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MARTIN BOUNDARY OF A UNIFORMLY JOHN DOMAIN

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ABSTRACT. A uniformly John domain is a domain intermediate between a John domain and a uniform domain. We determine the Martin boundary of a uniformly John domain D as an application of a boundary Harnack principle. Define the internal metric between two points in D by the infimum of the diameter of arcs in D connecting the points. The Martin boundary of D is the boundary with respect to the internal metric. We assume no exterior condition for D.

1. INTRODUCTION

Balogh and Volberg [5, 6] introduced a uniformly John domain in connection with conformal dynamics. The main aim of this paper is to determine the Martin boundary of a uniformly John domain. Let D be a domain in $\mathbb{R}^n$, $n \geq 2$. We define the internal metric $\rho_D(x, y)$ by

$$\rho_D(x,y) = \inf\{\operatorname{diam}(\gamma) : \ \gamma \text{ is an arc joining } x \text{ and } y \text{ in } D\}$$

for $x, y \in D$. Here $\operatorname{diam}(\gamma)$ denotes the diameter of γ. Obviously $|x-y| \leq \rho_D(x,y)$. We say that D is a uniformly John domain if there exist positive constants A_1 and A_2 such that each pair of points $x, y \in D$ can be joined by an arc $\gamma \subset D$ for which

$$\operatorname{diam}(\gamma) \leq A_1 \rho_D(x,y), \tag{1.1}$$

$$\min\{|x-z|, |z-y|\} \leq A_2 \delta_D(z) \quad \text{for all } z \in \gamma. \tag{1.2}$$

A uniformly John domain is a domain intermediate between a John domain and a uniform domain. By definition

$$\text{uniform} \subsetneqq \text{uniformly John} \subsetneqq \text{John}.$$

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In the previous paper [1], the first author showed that the Martin compactification of a bounded uniform domain is homeomorphic to the Euclidean closure. A Lipschitz domain and more generally an NTA domain are uniform domain, so that [1] is a generalization of Hunt and Wheeden [11] and Jerison and Kenig [12]. The Martin compactification of a uniformly John domain is more complicated. We shall show that it is homeomorphic to the completion D^* with respect to the internal metric. That is, D^* is the equivalence class of all ρ_D-Cauchy sequences with equivalence relation "$\sim$", where we say $\{x_j\} \sim \{y_j\}$ if $\{x_j\} \cup \{y_j\}$ is a ρ_D-Cauchy sequence. Let $\partial^* D = D^* \setminus D$, the boundary with respect to ρ_D. Take $\xi^* \in D^*$. Suppose ξ^* is represented by a ρ_D-Cauchy sequence $\{x_j\}$. Since $\{x_j\}$ is also a usual Cauchy sequence, it follows that x_j converges to some point $\xi \in \overline{D}$. The point ξ is independent of the representative $\{x_j\}$ and uniquely determined by ξ^*. We say that ξ^* lies over $\xi \in \overline{D}$. If $\xi \in D$, then ξ and ξ^* coincide. Define the projection $\pi : D^* \to \overline{D}$ by $\pi(\xi^*) = \xi$. It is easy to see that π is a continuous contraction mapping, i.e. $|\pi(\xi_1^*) - \pi(\xi_2^*)| \leq \rho_D(\xi_1^*, \xi_2^*)$. The main result of this paper is the following theorem.

Theorem 1. *Let D be a bounded uniformly John domain. Then the Martin compactification of D is homeomorphic to D^* and each boundary point $\xi^* \in \partial^* D$ is minimal. Moreover, for every boundary point $\xi \in \partial D$, the number of Martin boundary points over ξ is bounded by a constant depending only on D.*

The above theorem will be proved as a corollary to the boundary Harnack principle for a uniformly John domain. Balogh and Volberg [6] proved the boundary Harnack principle for a planar uniformly John domain with uniformly perfect boundary, an additional assumption. They also demonstrated that the harmonic measure satisfies the doubling condition with respect to the internal metric [6, Theorem 3.1].

The significant difference between [6] and the present paper is that we have no assumption on the boundary or the complement of the domain. In the present setting, the harmonic measure needs not satisfy the doubling condition, because of the lack of exterior condition. The argument of [6] is not applicable. Moreover, our domain may admit an irregular boundary point. Hence, we always consider a generalized Dirichlet problem, i.e. boundary values have meaning outside a polar set. For simplicity, we shall say that a property holds q.e. (quasi everywhere) if it holds outside a polar set.

We note that there are very precise results on the Martin boundary of Denjoy type domains and some specific domains. See Ancona [3, 4], Benedicks [8], Chevallier [9], Segawa [14] and references therein. Our Theorem 1 is not so precise but it is applicable to various domains. Conditions (1.1) and (1.2) are simple.

The plan of the paper is as follows: in the next section we shall give several geometrical notions and properties of a uniformly John domain. In Section 3 we shall state the boundary Harnack principle and prove it along a line similar to [1]. Our proof is inspired by the probabilistic work of Bass and Burdzy [7]. Section 4 will be devoted to the proof of Theorem 1, and some further properties, such as the Hölder continuity of the kernel function.

We shall use the following notation. By the symbol A we denote an absolute positive constant whose value is unimportant and may change from line to line. If necessary, we use $A_0, A_1, \dots,$ to specify them. We shall say that two positive functions f_1 and f_2 are comparable, written $f_1 \approx f_2$, if and only if there exists a constant $A \geq 1$ such that $A^{-1} f_1 \leq f_2 \leq A f_1$. The constant A will be called the constant of comparison. By $B(x,r)$, $C(x,r)$ and $S(x,r)$ we denote the open ball, the closed ball and the sphere with center at x and radius r, respectively.

2. Geometric properties of a uniformly John domain

Balogh and Volberg [5] proved a very deep property of a planar uniformly John domain; a geometric localization. In the course of the proof of Theorem 1 we shall not use their result. Instead, we shall need some elementary properties of a uniformly John domain. The purpose of this section is to show these properties with purely geometrical proofs. No potential theory will be involved in this section.

Hereafter we let D be a bounded uniformly John domain. In view of the equivalence between the distance, the diameter and the length cigar conditions ([13, Lemma 2.7] and [15, Theorem 2.18]), we observe that (1.1) and (1.2) can be replaced by the following stronger condition: there exist positive constants A_3 and A_4

such that

$$\ell(\gamma) \leq A_3 \rho_D(x,y), \tag{2.1}$$

$$\min\{\ell(\gamma(x,z)), \ell(\gamma(z,y))\} \leq A_4 \delta_D(z) \quad \text{for all } z \in \gamma, \tag{2.2}$$

where $\ell(\gamma)$ and $\gamma(x,z)$ denote the length of γ and the subarc $\gamma(x,z)$ of γ connecting x and z, respectively.

