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The growth theorem of spirallike mappings in several complex variables

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Abstract

Let $\mathbf{B}$ be the unit ball in an arbitrary complex Banach space X. Let $\alpha \in \mathbf{R}$, $|\alpha| < \pi/2$. First, we give the growth theorem for normalized spirallike mappings of type α on $\mathbf{B}$. We also show that the growth theorem does not hold for normalized spirallike mappings defined by Suffridge. Next, we give an alternate characterization of normalized spirallike mappings of type α on $\mathbf{B}$ in terms of subordination chains, when the dimension of X is finite.

1 Introduction

Let f be a univalent mapping in the unit disc Δ with $f(0) = 0$ and $f'(0) = 1$. Then the classical growth theorem is as follows:

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}.$$

It is well known that the above growth theorem cannot be generalized to normalized biholomorphic mappings on the Euclidean unit ball $\mathbf{B}^n$ in $\mathbf{C}^n$ $(n \geq 2)$. Barnard, FitzGerald and Gong [1] and Chuaqui [2] extended the above growth theorem to normalized starlike mappings on $\mathbf{B}^n$. Dong and Zhang [3] generalized the above result to normalized starlike mappings on the unit ball in complex Banach spaces.

In this paper, we will generalize the above growth theorem to spirallike mappings of type α on the unit ball $\mathbf{B}$ in an arbitrary complex Banach space. One might consider that the same growth theorem holds for all normalized spirallike mappings defined by Suffridge [12]. However, we can give an example of a normalized spirallike mapping such that the same growth theorem does not hold. This

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example also shows that the growth of normalized spirallike mappings cannot be estimated from above.

We also give an alternate characterization of normalized spirallike mappings of type α on the unit ball $\mathbf{B}$ with respect to an arbitrary norm on $\mathbf{C}^n$ in terms of subordination chains.

2 Introduction

For complex Banach spaces X, Y, let $\mathcal{L}(X,Y)$ be the space of all continuous linear operators from X into Y with the standard operator norm. By I we denote the identity in $\mathcal{L}(X,X)$. Let G be a domain in X and let $f : G \to Y$. f is said to be holomorphic on G, if for any $z \in G$, there exists a $Df(z) \in \mathcal{L}(X,Y)$ such that

$$\lim_{h\to 0} \frac{\|f(z+h) - f(z) - Df(z)h\|}{\|h\|} = 0.$$

Let $\mathcal{H}(G)$ be the set of holomorphic mappings from a domain $G \subset X$ into X.

A mapping $f \in \mathcal{H}(G)$ is said to be locally biholomorphic on G if its Fréchet derivative $Df(z)$ as an element of $\mathcal{L}(X,X)$ is nonsingular at each $z \in G$. Let $\mathbf{B}$ denote the unit ball with respect to the norm $\|\cdot\|$ on X. A mapping $f \in \mathcal{H}(\mathbf{B})$ is said to be normalized if $f(0) = 0$ and $Df(0) = I$. For each $z \in X \setminus \{0\}$, we define

$$T(z) = \{z^* \in \mathcal{L}(X, \mathbf{C}) : \|z^*\| = 1, z^*(z) = \|z\|\}.$$

By the Hahn-Banach theorem, $T(z)$ is nonempty. Let

$$\mathcal{N} = \Big\{g \in \mathcal{H}(\mathbf{B}) \,:\, g(0) = 0,\ \Re z^*(g(z)) > 0 \text{ for all } z \in \mathbf{B}\backslash\{0\}, z^* \in T(z)\Big\},$$

and also, let

$$\mathcal{M} = \{g \in \mathcal{N} \,:\, Dg(0) = I\}.$$

The following definition generalizes the notion of spirallike functions of type α on the unit disc to $\mathbf{B}$.

Definition 2.1 Let $f : \mathbf{B} \to X$ be a normalized biholomorphic mapping on $\mathbf{B}$. Let $\alpha \in \mathbf{R}$, $|\alpha| < \pi/2$. We say that f is a spirallike mapping of type α if the spiral $\exp(-e^{-i\alpha}t)f(z)$ $(t \geq 0)$ is contained in $f(\mathbf{B})$ for any $z \in \mathbf{B}$.

We obtain the following theorem from Corollary 2 and Theorem 6 of Gurganus [4] (cf. [6], see also [7]).

