数理解析研究所講究録
1112 巻 1999 年 13-16

SUFFICIENT CONDITIONS FOR MEROMORPHIC STARLIKENESS AND CLOSE-TO-CONVEXITY OF ORDER α

*NAK EUN CHO AND **SHIGEYOSHI OWA

ABSTRACT. The object of the present paper is to drive a property of certain meromorphic functions in the punctured unit disk. Our main theorem contains certain sufficient conditions for starlikeness and close-to-convexity of order α of meromorphic functions.

1. Introduction

Let Σ denote the class of functions of the form

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n$$

which are analytic in the punctured unit disk $\mathcal{D}=\{z:0<|z|<1\}$. A function $f\in\Sigma$ is said to be meromorphic starlike of order α if it satisfies

$$-\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha \quad (z\in\mathcal{U}=\mathcal{D}-\{0\})$$

for some $\alpha(0\leq\alpha<1)$. We denote $\Sigma^*(\alpha)$ the class of all meromorphic starlike functions of order α.

Let $MC(\alpha)$ be the subclass of Σ consisting of functions f which satisfy

$$-\text{Re}\{z^2f'(z)\}>\alpha \quad (z\in\mathcal{U})$$

for some $\alpha(0\leq\alpha<1)$. A function f in $MC(\alpha)$ is meromorphic close-to-convex of order α in $\mathcal{D}$ [1].

1991 Mathematics Subject Classification : 30C45.
Key words and phrases. meromorphic starlike of order α, meromorphic convex of order α.

2. Main result

In proving our main theorem, we need the following lemma due to Owa, Nunokawa, Saitoh and Fukui [2].

Lemma 2.1. *Let p be analytic in $\mathcal{U}$ with $p(0) = 1$. Suppose that there exists a point $z_0 \in \mathcal{U}$ such that $Rep(z) > 0(|z| < |z_0|)$, $Rep(z_0) = 0$, and $p(z) \neq 0$. Then we have $p(z) = ia(a \neq 0)$ and*

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\frac{k}{2}(a + \frac{1}{a}),$$

k is a real number with $k \geq 1$.

With the aid of above Lemma 2.1, we drive

Theorem 2.1. *If $f \in \Sigma$ satisfies $f(z)f'(z) \neq 0$ in $\mathcal{D}$ and*

$$\operatorname{Re}\left\{\alpha\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)}\right\} < 2(2-\alpha) - \beta \quad (z \in \mathcal{U}),$$

then

$$-\operatorname{Re}\left\{\frac{z^{2-\alpha}f'(z)}{f^{\alpha}(z)}\right\} > \frac{1}{1 + 2(2-\alpha) - 2\beta} \quad (z \in \mathcal{U}),$$

where $\alpha \leq 2$ and $\frac{2(2-\alpha)-1}{2} \leq \beta < 2 - \alpha$.

Proof. We define the function p in U by

$$-\frac{z^{2-\alpha}f'(z)}{f^{\alpha}(z)} = \gamma + (1-\gamma)p(z)$$

with $\gamma = \frac{1}{1+2(2-\alpha)-2\beta}$. Then p is analytic in $\mathcal{U}$ with $p(0) = 1$ and

$$\alpha\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} = 2 - \alpha - \frac{(1-\gamma)zp'(z)}{\gamma + (1-\gamma)p(z)}.$$

Suppose that there exists a point $z_0 \in \mathcal{U}$ such that

$$\operatorname{Re}p(z) > 0(|z| < |z_0|), \operatorname{Re}p(z_0) = 0, \quad \text{and} \quad p(z) \neq 0.$$

Then, applying Lemma 2.1, we have $p(z) = ia(a \neq 0)$ and

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\frac{k}{2}(a + \frac{1}{a}) \quad (k \geq 1).$$

It follows from this that

$$\begin{aligned}\alpha\frac{z_0 f'(z_0)}{f(z_0)} - \frac{z_0 f''(z_0)}{f'(z_0)} &= 2-\alpha-\frac{(1-\gamma)z_0 p'(z_0)}{\gamma+(1-\gamma)p(z_0)}\\ &= 2-\alpha+\frac{k(1-\gamma)(1+a^2)}{2(\gamma+i(1-\gamma)a)}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\mathrm{Re}\left\{\alpha\frac{z_0 f'(z_0)}{f(z_0)} - \frac{z_0 f''(z_0)}{f'(z_0)}\right\} &= 2-\alpha+\frac{k(1-\gamma)(1+a^2)}{2(\gamma^2+(1-\gamma)^2a^2)}\\ &\geq 2-\alpha+\frac{k(1-\gamma)}{2\gamma}\\ &\geq 2(2-\alpha)-\beta.\end{aligned}$$

This contracts our assumption. Thus, we conclude that $\mathrm{Re}p(z) > 0$ for all $z \in \mathcal{U}$, that is, that

$$-\mathrm{Re}\left\{\frac{z^{2-\alpha}f'(z)}{f^{\alpha}(z)}\right\} > \gamma = \frac{1}{1+2(2-\alpha)-2\beta} \quad (z \in U).$$

Putting $\beta = \frac{2(2-\alpha)-1}{2}$ in Theorem 2.1, we have

Corollary 2.1. *If $f \in \Sigma$ satisfies $f(z)f'(z) \neq 0$ in$\mathcal{D}$ and*

$$\mathrm{Re}\left\{\alpha\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)}\right\} < \frac{3}{2}-\alpha \quad (z \in \mathcal{U}),$$

then

$$-\mathrm{Re}\left\{\frac{z^{2-\alpha}f'(z)}{f^{\alpha}(z)}\right\} > \frac{1}{2} \quad (z \in \mathcal{U}),$$

where $\alpha \leq 2$.

Taking $\alpha = 1$ in Theorem 2.1, we have

Corollary 2.2. *If $f \in \Sigma$ satisfies $f(z)f'(z) \neq 0$ in $\mathcal{D}$ and*

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)}\right\} < 2-\beta \quad (z \in \mathcal{U}),$$

then

$$-\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1}{3-2\beta} \quad (z \in \mathcal{U}),$$

that is, $f \in \Sigma^*\left(\frac{1}{3-2\beta}\right)$, *where* $\frac{1}{2} \leq \beta < 1$.

Further, letting $\alpha = 0$ in Theorem, we have

Corollary 2.3. *If* $f \in \Sigma$ *satisfies* $f(z)f'(z) \neq 0$ *in* $\mathcal{D}$ *and*

$$-\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} < 4-\beta \quad (z \in \mathcal{U}),$$

then

$$-\mathrm{Re}\left\{z^2 f'(z)\right\} > 5-2\beta \quad (z \in \mathcal{U}),$$

that is, $f \in MC\left(\frac{1}{5-2\beta}\right)$, *where* $\frac{3}{2} \leq \beta < 2$.

Acknowledgement. This work was partially supported by the Korea Research Foundation(Project No.: 1998-015-D00039).

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Department of Applied Mathematics, Pukyong National University, Pusan 608-737, Korea

**Department of Mathematics, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan