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Voronoi 図を用いた多次元データの補間

Interpolation of Multivariate Data Using Voronoi Diagrams

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Abstract: This paper presents a general framework for constructing a variety of multidimensional interpolants based on Voronoi diagrams. This framework includes previously known methods such as Sibson's interpolant and Minkowski's interpolant; moreover it contains infinitely many new interpolants. Computational experiments suggest that the smoothness improves by the proposed generalization. Hence, this framework gives a new and promising direction of research on the interpolation based on the Voronoi diagrams.

Keywords: Voronoi diagram, multivariate data interpolation, spatial surface.

1 Introduction

Interpolation is an extremely important technique to solve various problems in engineering such as differential equations and geometric modeling. The finite element method is one of the most practical approaches to the interpolation problem, and is well-established today. However, there is an alternative approach to the interpolation problem, which utilizes Voronoi diagrams. In this paper, we call this approach the *Voronoi diagram approach.* See, for example, [1, 5, 6] for the theory of the Voronoi diagrams.

The interpolation problem is formulated in the following way. Let $\varphi : \mathbf{R}^d \to \mathbf{R}$ be a function, whose value is known only on the points $P_1, \ldots,$ P_n. These points are called the *data sites.* The (exact) interpolation problem is to find a *good* function $\tilde{\varphi}$ such that

$$\tilde{\varphi}(P_i) = \varphi(P_i)$$

for $i = 1, \ldots, n$. The meaning of the word "good" depends on the context.

Thiessen first applied Voronoi diagrams to the interpolation problem [11]. Let P be the target point the value on which is to be estimated. In his method, the Voronoi diagram for the data sites is constructed. Assume that the point P belongs to the Voronoi region of P_i. Then, the value at P is estimated at the value at the point P_i. By the definition, Thiessen's interpolant is a piecewise constant function.

Recently, Sibson found another interpolation method [8, 9]. In his method, the Voronoi diagram for $\{P_1, \ldots, P_n, P\}$ is constructed. If the Voronoi regions of P and P_i are adjacent via a $(d-1)$-dimensional facet, we call P_i a *neighbor* of P. The critical fact he found is that the position vector of P can be expressed as a convex combination of the position vectors of P's neighbors with the coefficients computed from the second-order Voronoi diagram [8]. Hence we can interpret the coefficients of this convex combination as the coordinates of P. Sibson constructed C^0 and C^1 interpolants based on this coordinate system. Sibson's interpolation method was further researched by Farin [2] and Piper [7]. Farin proposed another C^1 interpolant based on the Bernstein polynomial [2].

On the other hands, Hiyoshi and Sugihara found another coordinate system, and proposed a C^0 interpolant [3, 4, 10]. We call this system Minkowski's coordinate system, because it is based on the Minkowski's theorem on convex polytopes.

Compared with the finite element interpolation, the Voronoi diagram approach has a lot of virtues [9]. For example, interpolants based on the Voronoi diagram approach behaves continuously when the data sites move. However, the Voronoi diagram approach is less flexible in some points: for example, it is difficult to construct even C^2 interpolants by the Voronoi diagram approach, as explained in Section 2. The main reason, we guess, is that the number of the coordinate systems that the Voronoi diagram approach stands on is too few. In this paper, we generalize the coordinate systems in order to make the Voronoi diagram approach more flexible, concentrating on the case when $d = 2$. Indeed we construct a large class of interpolants based on the Voronoi diagram approach, which contains both Sibson' and Minkowski's interpolants, and which contains infinitely many other interpolants. Computational experiments suggest that the smoothness improves by the proposed generalization. Hence, this generalization gives a new and promising direction of research on the interpolation based on the Voronoi diagrams.

Section 2 reviews Sibson's coordinate system and Minkowski's coordinate system. Section 3 gives another proof to Sibson's result, which provides the basic idea of our generalization. Section 4 proposes the generalization. Section 5 gives some computational experiments. Section 6 concludes our research.