Theorem 2.1 *Let f be a normalized locally biholomorphic mapping on* $\mathbf{B}$. *If f is a spirallike mapping of type α, then $e^{-i\alpha}[Df(z)]^{-1}(f(z)) \in \mathcal{N}$. Moreover, when X is a finite dimensional complex Banach space, f is a spirallike mapping of type α if and only if $e^{-i\alpha}[Df(z)]^{-1}(f(z)) \in \mathcal{N}$.*

Remark 2.1 In Lemma 5 of Gurganus [4], he claimed that for each $h \in \mathcal{N}$ and for each $x \in \mathbf{B}$, the initial value problem

$$\frac{\partial v}{\partial t} = -h(v), \quad v(0) = x,$$

has a unique solution $v(t)$ for all $t \geq 0$. For the proof, he uses Theorem 2.1 of Pfaltzgraff [9]. One of the conditions on h in the theorem is that for each $r \in (0,1)$, there exists a constant $K(r)$ such that $\|h(z)\| \leq K(r)$ for all z with $\|z\| \leq r$. However, in general, holomorphic mappings on the unit ball is not necessarily bounded on $\|z\| \leq r$. We do not know whether the above condition is satisfied for all $h \in \mathcal{N}$ or not. So, we restrict ourselves to the finite dimensional case in Theorem 2.1.

The following definition is due to Suffridge [12].

Definition 2.2 Let $f : \mathbf{B} \to X$ be a normalized biholomorphic mapping. Let $A \in \mathcal{L}(X, X)$ such that

$$\inf\{\Re z^*(A(z)) : \|z\| = 1, z^* \in T(z)\} > 0.$$

We say that f is spirallike relative to A if $e^{-tA}f(\mathbf{B}) \subset f(\mathbf{B})$ for all $t \geq 0$, where

$$e^{-tA} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^k A^k.$$

Let $\mathbf{B}$ denote the unit ball with respect to an arbitrary norm $\|\cdot\|$ on $\mathbf{C}^n$. A mapping $v \in \mathcal{H}(\mathbf{B})$ is called a Schwarz mapping if $\|v(z)\| \leq \|z\|$ for all $z \in \mathbf{B}$. This condition is equivalent to the condition that $v(0) = 0$ and $\|v(z)\| \leq 1$ for $z \in \mathbf{B}$.

If $f, g \in \mathcal{H}(\mathbf{B})$, we say that f is subordinate to g $(f \prec g)$ if there exists a Schwarz mapping $v \in \mathcal{H}(\mathbf{B})$ such that $f(z) = g(v(z))$ for $z \in \mathbf{B}$.

Let $\{f(z,t)\}_{t \geq 0}$ be a family of mappings such that $f_t(z) = f(z,t) \in \mathcal{H}(\mathbf{B})$ and $f_t(0) = 0$ for each $t \geq 0$. We call $\{f(z,t)\}$ a subordination chain if $f(z,s) \prec f(z,t)$ for all $z \in \mathbf{B}$ and $0 \leq s \leq t$. Moreover, $f(z,t)$ is called univalent if $f(\cdot, t)$ is univalent on $\mathbf{B}$ for each $t \geq 0$.

3 The growth theorem

In this section, we will prove the following theorem.

Theorem 3.1 *Let f be a normalized spirallike mapping of type α from* $\mathbf{B}$ *to X. Then*

$$\frac{\|z\|}{(1+\|z\|)^2} \le \|f(z)\| \le \frac{\|z\|}{(1-\|z\|)^2}.$$

Proof. Let

$$h(z) = e^{-i\alpha}[Df(z)]^{-1}f(z).$$

Since $h \in \mathcal{N}$, we obtain the following inequalities from Lemma 4 of Gurganus [4].