Let us first show that the completion D^* is a compact space.

Proposition 1. *Let D be a bounded uniformly John domain. Then D^* is a compact space and each boundary point $\xi^* \in \partial^* D$ is accessible from D, i.e., there is an arc $\gamma \subset D$ converging to ξ^*. Moreover, for every boundary point $\xi \in \partial D$, the number of points in $\partial^* D$ over ξ is bounded by a constant depending only on D.*

Proof. Take a sequence $\{x_m^*\}$ in D^*. We need to show that there exists a subsequence of $\{x_m^*\}$ converging to some point in D^* with respect to ρ_D. Suppose that each x_m^* is represented by a ρ_D-Cauchy sequence $\{x_m^j\} \subset D$. Since $\{x_m^j\}$ is also a usual Cauchy sequence, it must converge to $x_m = \pi(x_m^*) \in \overline{D}$ with respect to the usual metric. Taking a subsequence, if necessary, we may assume that $\{x_m\}$ is a Cauchy sequence converging to some $\xi \in \overline{D}$ with respect to the usual metric. If $\xi \in D$, then it is easy to show that x_m^* converges to ξ with respect to ρ_D. So, we may assume that $\xi \in \partial D$.

Let $r > 0$. Then $D \cap B(\xi, r)$ consists of countably many open connected components $B_i(r)$. Obviously

$$\rho_D(x,y) \leq 2r \quad \text{for } x, y \in B_i(r). \tag{2.3}$$

Let us count the number $\nu(r)$ of components $B_i(r)$ having a point x_m with $|x_m - \xi| < r/2$. We claim that

$$\nu(r) \leq N, \tag{2.4}$$

where N is independent of r and ξ. Since D is connected, two distinct components are connected by a curve in D. This curve must get out $B(\xi, r)$. Hence each component $B_i(r)$ has a limit point on $S(\xi, r)$. On the other hand, our $B_i(r)$ has a point x_m with $|x_m - \xi| < r/2$, and hence $\operatorname{diam}(B_i(r)) \geq r/2$. It follows from the definition of a uniformly John domain that the Lebesgue measure of $B_i(r)$ is comparable to r^n. Therefore, (2.4) holds.

Now let $r_k = 2^{-k} \downarrow 0$. Then we infer from (2.4) that there exists a decreasing sequence of components $B_{i_k}(r_k)$ each of which contains infinitely many x_m. We find $\xi^* \in \partial^* D$ such that

$$B_{i_1}(r_1) \supset B_{i_2}(r_2) \supset \cdots \to \xi^* \in \partial^* D,$$

and a subsequence of $\{x_m^*\}$ converges along $\{B_{i_k}(r_k)\}$ to ξ^* with respect to ρ_D by (2.3). Obviously $\pi(\xi^*) = \xi$. This shows D^* is compact and ξ^* is accessible from D. Moreover, since every $\xi^* \in \partial^* D$ has a ρ_D-Cauchy sequence converging to ξ^*, the second assertion follows.

Finally let $\xi \in \partial D$ and suppose k points $\xi_1^*, \dots \xi_k^* \in \partial^* D$ lie over ξ. Then there is $\varepsilon > 0$ such that $\rho_D(\xi_i^*, \xi_j^*) > 2\varepsilon$ for $i \neq j$. By V_i we denote the component of $D \cap B(\xi, \varepsilon)$ from which ξ_i^* is accessible. Then $V_1, \dots, V_k$ are disjoint. In fact, if $V_i \cap V_j \neq \emptyset$ for $i \neq j$, then V_i and V_j would coincide and ξ_i^* and ξ_j^* would be accessible from the same component. That is, there would be an arc γ in $V_i = V_j$ connecting ξ_i^* and ξ_j^*. By definition, $\rho_D(\xi_i^*, \xi_j^*) \leq \mathrm{diam}(\gamma) \leq 2\varepsilon$; a contradiction would arise. Thus $V_1, \dots, V_k$ are disjoint. We may assume that $x_0 \in D \setminus B(\xi, \varepsilon)$. Then each ξ_i^* can be connected to x_0 by a curve, say γ_i, in D with (1.1) and (1.2). Let $x_i \in \gamma_i \cap V_i \cap S(\xi, \varepsilon/2)$. Then $B(x_i, A_2\varepsilon/2) \subset V_i$ by (1.2), so that the Lebesgue measure of V_i is comparable to ε^n. Since $V_1, \dots, V_k$ are disjoint subsets of $B(\xi, \varepsilon)$, it follows that the number k is bounded by a constant depending only on A_2 and the dimension. The proof is complete. □

Remark 1. In general, a minimal boundary point of the Martin boundary is accessible from the domain (e.g. [10, Satz 13.3]). Hence, if we have shown Theorem 1, the above proposition follows automatically. The above argument proves the accessibility without potential theoretic consideration.

We shall define 'balls' with respect to the internal metric. For this purpose it is convenient to modify the internal metric slightly. For $x \in D$ and $\gamma \subset D$ we let

$$r^*(x, \gamma) = \sup_{z \in \gamma} |z - x|$$

be the the infimum of radii r for which $\gamma \subset B(x, r)$. Observe that $r^*(x, \gamma) \leq \mathrm{diam}(\gamma) \leq 2r^*(x, \gamma)$ for $x \in \gamma$. Let

$$\rho_D^*(x, y) = \inf\{r^*(x, \gamma) : \ \gamma \text{ is an arc joining } x \text{ and } y \text{ in } D\}$$

for $x, y \in D$. The quantity ρ_D^* is not symmetric. It is related to the internal metric ρ_D as follows:

$$\rho_D^*(x,y) \le \rho_D(x,y) \le 2\rho_D^*(x,y).$$

Therefore the convergence with respect to ρ_D is equivalent to the convergence with respect to ρ_D^*. We can also show the following inequalities

$$\rho_D^*(x,z) \le \rho_D^*(x,y) + \rho_D^*(y,z),$$
$$\rho_D^*(x,z) \le \rho_D^*(x,y) + 2\rho_D^*(z,y)$$

