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ON THE QUOTIENT OF THE ANALYTIC REPRESENTATIONS OF CONVEX AND STARLIKE FUNCTIONS

MAMORU NUNOKAWA, SHIGEYOSHI OWA
HITOSHI SAITOH AND O. P. AHUJA

Abstract. *Silverman* [3] *investigated and obtained some results concerning the properties of the functions defined in terms of the quotient of the analytic representations of convex and starlike functions. Using Silverman's function, we obtain sufficient conditions for a function to be strongly starlike, strongly convex, or starlike in the open unit disk.*

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1. INTRODUCTION

Let N denote the class of functions f normalized by $f(0) = 0 = f'(0)-1$ that are analytic in the unit disk $\mathrm{E} = \{z: |z|<1\}$. Denote by S the family of functions f in N that are univalent in E. A function f in S is said to be starlike of order α, $0\le\alpha<1$, and is denoted by $S^*(\alpha)$ if $\mathrm{Re}\{zf'(z)/f(z)\}>\alpha$, $z\in \mathrm{E}$, and is said to be convex and is denoted by K if $\mathrm{Re}\{1+zf''(z)/f'(z)\}>0$, $z\in \mathrm{E}$. We write $S^* = S^*(0)$ and a function in S^* is called a starlike function.

A function f in S is said to be strongly starlike of order β and $f\in STS(\beta)$, $0<\beta\le1$ if $|\arg(zf'(z)/f(z))|<\pi\beta/2$ in E, and is said to be strongly convex of order β and $f\in STC(\beta)$ if $|\arg(1+ zf''(z)/f'(z))|<\pi\beta/2$ in E.

Recently, Silverman [3] investigated an expression involving the quotient of the analytic representations of convex and starlike functions. He found sufficient conditions for functions to be starlike of a positive order and convex. In particular, he studied the class

$$G_b = \{f \in N : |\Psi(z)-1| < b, z \in \mathrm{E}\}$$

where Ψ is the quotient function defined by

$$\Psi(z) = \frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)}$$

In this paper, we obtain sufficient conditions for a function to be strongly starlike of order β, strongly convex of order β, or starlike in the unit disk E in terms of the Silverman's quotient function Ψ.

Our main tools are the following results.

LEMMA 1. [1] *Let p be an analytic function in E, p(0) = 1, p(z)≠0 in E and suppose that there exists a point $z_0 \in E$ such that*

$$|\arg p(z)| < \frac{\pi\alpha}{2} \text{ for } |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi\alpha}{2}$$

where 0<α. Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad \textit{when } \arg p(z_0) = \frac{\pi\alpha}{2}$$

and

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \quad \textit{when } \arg p(z_0) = -\frac{\pi\alpha}{2},$$

where

$$p(z_0)^{1/\alpha} = \pm ia, \textit{ and } a > 0.$$

LEMMA 2. [2] *Let p be an analytic function in E which is normalized by* p(0)=0. *If there exists a point $z_0 \in E$ such that* Re(*p(z)*)>0 (|z| < |z_0|), Re(*p(z_0)*)=0 *and p(z_0)≠0, then*

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{k}{2}\left(a + \frac{1}{a}\right)i,$$

where p(z_0) = ia, k is real and k≥1.

2. MAIN RESULTS

THEOREM 1. *Let $f \in S$ and suppose that*

$$|\arg \Psi(z)| < \tan^{-1}\left(\frac{\beta\rho(\beta)\cos\frac{\pi\beta}{2}}{\sigma(\beta)+\beta\rho(\beta)\sin\frac{\pi\beta}{2}} \right)$$

in E, *where* $\sigma(\beta)=(1+\beta)^{(1+\beta)/2}$, $\rho(\beta)=(1-\beta)^{(\beta-1)/2}$ and $0<\beta<1$. *Then* $f \in STS(\beta)$.

PROOF. We define the function p by

$$p(z) = \frac{zf'(z)}{f(z)}.$$

Clearly, p is analytic in E, $p(0) = 1$, $p(z) \neq 0$ in E and

$$\Psi(z) = \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} = 1 + \frac{zp'(z)}{(p(z))^2}.$$

Let, if possible, $f \notin STS(\beta)$. Then, there must exist a point $z_0 \in E$ for which $|\arg(p(z))| < \pi\beta/2$ ($|z| < |z_0|$) and $|\arg(p(z_0))| = \pi\beta/2$. Therefore, by Lemma 1 we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta,$$

where

$$|k| \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad \textit{when } \arg p(z_0) = \pm\frac{\pi\beta}{2},$$

and

$$p(z_0) = (\pm ia)^\beta = a^\beta e^{\pm i\pi\beta/2} \quad (a > 0).$$

A straightforward computation shows that

$$|\arg \Psi'(z_0)| = \left|\arg\left(1 + \frac{z_0 p'(z_0)}{(p(z_0))^2}\right)\right|$$

$$= \left|\arg\left(1 + \frac{k\beta}{a^{\beta}} e^{i\frac{\pi}{2}(1 \mp \beta)}\right)\right|$$

$$= \left|\tan^{-1} \frac{\frac{k\beta}{a^{\beta}}\cos\frac{\pi}{2}\beta}{1 \pm \frac{k\beta}{a^{\beta}}\sin\frac{\pi}{2}\beta}\right|.$$