$$\cos\alpha\|z\|\frac{1-\|z\|}{1+\|z\|} \le \Re z^*(h(z)) \le \|z\|\frac{1+\|z\|}{1-\|z\|}\cos\alpha \tag{3.1}$$

for $z \in \mathbf{B} \setminus \{0\}$, $z^* \in T(z)$. Let $0 < r_1 < r_2 < 1$. Let z_2 be a point such that $\|z_2\| = r_2$. The curve $c(t) = \exp(-e^{-i\alpha}t)f(z_2)$ is contained in $f(\mathbf{B})$ for all $t \ge 0$. Also $c(t) \to 0$ as $t \to \infty$. Since f is biholomorphic, the curve $v(t) = f^{-1}(c(t))$ is well-defined and intersects the sphere $\|z\| = r_1$ at some point $z_1 = f^{-1}(c(t_1))$. Since

$$\frac{\partial v}{\partial t} = -h(v),$$

we can show that $\|v(t)\|$ is absolutely continuous. Therefore, $\|v(t)\|$ is differentiable a.e. on $[0,\infty)$ and

$$\frac{\partial\|v\|}{\partial t} = \Re v^*\left(\frac{\partial v}{\partial t}\right)$$

for $v^* \in T(v(t))$ a.e. on $[0,\infty)$ by Lemma 1.3 of Kato [8]. Then

$$\frac{\partial\|v\|}{\partial t} = -\Re v^*\left(h(v)\right) < 0 \tag{3.2}$$

for $v^* \in T(v(t))$. Let $F(t) = \|f(v(t))\| = e^{-t\cos\alpha}\|f(z_2)\|$. Then we have

$$\begin{aligned}\frac{1+\|v(t)\|}{\|v(t)\|(1-\|v(t)\|)}\frac{\partial\|v\|}{\partial t} &\le \frac{1}{F}\frac{dF}{dt} = -\cos\alpha \\ &\le \frac{1-\|v(t)\|}{\|v(t)\|(1+\|v(t)\|)}\frac{\partial\|v\|}{\partial t}\end{aligned}$$

from (3.1) and (3.2). Since $\|v(t)\|$ is strictly decreasing on $[0,t_1]$ by (3.2), we have

$$\begin{aligned}\log F(t_1)-\log F(0) &\geq \int_0^{t_1}\frac{1+\|v(t)\|}{\|v(t)\|(1-\|v(t)\|)}\frac{\partial\|v\|}{\partial t}dt\\ &= \int_{\|v(0)\|}^{\|v(t_1)\|}\frac{1+x}{x(1-x)}dx\\ &= \log\|v(t_1)\|-2\log(1-\|v(t_1)\|)\\ &\quad -\{\log\|v(0)\|-2\log(1-\|v(0)\|)\}\end{aligned}$$

and

$$\begin{aligned}\log F(t_1)-\log F(0) &\leq \log\|v(t_1)\|-2\log(1+\|v(t_1)\|)\\ &\quad -\{\log\|v(0)\|-2\log(1+\|v(0)\|)\}.\end{aligned}$$

Then

$$\frac{(1-\|v(0)\|)^2}{\|v(0)\|(1-\|v(t_1)\|)^2}F(0)\leq\frac{F(t_1)}{\|v(t_1)\|}\leq\frac{(1+\|v(0)\|)^2}{\|v(0)\|(1+\|v(t_1)\|)^2}F(0).$$

Namely,

$$\frac{(1-\|z_2\|)^2}{\|z_2\|(1-\|v(t_1)\|)^2}\|f(z_2)\|\leq\frac{\|f(v(t_1))\|}{\|v(t_1)\|}\leq\frac{(1+\|z_2\|)^2}{\|z_2\|(1+\|v(t_1)\|)^2}\|f(z_2)\|. \tag{3.3}$$

Letting $r_1\to 0$, we obtain that

$$\frac{(1-\|z_2\|)^2}{\|z_2\|}\|f(z_2)\|\leq 1\leq\frac{(1+\|z_2\|)^2}{\|z_2\|}\|f(z_2)\|,$$

since

$$\lim_{z\to 0}\frac{\|f(z)\|}{\|z\|}=\lim_{z\to 0}\frac{\|Df(0)z\|}{\|z\|}=1.$$

This completes the proof.

When $\alpha=0$, we obtain the growth theorem of normalized starlike mappings on the unit ball in complex Banach spaces [3] (cf. [1],[2]).

Let

$$M_\infty(r,f)=\sup_{\|z\|=r}\|f(z)\|.$$

Then we obtain the following corollary (cf. Tsurumi [13]) from (3.3) and Theorem 3.1.