for $x, y, z \in D$. We extend $\rho_D(x,y)$ and $\rho_D^*(x,y)$ for $x, y \in D^*$ by $\rho_D(x,y) = \lim \rho_D(x_j, y_j)$ and $\rho_D^*(x,y) = \lim \rho_D^*(x_j, y_j)$ if x and y are represented by ρ_D-Cauchy sequences $\{x_j\}$ and $\{y_j\}$ in D. It is easy to see that the quantities $\rho_D(x,y)$ and $\rho_D^*(x,y)$ are independent of the choice of the ρ_D-Cauchy sequences $\{x_j\}$ and $\{y_j\}$. Let $\xi^* \in \partial^* D$ and put

$$B_\rho(\xi^*, r) = \{x \in D : \rho_D^*(\xi^*, x) < r\}.$$

Moreover, let $S_\rho(\xi^*, r) = D \cap \partial B_\rho(\xi^*, r)$ and $C_\rho(\xi^*, r) = D \cap \overline{B_\rho(\xi^*, r)}$. Here, '$\partial$' and '$\overline{\quad}$' mean the boundary and the closure in the Euclidean space, respectively. These sets correspond to $D \cap B(x,r)$, $D \cap C(x,r)$ and $D \cap S(x,r)$. The following observation enables us to use many arguments in [1].

Lemma 1. *The set $B_\rho(\xi^*, r)$ is the open connected component of $D \cap B(\pi(\xi^*), r)$ which can be connected to ξ^* in itself, i.e. there is an arc $\gamma \subset B_\rho(\xi^*, r)$ converging to ξ^*.*

Proof. It is sufficient to show the following (i)–(iv).

(i) $B_\rho(\xi^*, r) \subset D \cap B(\pi(\xi^*), r)$.

(ii) $B_\rho(\xi^*, r)$ is open.

(iii) Every point $x \in B_\rho(\xi^*, r)$ is connected to ξ^* by an arc in itself.

(iv) $B_\rho(\xi^*, r)$ is the maximal set with the above properties (i)–(iii).

Let ξ^* be represented by a ρ_D-Cauchy sequence $\{x_j\}$. First, we prove (i), (ii) and (iii). Suppose $x \in B_\rho(\xi^*, r)$. Then $\varepsilon = r - \rho_D^*(\xi^*, x) > 0$. Since $\rho_D^*(\xi^*, x) = \lim_{j\to\infty} \rho_D^*(x_j, x) < r - \varepsilon$, there exists a positive integer j_0 such that $\rho_D^*(x_j, x) < r - \varepsilon/2$ for $j \ge j_0$. By the definition of ρ_D^* we find an arc $\widetilde{x_j x} \subset D$ joining x_j and x

with

$$|x_j - x| \le r^*(x_j, \widetilde{x_j x}) < r - \varepsilon/2 \tag{2.5}$$

for $j \ge j_0$. Hence

$$|\pi(\xi^*) - x| = \lim_{j\to\infty} |x_j - x| \le r - \varepsilon/2 < r.$$

Therefore, $x \in D \cap B(\pi(\xi^*), r)$ and (i) follows. Now x lies in the open set $D \cap B(\pi(\xi^*), r)$, whence we find r_0, $0 < r_0 < \varepsilon/2$, such that $B(x, r_0) \subset D \cap B(\pi(\xi^*), r)$. For (ii) it suffices to show that $B(x, r_0) \subset B_\rho(\xi^*, r)$. In fact, every $y \in B(x, r_0)$ can be connected to x_j by $\widetilde{x_j x} \cup \overline{xy}$ for $j \ge j_0$, where $\overline{xy}$ denotes the line segment between x and y. Hence, (2.5) yields

$$\rho_D^*(\xi^*, y) = \lim_{j\to\infty} \rho_D^*(x_j, y) \le \limsup_{j\to\infty} r^*(x_j, \widetilde{x_j x} \cup \overline{xy}) \le r - \frac{\varepsilon}{2} + r_0 < r,$$

so that $B(x, r_0) \subset B_\rho(\xi^*, r)$ and (ii) follows. In order to prove (iii) we may assume that

$$\rho_D(x_j, x_{j+1}) < 2^{-j}\varepsilon, \tag{2.6}$$

by taking a subsequence of $\{x_j\}$. Then each pair of points x_j and x_{j+1} can be connected by an arc $\widetilde{x_j x_{j+1}} \subset D$ with $\operatorname{diam}(\widetilde{x_j x_{j+1}}) < 2^{-j}\varepsilon$. Let

$$\gamma = \widetilde{x x_{j_0}} \cup \left(\bigcup_{j=j_0}^{\infty} \widetilde{x_j x_{j+1}} \right).$$

Then, by (2.5) and (2.6), γ is an arc in D connecting x and ξ^* such that

$$r^*(\xi^*, \gamma) \le r^*(x_{j_0}, \widetilde{x x_{j_0}}) + \sum_{j=j_0}^{\infty} \operatorname{diam}(\widetilde{x_j x_{j+1}}) < r - \frac{\varepsilon}{2} + \sum_{j=j_0}^{\infty} 2^{-j}\varepsilon.$$

Without loss of generality, we may assume that $j_0 \ge 2$, so that $r^*(\xi^*, \gamma) < r$. Hence $\gamma \subset B_\rho(\xi^*, r)$ and (iii) follows. We remark that (iii) implies that $B_\rho(\xi^*, r)$ is connected.

Finally we prove (iv). Suppose that D_1 is a subset of $D \cap B(\pi(\xi^*), r)$ such that every $x \in D_1$ is connected to ξ^* by an arc in itself. We have to show that $\rho_D^*(\xi^*, x) < r$ for $x \in D_1$. Suppose $x \in D_1$. Then there is an arc $\gamma \subset D_1$ connecting ξ^* and x. By the compactness of γ we see that $\gamma \subset B(\pi(\xi^*), r - \eta)$ for some $\eta > 0$.

By definition

$$\rho_D^*(\xi^*,x) = \lim_{\substack{y\to\xi^* \\ y\in\gamma}} \rho_D^*(y,x) \le \limsup_{\substack{y\to\xi^* \\ y\in\gamma}} r^*(y,\gamma) \le \limsup_{\substack{y\to\xi^* \\ y\in\gamma}} |y-\pi(\xi^*)| + r - \eta = r-\eta < r.$$

Hence (iv) follows. □

As a corollary to Lemma 1 we have the following.