2 Previous Works

In this section, we review the previous works briefly.

2.1 Notations

First, let us introduce some notations.

Assume that a finite number of points $P_1, \dots, P_n \in \mathbf{R}^2$ are given. P_i's are called the *generators.* Let $V(P_i)$ be the set of all the points $Q \in \mathbf{R}^2$ such that $\mathrm{d}(Q, P_i) < \mathrm{d}(Q, P_j)$ for $j \neq i$, where $d(P, Q)$ denotes the Euclidean distance between two points P and Q. We call $V(P_i)$ the *Voronoi region* of the generator P_i. By the definition, the boundary of each $V(P_i)$ is a convex polygon. The Euclidean plane $\mathbf{R}^2$ is almost partitioned into $V(P_1), \dots, V(P_n)$, that is, the measure of the set of all the point that does not belong to any $V(P_i)$ is zero. The collection of all $V(P_i)$'s is denoted by $\mathcal{V}(P_1, \dots, P_n)$ and is called the *Voronoi diagram* for the generator set $\{P_1, \dots, P_n\}$. When two Voronoi regions $V(P_i)$ and $V(P_j)$ are adjacent via some open line segment, this line segment is called the *Voronoi edge* between P_i and P_j, and is denoted by $E(P_i, P_j)$. If $E(P_i, P_j)$ exists, P_j is called a *neighbor* of P_i (and vice versa).

For $i \neq j$, let $V(P_i, P_j)$ be the set of all the points $Q \in \mathbf{R}^2$ such that

$$\mathrm{d}(Q, P_i) < \mathrm{d}(Q, P_j) < \mathrm{d}(Q, P_k)$$

for $k \neq i, j$. $V(P_i, P_j)$ is called the *second-order Voronoi region* of the ordered pair (P_i, P_j). By the definition, each $V(P_i)$ is almost partitioned into $V(P_i, P_j)$, $i \neq j$. The collection of all $V(P_i, P_j)$'s is called the *second-order Voronoi diagram* for the generator set $\{P_1, \dots, P_n\}$.

See, for example, [1, 5, 6] for the detail of the Voronoi diagram theory.

2.2 Sibson coordinates

Now let us describe Sibson's interpolation method [8, 9] briefly.

Let $P_1, \dots, P_n \in \mathbf{R}^2$ be the given data sites, and let $y_1, \dots, y_n \in \mathbf{R}$ be the data value associated with $P_1, \dots, P_n$, respectively. Assume that we want to evaluate the value on the target point P. Here, we require that P is an inner point of

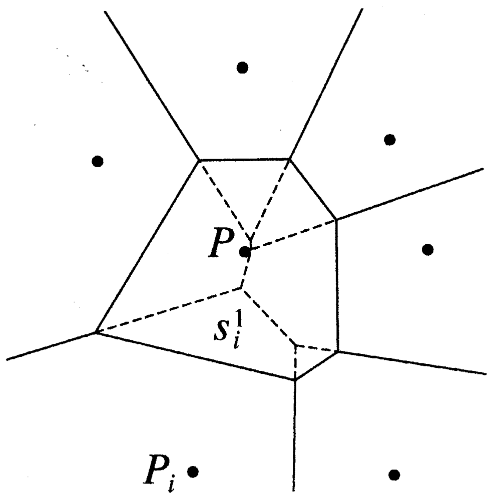


Figure 1. The Voronoi region $V(P)$ is partitioned into the subregions $V(P, P_i)$.

the convex hull of $P_1, \ldots, P_n$, but does not equal to any of $P_1, \ldots, P_n$.

According to Sibson [8], the Voronoi diagram $\mathcal{V}(P_1, \ldots, P_n, P)$ is constructed. Each $V(P_i)$ is not necessarily bounded, but $V(P)$ is always bounded because P is an inner point of the convex hull of $P_1, \ldots, P_n$. Now partition $V(P)$ into $V(P, P_i)$'s, and let $s_i^1(P)$ denote the measure of $V(P, P_i)$. Note that $s_i^1(P) > 0$ if and only if P_i is a neighbor of P.