Corollary 3.1 *Let f be a normalized spirallike mapping of type α from* $\mathbf{B}$ *to* X. *Then the limit*

$$\beta = \lim_{r\to 1}(1-r)^2 M_\infty(r,f)$$

exists. Moreover, we have $0 \le \beta \le 1$.

Example 3.1 Consider the holomorphic mapping $f(z_1, z_2) = (z_1 + az_2^2, z_2)$ on the Euclidean unit ball in $\mathbf{C}^2$. Let A be a linear mapping such that $A(z_1, z_2) = (2z_1, z_2)$. Then $[Df(z)]^{-1}Af(z_1, z_2) = (2z_1, z_2)$. Therefore, f is a normalized spirallike mapping relative to A for any $a \in \mathbf{C}$. Let $a \in \mathbf{R}$ and $a > 2\sqrt{15}$. Let $z^0 = (0, 1/2)$. Then $f(z^0) = (a/4, 1/2)$ and $\|f(z^0)\| > 2$. On the other hand,

$$\frac{\|z^0\|}{(1-\|z^0\|)^2} = 2.$$

Therefore,

$$\|f(z^0)\| > \frac{\|z^0\|}{(1-\|z^0\|)^2}.$$

Also, $\|f(z^0)\| \to \infty$ as $a \to \infty$. Therefore, the growth of normalized spirallike mappings cannot be estimated from above.

4 Subordination chains

In this section, we will give an alternate characterization of normalized spirallike mappings of type α on $\mathbf{B}$ in terms of subordination chains, where $\mathbf{B}$ is the unit ball in $\mathbf{C}^n$ with respect to an arbitrary norm on $\mathbf{C}^n$.

Let $f : \mathbf{B} \to \mathbf{C}^n$ be a holomorphic mapping on $\mathbf{B}$ and let $\alpha \in \mathbf{R}$, $|\alpha| < \pi/2$.

Let

$$f(z,t) = e^{(1-ia)t} f(e^{iat}z), \quad z \in \mathbf{B}, \ t \ge 0,$$

where $a = \tan\alpha$. Then, we have the following theorem(cf. [11, Theorem 6.6], [5, Theorem 2.4]).

Theorem 4.1 *Let $f : \mathbf{B} \to \mathbf{C}^n$ be a normalized locally biholomorphic mapping on* $\mathbf{B}$ *and let $\alpha \in \mathbf{R}$, $|\alpha| < \pi/2$. Then $\{f(z,t)\}$ is a univalent subordination chain if and only if f is a spirallike mapping of type α.*

Proof. First, assume that f is a spirallike mapping of type α. Then f_t is univalent. It is easy to check that $f_t(z) \in \mathcal{H}(\mathbf{B})$, $Df_t(0) = e^t I$, $t \ge 0$ and also,

$f(z,t)$ satisfies the absolute continuity hypothesis of Theorem 2.2 of Pfaltzgraff [9].

On the other hand, we have

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)g(z,t), \ \ z \in \mathbf{B}, \ \ t \geq 0, \tag{4.1}$$

where

$$g(z,t) = iaz + (1-ia)e^{-iat}[Df(e^{iat}z)]^{-1}f(e^{iat}z).$$

Clearly $g(z,t)$ is a measurable function for each $z \in \mathbf{B}$, $g(0,t) = 0$ and $Dg(0,t) = I$. For any $z \in \mathbf{B} \setminus \{0\}$, $z^* \in T(z)$, we have

$$\begin{aligned} \Re z^*(g(z,t)) &= \Re(ia\|z\|) + \frac{1}{\cos\alpha}\Re\left(e^{-i\alpha}e^{-iat}z^*[Df(e^{iat}z)]^{-1}f(e^{iat}z)\right) \\ &> 0, \end{aligned}$$

since f is a spirallike mapping of type α and $e^{-iat}z^* \in T(e^{iat}z)$. Therefore $g_t(z) \in \mathcal{M}$. It is easy to show that, for each $r \in (0,1)$, there exists a constant $M = M(r) > 0$ such that

$$\|g(z,t)\| \leq M(r),$$

for all $z \in \overline{\mathbf{B}}_r$ and $t \geq 0$.

Let $t_m = m$ if $a = 0$ and $t_m = 2\pi m/a$ if $a \neq 0$. Then $e^{-t_m}f(z,t_m) = f(z)$ holds for any $m \in \mathbf{Z}$.