Lemma 2. *Let V be a connected open subset of $D\cap B(\pi(\xi^*),r)$. If $V\cap B_\rho(\xi^*,r)\neq\emptyset$, then $V\subset B_\rho(\xi^*,r)$. In particular, if $\xi_1^*\in\partial^*D$ is accessible from $B_\rho(\xi^*,r)$ and $r_1+|\pi(\xi^*)-\pi(\xi_1^*)|<r$, then $B_\rho(\xi_1^*,r_1)\subset B_\rho(\xi^*,r)$.*

For a moment let D be a general proper subdomain of $\mathbb{R}^n$. We define the quasi-hyperbolic metric $k_D(x,y)$ by

$$k_D(x,y) = \inf_\gamma \int_\gamma \frac{ds(z)}{\delta_D(z)},$$

where the infimum is taken over all rectifiable arcs γ joining x to y in D. Observe that $k_D(x,y)$ is monotone decreasing with respect to D, i.e., if $x,y\in D_1\subset D$, then $k_{D_1}(x,y)\ge k_D(x,y)$. The converse estimate will be needed in the sequel. Observe that if $z\in D$, then

$$k_D(x,y) \le k_{D\setminus\{z\}}(x,y) \le k_D(x,y) + A \quad \text{for } x,y\in D\setminus B(z,2^{-1}\delta_D(z)). \tag{2.7}$$

This observation will be useful to estimate the Green function with pole at z.

Now let D be a bounded uniformly John domain. Then the following uniform quasihyperbolic boundary condition holds.

Lemma 3. *Let D be a bounded uniformly John domain. Then*

$$k_D(x,y) \le A\log\frac{\rho_D(x,y)}{\min\{\delta_D(x),\delta_D(y)\}} + A',$$

where A and A' depend only on D.

Proof. If $y\in B(x,\delta_D(x)/2)$ or $x\in B(y,\delta_D(y)/2)$, then the lemma is obvious. Hence, suppose $|x-y|\ge\frac{1}{2}\max\{\delta_D(x),\delta_D(y)\}$. Let γ be a curve joining x to y with (2.1) and (2.2). Then

$$\begin{aligned}\int_\gamma \frac{ds(z)}{\delta_D(z)} &\le \int_0^{\delta_D(x)/2} \frac{ds}{\delta_D(x)/2} + \int_{\delta_D(x)/2}^{\ell(\gamma)/2} \frac{A_4 ds}{s} + \int_{\ell(\gamma)/2}^{\ell(\gamma)-\delta_D(y)/2} \frac{A_4 ds}{s} + \int_0^{\delta_D(y)/2} \frac{ds}{\delta_D(y)/2} \\ &\le 2 + 2A_4\log\frac{A_3\rho_D(x,y)}{\min\{\delta_D(x),\delta_D(y)\}}.\end{aligned}$$

Thus the lemma follows. $\square$

Let $x_0 \in D$ be fixed. Then every point $x \in D$ can be connected to x_0 by γ along which the distance to the boundary increases as in (1.2). Hence, there is A_5, $0 < A_5 < 1$ such that

$$A_5 R \leq \sup_{x \in S_\rho(\xi^*, R)} \delta_D(x) \leq R$$

for sufficiently small R, say $0 < R < \delta_D(x_0)/2$. Let us take $\xi_R \in S_\rho(\xi^*, 4R)$ with $4A_5 R \leq \delta_D(\xi_R) \leq 4R$. Then, we have the following.

Lemma 4. *Let D be a bounded uniformly John domain. Then there exists a constant $A_6 > 9$ depending only on D such that*

$$k_{B_\rho(\xi^*, A_6 R)}(x, y) \leq A \log \frac{\rho_D(x, y)}{\min\{\delta_D(x), \delta_D(y)\}} \quad \text{for } x, y \in B_\rho(\xi^*, 9R). \tag{2.8}$$

where $\xi^* \in \partial^* D$, $R > 0$ is sufficiently small and A depends only on D. In particular,

$$k_{B_\rho(\xi^*, A_6 R)}(x, \xi_R) \leq A \log \frac{18R}{\delta_D(x)} \quad \text{for } x \in B_\rho(\xi^*, 9R), \tag{2.9}$$

where A is independent of the choice of ξ_R. In the sequel, estimates will be independent of the choice of ξ_R.

Proof. Let $x, y \in B_\rho(\xi^*, 9R)$. Suppose γ is a curve joining x to y with (2.1) and (2.2). Then

$$\rho_D^*(\xi^*, z) \leq \rho_D^*(\xi^*, x) + \rho_D^*(x, z) < 9R + \operatorname{diam}(\gamma) \leq AR \quad \text{for } z \in \gamma.$$

Let A_6 be the twice of the above A. Then $\gamma \subset B_\rho(\xi^*, \frac{1}{2} A_6 R)$ and $\delta_{B_\rho(\xi^*, A_6 R)}(z) = \delta_D(z)$ for $z \in \gamma$. Hence the proof of the preceding lemma yields (2.8). Since $\rho_D(x, \xi_R) < 18R$ and $\delta_D(\xi_R) \geq 4A_5 R$, we have (2.9) from (2.8). $\square$

3. Boundary Harnack Principle

The main aim of this section is to show the following boundary Harnack principle.

Theorem 2. *Let D be a bounded uniformly John domain. Then there exists a constant $A_7 > 1$ depending only on D with the following property: Let $\xi^* \in \partial^* D$*

and let $R > 0$ be sufficiently small. Suppose u and v are positive bounded harmonic functions on $B_\rho(\xi^*, A_7R)$ vanishing q.e. on $\partial D \cap \overline{B_\rho(\xi^*, A_7R)}$. Then

$$\frac{u(x)}{v(x)} \approx \frac{u(x')}{v(x')} \quad \textit{uniformly for } x, x' \in B_\rho(\xi^*, R),$$

where the constant of comparison depends on D.

Theorem 2 can be proved in a way similar to that of [1, Theorem 1] with the aid of Lemma 1. However, we must be careful about the fact that D^* is the completion of D with respect to the internal metric. It is, in general, different from the Euclidean closure.

We say that $x, y \in D$ is connected by a Harnack chain $\{B(x_j, \frac{1}{2}\delta_D(x_j))\}_{j=1}^k$ if $x \in B(x_1, \frac{1}{2}\delta_D(x_1))$, $y \in B(y_k, \frac{1}{2}\delta_D(y_k))$, and $B(x_j, \frac{1}{2}\delta_D(x_j)) \cap B(x_{j+1}, \frac{1}{2}\delta_D(x_{j+1})) \neq \emptyset$ for $j = 1, \dots, k-1$. The number k is called the length of the Harnack chain. We observe that the shortest length of the Harnack chain connecting x and y is comparable to $k_D(x, y)$. Therefore, the Harnack inequality yields that there is a positive constant A depending only on n such that

$$\exp(-Ak_D(x, y)) \le \frac{h(x)}{h(y)} \le \exp(Ak_D(x, y))$$

for every positive harmonic function h on D.

Our proof of Theorem 2 will be based on a certain estimate of harmonic measure. By $\omega(x, E, U)$ we denote the harmonic measure of E for an open set U evaluated at x. For $r > 0$ let $U(r) = \{x \in D : \delta_D(x) < r\}$. Since every point $x \in U(r)$ can be connected to x_0 by an arc γ along which the distance to the boundary increases as in (1.2), it follows that if $r > 0$ is sufficiently small, then there is a point $z \in D \cap S(x, A_8r)$ with $\delta_D(z) > 2r$, where $A_8 > 1$ is a constant depending only on D. Hence there is a ball $B(z, r) \subset B(x, A_8r) \setminus U(r)$. This implies that

$$\omega(x, \overline{U(r)} \cap S(x, A_8r), U(r) \cap B(x, A_8r)) \le 1 - \varepsilon_0 \quad \text{for } x \in U(r)$$

with $0 < \varepsilon_0 < 1$ depending only on A_8 and the dimension. Let $R \ge r$ and repeat this argument with the maximum principle. Then there exist positive constants A_9 and A_{10} such that

$$\omega(x, \overline{U(r)} \cap S(x, R), U(r) \cap B(x, R)) \le \exp(A_9 - A_{10}R/r). \tag{3.1}$$

See [1, Lemma 1] for details.

Let us compare the Green function and the harmonic measure. For simplicity we let $D_R = B_\rho(\xi^*, (A_6+7)R)$ and $D'_R = B_\rho(\xi^*, A_6R)$ with A_6 as in Lemma 4. By G_R and G'_R we denote the Green functions for D_R and D'_R, respectively.

Lemma 5. *If $R > 0$ is sufficiently small, then*

$$\omega(\cdot, S_\rho(\xi^*, 2R), B_\rho(\xi^*, 2R)) \le AR^{n-2}G'_R(\cdot, \xi_R) \le AR^{n-2}G_R(\cdot, \xi_R) \qquad \text{on } B_\rho(\xi^*, R),$$

where A depends only on D.

Proof. It is sufficient to show the first inequality. We follow the idea of [7] and [1]. We find $A_{11} > 0$ depending only on D such that $A_{11}R^{n-2}G'_R(\cdot, \xi_R) < 1/e$ on $B_\rho(\xi^*, 2R)$. Then

$$B_\rho(\xi^*, 2R) = \bigcup_{j\ge 0} D_j \cap B_\rho(\xi^*, 2R), \tag{3.2}$$

where

$$D_j = \{x \in D : \exp(-2^{j+1}) \le A_{11}R^{n-2}G'_R(x, \xi_R) < \exp(-2^j)\}.$$

Let $U_j = (\cup_{k\ge j} D_k) \cap B_\rho(\xi^*, 2R) = \{x \in B_\rho(\xi^*, 2R) : A_{11}R^{n-2}G'_R(x, \xi_R) < \exp(-2^j)\}$. First we observe

$$U_j \subset \{x \in D : \delta_D(x) < AR\exp(-2^j/\lambda)\} \tag{3.3}$$

with some $\lambda > 0$ depending only on D. For a moment fix $z \in S(\xi_R, \frac{1}{2}\delta_D(\xi_R))$. Then $G'_R(z, \xi_R) \approx R^{2-n}$ and

$$k_{D'_R\setminus\{\xi_R\}}(x, z) \le k_{D'_R}(x, \xi_R) + A \le A\log\frac{18R}{\delta_D(x)}$$

for $x \in B_\rho(\xi^*, 9R) \setminus B(\xi_R, \frac{1}{2}\delta_D(\xi_R))$ by (2.7) and (2.9). We see from the Harnack inequality that there is $\lambda > 0$ such that

$$\begin{aligned}\exp(-2^j) > A_{11}R^{n-2}G'_R(x, \xi_R) &\ge AR^{n-2}G'_R(z, \xi_R)\exp(-Ak_{D'_R\setminus\{\xi_R\}}(x, z))\\ &\ge A\exp\left(-\lambda\log\frac{18R}{\delta_D(x)}\right) = A\left(\frac{\delta_D(x)}{18R}\right)^\lambda\end{aligned}$$

for $x \in U_j$. Thus (3.3) follows.

Let $r_j = AR\exp(-2^j/\lambda)$ with A in (3.3). We take a slowly decreasing sequence $\{R_j\}$ converging to R such that

$$\sum_{j=1}^\infty \exp\left(2^{j+1} - \frac{A_{10}(R_{j-1} - R_j)}{r_j}\right) < \infty, \tag{3.4}$$

where the value of the summation is independent of R. In fact, if we let $R_0 = 2R$ and $R_j = \left(2 - \frac{6}{\pi^2}\sum_{k\le j}\frac{1}{k^2}\right)R$ for $j \ge 1$, then (3.4) holds. For simplicity we let $\omega_0 = \omega(\cdot, S_\rho(\xi^*, 2R), B_\rho(\xi^*, 2R))$ and

$$d_j = \begin{cases} \displaystyle\sup_{x\in D_j\cap B_\rho(\xi^*,R_j)} \frac{\omega_0(x)}{R^{n-2}G'_R(x,\xi_R)} & \text{if } D_j\cap B_\rho(\xi^*,R_j)\ne\emptyset, \\ 0 & \text{if } D_j\cap B_\rho(\xi^*,R_j)=\emptyset. \end{cases}$$

In view of (3.2) it is sufficient to show that

$$\sup_{j\ge 0} d_j \le A < \infty, \tag{3.5}$$

where A is independent of R.

Let $j > 0$. Let us apply the maximum principle over $U_j \cap B_\rho(\xi^*, R_{j-1})$. Observe that $D \cap \partial(U_j \cap B_\rho(\xi^*, R_{j-1}))$ is included in the union of $\overline{U_j} \cap S_\rho(\xi^*, R_{j-1})$ and $\{x \in B_\rho(\xi^*, R_{j-1}) : A_{11}R^{n-2}G'_R(x, \xi_R) = \exp(-2^j)\}$. By definition the last set is included in $D_{j-1} \cap B_\rho(\xi^*, R_{j-1})$, on which $\omega_0 \le d_{j-1}R^{n-2}G'_R(\cdot, \xi_R)$ holds. Hence the maximum principle yields that

$$\omega_0(x) \le \omega(x, \overline{U_j} \cap S_\rho(\xi^*, R_{j-1}), U_j \cap B_\rho(\xi^*, R_{j-1})) + d_{j-1}R^{n-2}G'_R(x, \xi_R). \tag{3.6}$$

for $x \in U_j \cap B_\rho(\xi^*, R_{j-1})$.