Figure 1 shows this situation. In this figure, the Voronoi region $V(P)$ is partitioned into the subregions $V(P, P_i)$. The boundaries of this partition are drawn by the dashed lines.

Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ and $\boldsymbol{x}$ denote the position vectors of $P_1, \ldots, P_n$ and P, respectively. Sibson [8] proved the following identity:

$$\sum_{i=1}^{n} s_i^1(P)\boldsymbol{x} = \sum_{i=1}^{n} s_i^1(P)\boldsymbol{x}_i. \qquad (1)$$

We will give another proof of this identity in Section 3, which gives the basic idea of the generalization we propose.

Now define that

$$\hat{s}_i^1(P) = s_i^1(P) \Big/ \sum_{j=1}^{n} s_j^1(P).$$

It follows from the definition that

$$\sum_{i=1}^{n} \hat{s}_i^1(P) = 1,$$

$$0 \leq \hat{s}_i^1(P) \leq 1.$$

Thus $\boldsymbol{x}$ is expressed as the following convex combination:

$$\boldsymbol{x} = \sum_{i=1}^{n} \hat{s}_i^1(P)\boldsymbol{x}_i.$$

We call $\hat{s}_i^1(P)$'s the *Sibson coordinates* of the point P.[1] Note that $\hat{s}_j(P) \to \delta_{ij}$ as P approaches P_i. Therefore we define

$$\hat{s}_j^1(P) = \delta_{ij}$$

when P coincides with P_i.

From the Sibson coordinates, Sibson constructed the following C^0 interpolant:

$$\tilde{\varphi}^1(P) = \sum_{i=1}^{n} y_i \hat{s}_i^1(P).$$

Figure 2 shows an example of a surface obtained from the above interpolant.

Although C^1 interpolants has been also proposed by Sibson [9] and Farin [2], it is difficult to construct interpolants that have higher-order continuity. This difficulty comes from a global property of $\hat{s}_i^1$. Piper showed that $\hat{s}_i^1$'s are differentiable everywhere except the points P_i, but C^2-continuity fails on the Delaunay circles of the data sites [7]. Devising a technique to avoid this non-smoothness is difficult.

Figure 3 shows where C^2-continuity fails. In this figure, the Voronoi diagram for the 11 data sites is drawn by dashed lines. In particular, the generator P_1 has five neighbors $P_2, \ldots, P_6$. In other words, the boundary of the Voronoi region of P_1 is a pentagon. Hence, there are five Delaunay circles that are concerned with P_1, on which the C^2-continuity of $\hat{s}_1^1$.

[1] we also call $s_i^1(P)$'s the Sibson coordinates when there is no confusion. In fact, $(s_1^1, \ldots, s_n^1)$ and $(\hat{s}_1^1, \ldots, \hat{s}_n^1)$ denote the same point in the $(n-1)$-dimensional real projective space.

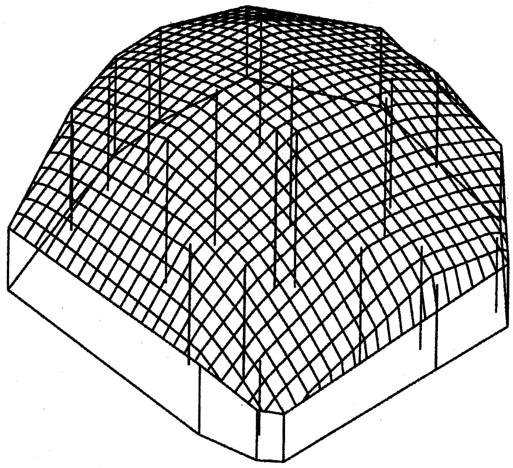


Figure 2. An example of a surface obtained from Sibson's interpolant.