Note that

$$\frac{|k|\beta}{a^{\beta}} \geq \frac{1}{2}\beta\left(a^{1-\beta} + a^{-1-\beta}\right)$$

and if

$$g(a) = \frac{1}{2}\beta\left(a^{1-\beta} + a^{-1-\beta}\right),$$

then

$$g'(a) = \frac{1}{2}\beta\left\{(1-\beta)a^{-\beta} - (1+\beta)a^{-2-\beta}\right\}$$

and the function g takes the minimum value at

$$a = \sqrt{(1+\beta)/(1-\beta)}$$

so that

$$\frac{|k|\beta}{a^{\beta}} \geq g(a) \geq g\left(\sqrt{\frac{1+\beta}{1-\beta}}\right) = \left(\frac{\beta}{1-\beta}\right)\left(\frac{1-\beta}{1+\beta}\right)^{\frac{1+\beta}{2}}$$

We, therefore, have

$$|\arg \Psi(z_0)| \geq \tan^{-1} \frac{\frac{|k|\beta}{a^\beta}\cos\frac{\pi}{2}\beta}{1+\frac{|k|\beta}{a^\beta}\sin\frac{\pi}{2}\beta}$$

$$\geq \tan^{-1} \frac{\left(\frac{\beta}{1-\beta}\right)\left(\frac{1-\beta}{1+\beta}\right)^{\frac{1+\beta}{2}}\cos\frac{\pi}{2}\beta}{1+\left(\frac{\beta}{1-\beta}\right)\left(\frac{1-\beta}{1+\beta}\right)^{\frac{1+\beta}{2}}\sin\frac{\pi}{2}\beta}$$

$$= \tan^{-1} \frac{\beta\rho(\beta)\cos\frac{\pi}{2}\beta}{\sigma(\beta)+\beta\rho(\beta)\sin\frac{\pi}{2}\beta}.$$

This contradicts our hypothesis and the proof is complete.

COROLLARY. *Let f satisfy the conditions of Theorem 1 and suppose that*

$$\alpha(\beta) = \beta + \frac{2}{\pi}\tan^{-1}\left(\frac{\beta\rho(\beta)\cos\frac{\pi\beta}{2}}{\sigma(\beta)+\beta\rho(\beta)\sin\frac{\pi\beta}{2}}\right)$$

Then $f \in \mathrm{STC}(\alpha(\beta))$.

PROOF. From Theorem 1, we have

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi\beta}{2}, \quad z \in E.$$

Then it follows that

$$\left|\arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right| \leq \left|\arg\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}\right| + \left|\arg\frac{zf'(z)}{f(z)}\right|$$

$$\leq \tan^{-1}\frac{\beta\rho(\beta)\cos\frac{\pi\beta}{2}}{\sigma(\beta)+\beta\rho(\beta)\sin\frac{\pi\beta}{2}} + \frac{\pi\beta}{2} = \frac{\pi}{2}\alpha(\beta).$$

This proves that $f \in STC(\alpha(\beta))$.

REMARK. The above Corollary shows that if the modulus of the Silverman's quotient function Ψ satisfies the conditions of Theorem 1, then f is strongly convex of order $\alpha(\beta)$.

The next theorem exhibits a sufficient condition in terms of a function to be starlike in E.

THEOREM 2. *Let $f \in S$ and suppose that*

$$\operatorname{Re}\Psi(z) < \frac{3}{2} + \frac{1}{2}\left|\frac{f(z)}{zf'(z)}\right|^2$$

in E. Then $f \in S^*$.

PROOF. Letting $p(z) = zf'(z)/f(z)$, we notice that p is analytic in E, $p(0)=1$, $p(z) \neq 0$ in E and

$$\Psi(z) = \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} = 1 + \frac{zp'(z)}{(p(z))^2}.$$

If $f \notin S^*$, then there must exist a point $z_0 \in E$ such that $\operatorname{Re}(p(z))>0$ $(|z| < |z_0|)$, $\operatorname{Re} p(z_0) = 0$ *and* $p(z_0) \neq 0$. Then, by Lemma 2

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{k}{2}\left(a + \frac{1}{a}\right)i,$$

where $p(z_0) = ia = | z_0 f'(z_0)/f(z_0)|$, where k is real and $k \geq 1$. Therefore, we have

$$\begin{aligned}\operatorname{Re}\Psi(z_0) &= \operatorname{Re}\left(1 + \frac{z_0 p'(z_0)}{(p(z_0))^2}\right) = \operatorname{Re}\left(1 + \frac{k}{2a}\left(a + \frac{1}{a}\right)\right)\\ &= 1 + \frac{k}{2}\left(1 + \frac{1}{a^2}\right)\\ &\geq \frac{3}{2} + \frac{1}{2a^2}\\ &= \frac{3}{2} + \frac{1}{2}\left|\frac{f(z_0)}{z_0 f'(z_0)}\right|^2\end{aligned}$$

which contradicts the hypothesis. Thus $\operatorname{Re} p(z) > 0$ $(z \in E)$ and hence $f \in S^*$.

COROLLARY. Let $f \in S$ and suppose that

$$\operatorname{Re} \Psi(z) < \sqrt{3}\left|\frac{f(z)}{zf'(z)}\right|$$

in E. Then $f \in S^*$.

PROOF. This follows from the proof of Theorem 2 because

$$\operatorname{Re} \Psi(z_0) \geq \frac{1}{2}\left(3 + \left|\frac{f(z_0)}{z_0 f'(z_0)}\right|^2\right) \geq \sqrt{3}\left|\frac{f(z_0)}{z_0 f'(z_0)}\right|.$$

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M. Nunokawa
Department of Mathematics
Gunma University
Aramaki 371-8510
Meabashi
Japan

H. Saitoh
Department of Mathematics
Gunma National College of Technology
Toriba
Maeashi
Japan.

S. Owa
Department of Mathematics
Kinki University
Higashi-Osaka
Osaka 577, Japan

O. P. Ahuja
School of Science
Nanyang Technological University
469 Bukit Timah Road
Singapore 259756