Now let $x \in U_j \cap B_\rho(\xi^*, R_j)$. We apply the maximum principle over the connected component V_x of $U_j \cap B(x, R_{j-1} - R_j)$ containing x. In view of Lemma 1 we have $|x - \pi(\xi^*)| < R_j$, so that $V_x \subset B(\pi(\xi^*), R_{j-1})$. Hence Lemma 2 yields that $V_x \subset B_\rho(\xi^*, R_{j-1})$. Moreover, we have

$$D \cap \partial V_x \subset (D \cap \overline{V_x} \cap S(x, R_{j-1} - R_j)) \cup (B_\rho(\xi^*, R_{j-1}) \cap \partial U_j). \tag{3.7}$$

In fact, suppose $y \in D \cap \partial V_x$ and $|y - x| < R_{j-1} - R_j$. Then there is $\varepsilon > 0$ such that $B(y, \varepsilon) \subset D \cap B(\pi(\xi^*), R_{j-1})$. By definition $V_x \cap B(y, \varepsilon) \ne \emptyset$, and hence $y \in B(y, \varepsilon) \subset B_\rho(\xi^*, R_{j-1})$ by Lemma 2. It is easy to see that $y \in \partial U_j$, so that (3.7) follows.

Since $\omega(\cdot, \overline{U_j} \cap S_\rho(\xi^*, R_{j-1}), U_j \cap B_\rho(\xi^*, R_{j-1}))$ vanishes q.e. on $\partial D \cup (B_\rho(\xi^*, R_{j-1}) \cap \partial U_j)$, it is less than or equal to

$$\omega(x, \overline{V_x} \cap S(x, R_{j-1} - R_j), V_x) \le \omega(x, \overline{U_j} \cap S(x, R_{j-1} - R_j), U_j \cap B(x, R_{j-1} - R_j))$$

by the maximum principle and (3.7). The last harmonic measure is less than or equal to $\exp(A_9 - A_{10}(R_{j-1}-R_j)/r_j)$ by (3.1) and (3.3). Since $A_{11}R^{n-2}G'_R(x,\xi_R) \geq \exp(-2^{j+1})$ for $x \in D_j$ by definition, (3.6) now becomes

$$\omega_0(x) \leq \left\{A_{11}\exp\left(2^{j+1}+A_9-\frac{A_{10}(R_{j-1}-R_j)}{r_j}\right)+d_{j-1}\right\}R^{n-2}G'_R(x,\xi_R)$$

for $x \in D_j \cap B_\rho(\xi^*, R_j)$. Dividing both sides by $R^{n-2}G'_R(x,\xi_R)$ and taking the supremum over $x \in D_j \cap B_\rho(\xi^*; R_j)$, we obtain

$$d_j \leq A_{11}\exp\left(2^{j+1}+A_9-\frac{A_{10}(R_{j-1}-R_j)}{r_j}\right)+d_{j-1}.$$

Hence (3.5) follows from (3.4). □

Lemma 6. *If $R>0$ is sufficiently small, then*

$$\frac{G_R(x,y)}{G_R(x',y)} \approx \frac{G_R(x,y')}{G_R(x',y')} \quad \text{for } x,x' \in B_\rho(\xi^*,R) \text{ and } y,y' \in S_\rho(\xi^*,6R)$$

with constant comparison depending only on D.

Proof. Let us take $x_R \in S_\rho(\xi^*,R)$ and $y_R \in S_\rho(\xi^*,6R)$ such that $A_5R \leq \delta_D(x_R) \leq R$ and $6A_5R \leq \delta_D(y_R) \leq 6R$. It is sufficient to show

$$G_R(x,y) \approx \frac{G_R(x_R,y)}{G_R(x_R,y_R)}G_R(x,y_R) \tag{3.8}$$

for $x \in B_\rho(\xi^*,R)$ and $y \in S_\rho(\xi^*,6R)$. For simplicity we fix $y \in S_\rho(\xi^*,6R)$ and let $u(x)$ (resp. $v(x)$) be the left (resp. right) hand side of (3.8).

First we show that $u \geq Av$ on $B_\rho(\xi^*,R)$ with A independent of y. Observe that

(i) u is a positive harmonic function on $D_R \setminus \{y\}$ with vanishing q.e. on ∂D_R;

(ii) v is a positive harmonic function on $D_R \setminus \{y_R\}$ with vanishing q.e. on ∂D_R.

Since u is superharmonic on D_R and $B_\rho(\xi^*,R) \subset D_R \setminus B(y_R, A_5R)$, it is sufficient to show that $u \geq Av$ on $S(y_R,A_5R)$ by the maximum principle. Take $z \in S(y_R, A_5R)$. Then $k_{D_R\setminus\{y_R\}}(z,x_R) \leq A$ by (2.7), and hence

$$v(z) \approx \frac{G_R(x_R,y)}{G_R(x_R,y_R)}G_R(x_R,y_R) = G_R(x_R,y) \leq AR^{2-n}. \tag{3.9}$$

If $y \in B(y_R, 2A_5R)$, then $u(z) = G_R(z,y) \geq AR^{2-n}$, so that $u(z) \geq Av(z)$. If $y \in D \setminus B(y_R, 2A_5R)$, then (2.7) and Lemma 4 yield

$$k_{D_R\setminus\{y\}}(z,x_R) \leq k_{D_R}(z,x_R) + A \leq A,$$

whence $v(z) \approx G_R(x_R, y) \approx G_R(z, y) = u(z)$ by (3.9). Hence we have $u \geq Av$ on $S(y_R, A_5R)$ in any case.

In order to show that $u(x) \leq Av(x)$, we make use of Lemma 5. It is clear that $G_R(x,z) \leq AR^{2-n} \approx G_R(x_R, y_R)$ for $x \in C_\rho(\xi^*, 2R)$ and $z \in B_\rho(\xi^*, 9R) \setminus B(\xi, 3R)$, where $\xi = \pi(\xi^*)$. Since $S_\rho(\xi^*, 2R) \subset C_\rho(\xi^*, 2R)$, it follows from the maximum principle that

$$G_R(\cdot, z) \leq AG_R(x_R, y_R)\omega(\cdot, S_\rho(\xi^*, 2R), B_\rho(\xi^*, 2R)) \quad \text{on } B_\rho(\xi^*, 2R).$$

Since $G_R(x_R, y_R) \approx R^{2-n}$ and $G_R(x, \xi_R) \approx G_R(x, y_R)$, it follows from Lemma 5 and the Harnack inequality that

$$G_R(x,z) \leq AG_R(x_R, y_R)R^{n-2}G_R(x, \xi_R) \leq AG_R(x, y_R) \tag{3.10}$$

for $x \in B_\rho(\xi^*, R)$ and $z \in B_\rho(\xi^*, 9R) \setminus B(\xi, 3R)$.

Now fix $x \in B_\rho(\xi^*, R)$ and $y \in S_\rho(\xi^*, 6R)$. If $\delta_D(y) \geq 2^{-1}A_5R$, then $k_{D_R}(y, y_R) \leq A$ by Lemma 4, so that $G_R(x, y) \approx G_R(x, y_R)$ and $G_R(x_R, y) \approx G_R(x_R, y_R)$ by the Harnack inequality. Hence (3.8) follows. Therefore, we may assume that $\delta_D(y) < 2^{-1}A_5R$. Then there is $\xi_1 \in \partial D$ such that $|y - \xi_1| = \delta_D(y) < 2^{-1}A_5R$. In view of Lemma 1, we find $\xi_1^* \in \partial^* D$ such that $\pi(\xi_1^*) = \xi_1$ and $y \in B_\rho(\xi_1^*, 2^{-1}A_5R)$ since $B(y, \delta_D(y)) \subset D$. Since $5R < 6R - 2^{-1}A_5R \leq |\xi - \xi_1| \leq 6R + 2^{-1}A_5R < 7R$, it follows from Lemmas 1 and 2 that $B_\rho(\xi_1^*, 2R) \subset B_\rho(\xi^*, 9R) \setminus B(\xi, 3R)$, and hence from (3.10) that $G_R(x, z) \leq AG_R(x, y_R)$ for $z \in B_\rho(\xi_1^*, 2R)$. Hence the maximum principle yields that

$$G_R(x,y) \leq AG_R(x, y_R)\omega(y, S_\rho(\xi_1^*, 2R), B_\rho(\xi_1^*, 2R)). \tag{3.11}$$

Using Lemma 5 with replacing ξ^* by ξ_1^*, we obtain

$$\omega(y, S_\rho(\xi_1^*, 2R), B_\rho(\xi_1^*, 2R)) \leq AR^{n-2}G_{B_\rho(\xi_1^*, A_6R)}(y, \xi_R')$$

with $\xi_R' \in S_\rho(\xi_1^*, 4R)$ such that $4A_5R \leq \delta_D(\xi_R') \leq 4R$. Since $|\xi - \xi_1| < 7R$, it follows from Lemma 2 that $B_\rho(\xi_1^*, A_6R) \subset B_\rho(\xi^*, (A_6 + 7)R) = D_R$, so that

$$\omega(y, S_\rho(\xi_1^*, 2R), B_\rho(\xi_1^*, 2R)) \leq AR^{n-2}G_R(y, \xi_R') = AR^{n-2}G_R(\xi_R', y).$$

Hence (3.11) becomes

$$G_R(x,y) \leq AG_R(x, y_R)R^{n-2}G_R(\xi_R', y) \leq AG_R(x, y_R)R^{n-2}G_R(x_R, y)$$

by the Harnack inequality. Since $G_R(x_R, y_R) \approx R^{2-n}$, we have $u(x) \le Av(x)$. Thus (3.8) is proved. The proof is complete. □

Proof of Theorem 2. We prove the theorem with $A_7 = A_6 + 7$. Since u is a positive harmonic function on D_R, we can consider the regularized reduced function $\widehat{R}_u^{S_\rho(\xi^*,6R)}$ of u to $S_\rho(\xi^*, 6R)$ with respect to D_R. This regularized reduced function is a superharmonic function on D_R such that $\widehat{R}_u^{S_\rho(\xi^*,6R)} = u$ q.e. on $S_\rho(\xi^*, 6R)$ and harmonic on $D_R \setminus S_\rho(\xi^*, 6R)$. Moreover, $\widehat{R}_u^{S_\rho(\xi^*,6R)} = 0$ q.e. on ∂D_R by assumption. Since u is bounded on D_R, it follows from the maximum principle that $u = \widehat{R}_u^{S_\rho(\xi^*,6R)}$ on $B_\rho(\xi^*, 6R)$. It is easy to see that $\widehat{R}_u^{S_\rho(\xi^*,6R)}$ is a Green potential of a measure μ supported on $S_\rho(\xi^*, 6R)$, i.e.

$$u(x) = \int_{S_\rho(\xi^*,6R)} G_R(x,y)d\mu(y) \quad \text{for } \in B_\rho(\xi^*, 6R).$$

Let $x, x' \in B_\rho(\xi^*, R)$ and $y, y' \in S_\rho(\xi^*, 6R)$. Then

$$G_R(x,y) \approx \frac{G_R(x,y')}{G_R(x',y')} G_R(x',y)$$

by Lemma 6. Hence

$$u(x) \approx \frac{G_R(x,y')}{G_R(x',y')} \int_{S_\rho(\xi^*,6R)} G_R(x',y)d\mu(y) = \frac{G_R(x,y')}{G_R(x',y')} u(x').$$

Therefore,

$$\frac{u(x)}{u(x')} \approx \frac{G_R(x,y')}{G_R(x',y')} \quad \text{uniformly for } y' \in S_\rho(\xi^*, 6R).$$

Similarly,

$$\frac{v(x)}{v(x')} \approx \frac{G_R(x,y')}{G_R(x',y')}.$$

Hence the theorem follows. □

Remark 2. In view of the above proof, the assertion of Theorem 2 holds for an unbounded uniformly John domain if ξ^* lies over a finite boundary point ξ of D.

4. Proof of Theorem 1

Let $\mathscr{H}_{\xi^*}$ be the family of all positive harmonic functions h on D vanishing q.e. on ∂D, bounded on $D \setminus B_\rho(\xi^*, r)$ for each $r > 0$ and taking value $h(x_0) = 1$. A function h in $\mathscr{H}_{\xi^*}$ is called a *kernel function* at ξ normalized at x_0.