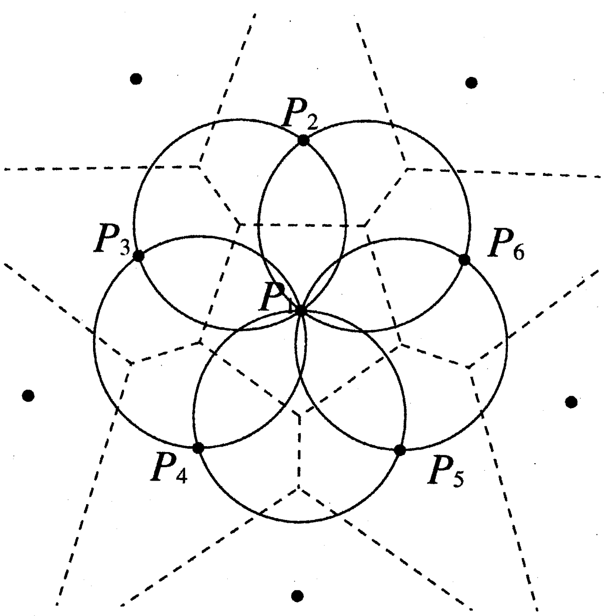


Figure 3. Circles on which C^2-continuity of the Sibson coordinates fails.

2.3 Minkowski coordinates

Recently, Hiyoshi and Sugihara found another coordinate system [3, 4, 10]. In their method, the Voronoi diagram $\mathcal{V}(P_1, \ldots, P_n, P)$ is also constructed. For $i = 1, \ldots, n$, define that

$$s_i^0(P) = l_i(P)/d_i(P),$$

where $l_i(P)$ is the length of the Voronoi edge $E(P, P_i)$ if any, and $d_i(P)$ is the Euclidean distance between P and P_i. Note that $s_i^0(P) > 0$ if and only if P_i is a neighbor of P. Then, the following identity follows, as proved in Section 3.1.

$$\sum_{i=1}^{n} s_i^0(P)\boldsymbol{x} = \sum_{i=1}^{n} s_i^0(P)\boldsymbol{x}_i. \tag{2}$$

Now let us define that

$$\hat{s}_i^0(P) = s_i^0(P) / \sum_{j=1}^{n} s_j^0(P).$$

We call $s_j^0(P)$'s the *Minkowski coordinates* (The name "Minkowski coordinates" is after the underlying theorem. See Section 3.1). Then, $\boldsymbol{x}$ can be expressed as another convex combination:

$$\boldsymbol{x} = \sum_{i=1}^{n} \hat{s}_i^0(P)\boldsymbol{x}_i.$$

It is clear that $\hat{s}_j^0(P) \to \delta_{ij}$ as P approaches P_i.

Hiyoshi and Sugihara proposed the following interpolant from the Minkowski coordinates [3, 4, 10]:

$$\tilde{\varphi}^0(P) = \sum_{i=1}^{n} y_i \hat{s}_i^0(P).$$

$\tilde{\varphi}^0$ does not have C^1-continuity on the Delaunay circles of $P_1, \ldots, P_n$ [3] (*cf* Figure 3).

Figure 4 shows an example of a surface obtained from $\tilde{\varphi}^0$.

3 Another Proof of Sibson's Identity

In this section, we give another proof of Sibson's identity. Observing this proof closely leads to the generalization proposed in Section 4.

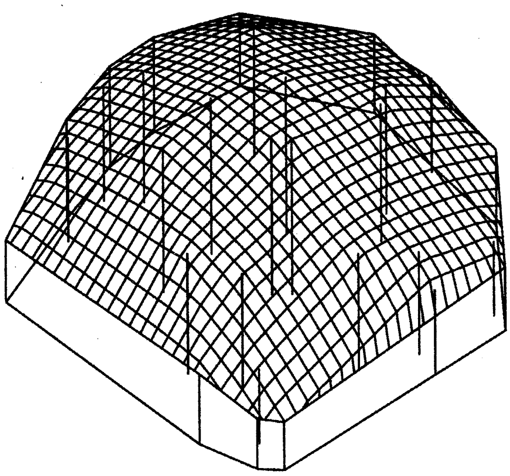


Figure 4. An example of a surface obtained from the Minkowski's interpolant.