Hence, from Theorem 2.2 of Pfaltzgraff [9], it follows that $\{f(z,t)\}$ is a subordination chain.

Conversely, assume that $\{f(z,t)\}$ is a univalent subordination chain. Then there exist Schwarz mappings $v(z,s,t)$ such that

$$f(z,s) = f(v(z,s,t),t), \ \ z \in \mathbf{B}, \ \ t \geq s \geq 0.$$

Then $v(z,s,t) = e^{-iat}f^{-1}(e^{(1-ia)(s-t)}f(e^{ias}z))$ and therefore $v(z,s,t)$ is differentiable with respect to t for each $z \in \mathbf{B}$, $s \geq 0$ and $t \geq s$. Hence, differentiating this equality with respect to t, we obtain that

$$Df(v(z,s,t),t)\frac{\partial v}{\partial t}(z,s,t) + \frac{\partial f}{\partial t}(v(z,s,t),t) = 0, \ \ z \in \mathbf{B}, \ \ t \geq s \geq 0. \tag{4.2}$$

If we compare the relations (4.1) and (4.2) and use the univalence of $f(\cdot,t)$ for $t \geq 0$, we obtain that

$$g(v(z,s,t),t) = -\frac{\partial v}{\partial t}(z,s,t), \ \ z \in \mathbf{B}, \ \ t \geq s \geq 0. \tag{4.3}$$

Since $v(z,s,t)$ is a Schwarz mapping, we obtain that

$$\Re z^*(v(z,s,t)) \le \|v(z,s,t)\| \le \|z\|$$

for all $z \in \mathbf{B}\setminus\{0\}$, $t \ge s$ and $z^* \in T(z)$.

Therefore,

$$\begin{aligned}\Re z^*\left(\frac{\partial v}{\partial t}(z,0,0)\right) &= \lim_{t\to 0+} \Re z^*\left(\frac{v(z,0,t)-z}{t}\right)\\ &\le \lim_{t\to 0+} \frac{\|z\|-\|z\|}{t}\\ &= 0.\end{aligned}$$

Now, from (4.3) and the above inequality, we obtain that

$$\Re\Big[z^*\left(e^{-i\alpha}[Df(z)]^{-1}f(z)\right)\Big] \ge 0, \ \ z \in \mathbf{B}\backslash\{0\}, z^* \in T(z).$$

Moreover, for fixed $z \in \partial\mathbf{B}$, $0 < r < 1$, $z^* \in T(rz)$, let

$$\phi(\zeta) = \Re\left[z^*\left(e^{-i\alpha}\frac{[Df(\zeta z)]^{-1}f(\zeta z)}{\zeta}\right)\right]$$

for $\zeta \in \Delta$, where Δ denotes the unit disc in $\mathbf{C}$. Then ϕ is harmonic on Δ. Since $(|\zeta|/\zeta)z^* \in T(\zeta z)$, $|\zeta|\phi(\zeta) \ge 0$ for $\zeta \in \Delta$. Since

$$\phi(0) = \Re\Big[z^*\left(e^{-i\alpha}z\right)\Big] = \Re\Big[e^{-i\alpha}\|z\|\Big] > 0,$$

we have $\phi(\zeta) > 0$ for all $\zeta \in \Delta$ by the minimum principle for harmonic functions. Considering $\phi(r)$, we obtain that f is a spirallike mapping of type α from Theorem 2.1. This completes the proof.

If we consider the case $\alpha = 0$ in Theorem 4.1, we obtain the following result due to Pfaltzgraff and Suffridge [10, Corollary 2].

Corollary 4.1 *Let $f : \mathbf{B} \to \mathbf{C}^n$ be a normalized locally biholomorphic mapping on $\mathbf{B}$. Then f is starlike if and only if $\{e^t f(z)\}$ is a univalent subordination chain.*

Remark 4.1 Let D be a bounded balanced pseudoconvex domain with C^1 plurisubharmonic defining functions in $\mathbf{C}^n$. Namely, for any $\zeta \in \partial D$, there exist a neighborhood U of ζ in $\mathbf{C}^n$ and a C^1 plurisubharmonic function r on U such that $D \cap U = \{z \in U : r(z) < 0\}$. In [5], the authors obtained similar results as in this section for normalized locally biholomorphic mappings on D.

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