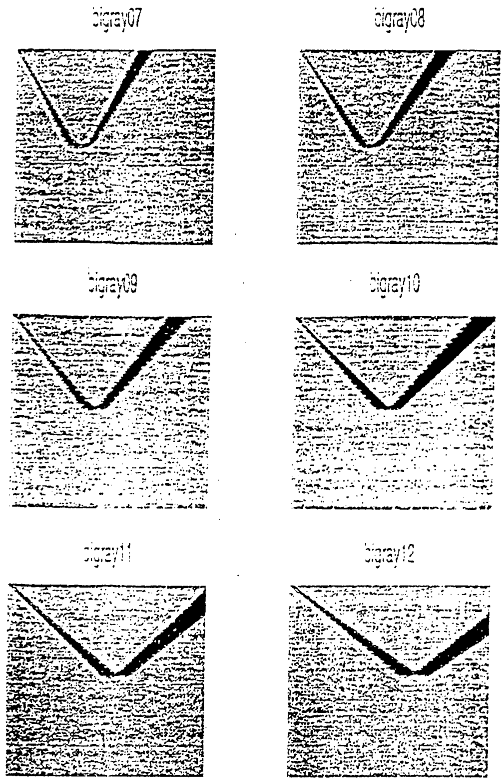

Fig 11 Big rays: in white the two rays around which the big ray envelope is generated (in black).

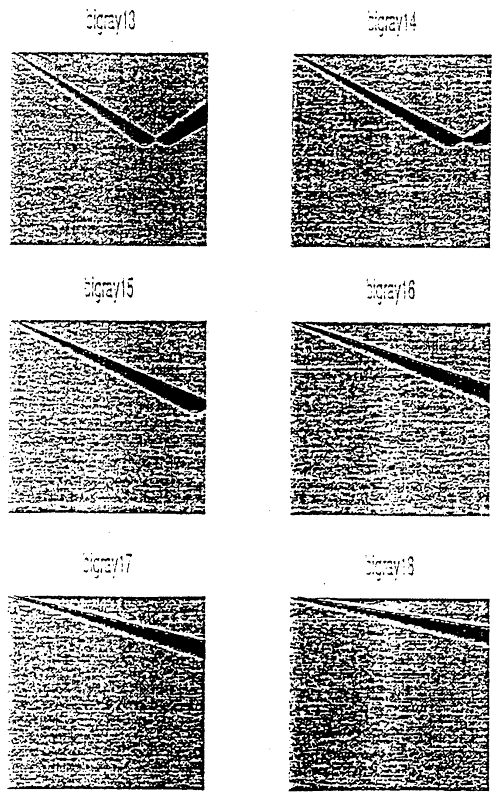

FIG. 12 Big rays: in white the two rays around which the big ray envelope is generated (in black).

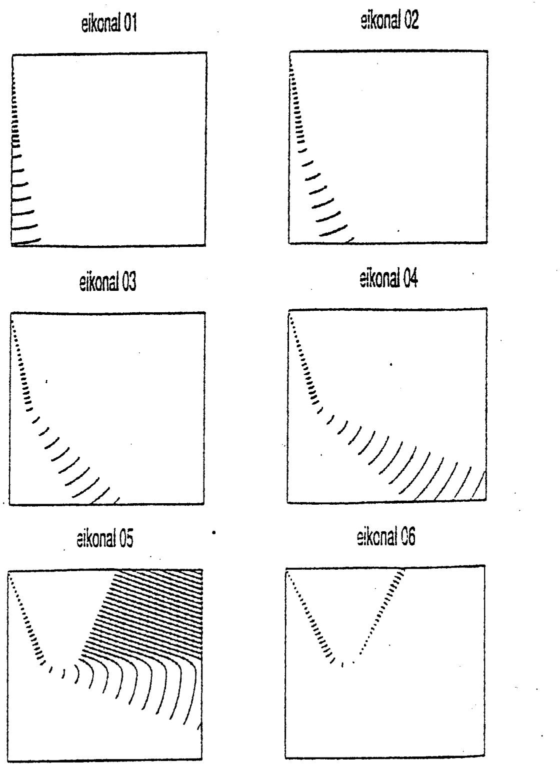

FIG. 13 Travel times computed in each big ray. Contour lines every 0.025 s.

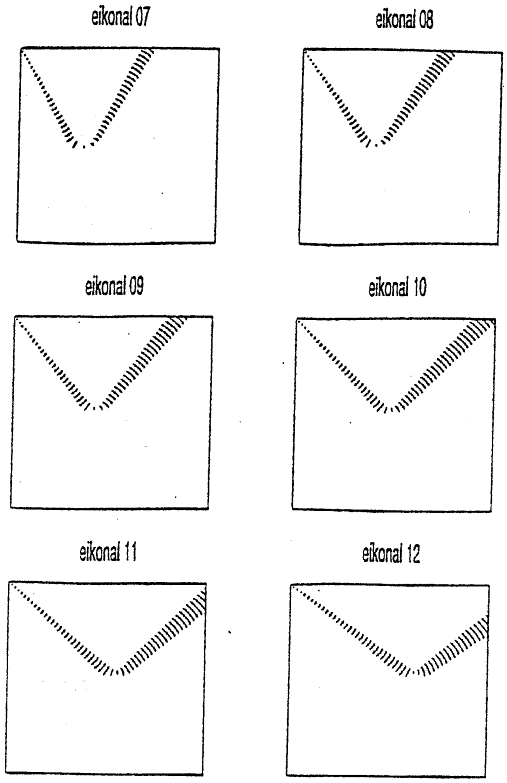

FIG. 14 Travel times computed in each big ray. Contour lines every 0.025 s.

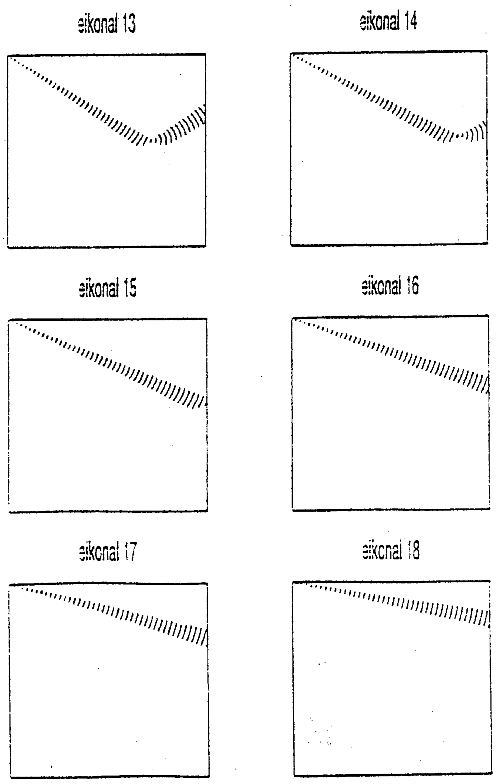

FIG. 15 Travel times computed in each big ray. Contour lines every 0.025 s.

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