Lemma 7. *There is a constant $A \geq 1$ depending only on D such that*

$$A^{-1} \leq \frac{u}{v} \leq A \quad \text{for } u, v \in \mathscr{H}_{\xi^*}.$$

Proof. Let $u, v \in \mathscr{H}_{\xi^*}$ and let $r > 0$. Then u and v be bounded on $B_\rho(\xi_1^*, 2^{-1}r)$ for $\xi_1^* \in \partial D \cap \overline{S_\rho(\xi^*, r)}$. Hence Theorem 2 yields

$$\frac{u(x)}{v(x)} \approx \frac{u(x')}{v(x')} \quad \text{for } x, x' \in B_\rho(\xi_1^*, 2^{-1}r/A_7),$$

where A_7 is as in Theorem 2. This, together with the Harnack inequality, shows that

$$\frac{u(x)}{v(x)} \approx \frac{u(x')}{v(x')} \quad \text{for } x, x' \in S_\rho(\xi^*, r),$$

where the constant of comparison is independent of r. Then the same comparison holds for $x, x' \in D \setminus B_\rho(\xi^*, r)$ by the maximum principle. Since $u(x_0) = v(x_0) = 1$, it follows that

$$\frac{u(x)}{v(x)} \approx 1 \quad \text{for } x \in D \setminus B_\rho(\xi^*, r).$$

Since $r > 0$ is arbitrary small and the constant of comparison is independent of r, the lemma follows. □

Proof of Theorem 1. Lemma 7 actually shows that $\mathscr{H}_{\xi^*}$ is a singleton and that the function $u \in \mathscr{H}_{\xi^*}$ is minimal. This is proved by Ancona [2, Lemma 6.2]. For the reader's convenience we give a short proof below. Let

$$c = \sup_{\substack{u,v \in \mathscr{H}_{\xi^*} \\ x \in D}} \frac{u(x)}{v(x)}.$$

Then $1 \leq c < \infty$ by Lemma 7. It is sufficient to show that $c = 1$. Suppose to the contrary $c > 1$. Take arbitrary u, $v \in \mathscr{H}_{\xi^*}$. Then $v_1 = (cv - u)/(c-1) \in \mathscr{H}_{\xi^*}$, so that $u \leq cv_1 = c(cv-u)/(c-1)$, whence $(2c-1)u \leq c^2 v$ on D. This would imply

$$c = \sup_{\substack{u,v \in \mathscr{H}_{\xi^*} \\ x \in D}} \frac{u(x)}{v(x)} \leq \frac{c^2}{2c-1} < c,$$

a contradiction. Thus $c = 1$ and $\mathscr{H}_{\xi^*}$ is a singleton. Moreover, the function $u \in \mathscr{H}_{\xi^*}$ is minimal. For if h is a positive harmonic function not greater than u, then $h/h(x_0) \in \mathscr{H}_{\xi^*}$, so that $h = h(x_0)u$. Let $G(x, y)$ be the Green function for D. Put $K(x, y) = G(x, y)/G(x_0, y)$ for $x \in D$ and $y \in D \setminus \{x_0\}$. The Martin kernel is given as the limit of $K(x,y)$ when y tends to a ideal boundary point. If $y \to \xi^* \in \partial^* D$, then some subsequence of $\{K(\cdot,y)\}$ converges to a positive harmonic function in $\mathscr{H}_{\xi^*}$. However, since $\mathscr{H}_{\xi^*}$ is a singleton, it follows that all sequences $\{K(\cdot,y)\}$ must converge to the same positive harmonic function, the Martin kernel $K(\cdot,\xi^*)$ at ξ^*. Therefore $K(x,\cdot)$ extends continuously to $\overline{D}\setminus\{x_0\}$. The kernel function $K(\cdot,\xi^*)$ should be minimal. It is easy to see that distinct ideal boundary points on $\partial^* D$ have different kernel functions. Hence the Martin compactification of D is homeomorphic to D^*. The last assertion now follows from Proposition 1. The theorem is proved. □

Using Theorem 2, we can show the following theorems in the same way as in [1, Section 4]. We omit the details.

Theorem 3. *Let D be a uniformly John domain and let V be an open set and K a compact subset of V intersecting ∂D. Then there are $A>0$ and $\varepsilon>0$ depending on D, V and K such that*

$$\left|\frac{u(x)/v(x)}{u(y)/v(y)}-1\right| \le A\rho_D(x,y)^\varepsilon \quad \text{for } x,y\in D\cap K,$$

whenever u and v are positive harmonic functions on D, bounded on $D\cap V$ and vanishing q.e. on $\partial D\cap V$. Moreover, the ratio u/v extends to $D^*\cap\pi^{-1}(K)$ as a Hölder continuous function with respect to ρ_D.

This theorem is deduced from the following local version.

Theorem 4. *Let D be a uniformly John domain. Then there exist positive constants A and ε depending only on D with the following property: Let $\xi^*\in\partial^* D$ and $R>0$ be sufficiently small. Suppose u and v are positive bounded harmonic functions on $B_\rho(\xi^*, A_7 R)$ vanishing q.e. on $\partial D\cap\overline{B_\rho(\xi^*, A_7R)}$. Then*

$$\operatorname*{osc}_{B_\rho(\xi^*,r)}\frac{u}{v} \le A'\left(\frac{r}{R}\right)^\varepsilon \operatorname*{osc}_{B_\rho(\xi^*,R)}\frac{u}{v} \quad \text{for } 0<r\le R.$$

Similarly, the Martin kernel $K(x,\xi^*)$ for D is Hölder continuous function with respect to ρ_D.

Theorem 5. *Let D be a bounded uniformly John domain. If $\xi_1^*,\xi_2^*\in\partial^* D$ and $R\ge 4\rho_D(\xi_1^*,\xi_2^*)$, then*

$$\operatorname*{osc}_{D\setminus B_\rho(\xi^*,R)}\frac{K(\cdot,\xi_1^*)}{K(\cdot,\xi_2^*)} \le A\left(\frac{\rho_D(\xi_1^*,\xi_2^*)}{R}\right)^\varepsilon.$$

Moreover, if $x \in D \setminus B_\rho(\xi_1^*, R)$*, then*

$$\left| \frac{K(x,\xi_1^*)}{K(x,\xi_2^*)} - 1 \right| \le A \left(\frac{\rho_D(\xi_1^*,\xi_2^*)}{R} \right)^{\varepsilon}.$$

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