3.1 Proof of the identity for the Minkowski coordinates

At first, let us give a proof of the identity (2), which comes from Hiyoshi-Sugihara [4]. For this purpose, the following lemma is used as a fundamental tool, which is known as Minkowski's theorem:

Lemma 1 *For any region $V \subset \mathbf{R}^d$, the following equation holds:*

$$\int_{Q\in\partial V} \boldsymbol{n}\mathrm{d}S = \mathbf{0}, \tag{3}$$

where ∂V denotes the boundary of V, $\boldsymbol{n}$ denotes the unit outer normal vector to ∂V at Q, and $\mathrm{d}S$ denotes the infinitesimal surface element at Q. □

Let $P_{\iota(1)}, \dots, P_{\iota(k)}$ be the neighbors of the target point P. In general, the boundary of the Voronoi region of some generator Q is a (possibly unbounded) polygon each edge of which is a part of the perpendicular bisector of the line segment $\overline{QQ'}$ with another generator Q'. Therefore, the unit outer normal vector to the Voronoi edge $E(P, P_{\iota(i)})$ is denoted by $(1/d_{\iota(i)})\overrightarrow{PP_{\iota(i)}}$. From this fact and Minkowski's theorem, we get

$$\sum_{i=1}^{k} \frac{l_{\iota(i)}}{d_{\iota(i)}} \overrightarrow{PP_{\iota(i)}} = \mathbf{0}.$$

Note that $l_i = 0$ if P_i is not a neighbor of P. Hence we get

$$\sum_{i=1}^{n} \frac{l_i}{d_i}\boldsymbol{x} = \sum_{i=1}^{n} \frac{l_i}{d_i}\boldsymbol{x}_i,$$

which proves (2).

3.2 Proof of Sibson's identity

Observing the proof we gave in Section 3.1 closely, we notice that the crucial fact is that for each P's neighbor P_i, the Voronoi edge $E(P, P_i)$ is perpendicular to the vector $\overrightarrow{PP_i}$. Therefore, if we find a polygon each of whose edges is perpendicular to the vector $\overrightarrow{PP_i}$ with some neighbor P_i, we can obtain another coordinate system from that polygon.

At first, we describe a procedure for constructing such polygons. For this purpose, the Voronoi diagram $\mathcal{V}(P_1, \dots, P_n, P)$ is constructed. We denote P's neighbors by $P_{\iota(1)}, \dots, P_{\iota(k)}$ in the counterclockwise order. Without loss of generality, we assume that $P_{\iota(1)}$ is the nearest generator to P (choose arbitrary one if not unique). Partition $V(P)$ into the subregions $V(P, P_{\iota(1)}), \dots, V(P, P_{\iota(k)})$. Let $S_i \in \overline{V(P, P_{\iota(i)})}$ be the point furthest from the line containing $E(P, P_{\iota(i)})$, and let h_i be the distance of S_i from the line containing $E(P, P_{\iota(i)})$.

Let us pay attention to $V(P, P_{\iota(1)})$. Since $P_{\iota(1)}$ is the nearest generator to P, $\overline{V(P, P_{\iota(1)})}$ contains the target point P. We start constructing the polygon by stroking its first edge inside $V(P, P_{\iota(1)})$. The following procedure outputs the vertices of a desired polygon.

1. Let L_1 be the line perpendicular to $\overrightarrow{PP_{\iota(1)}}$ such that $L_1 \cap V(P, P_{\iota(1)})$ is not empty.

2. Let $\{Q_1, Q_2\} = \partial V(P, P_{\iota(1)}) \cap L_1$ such that S_1 is located on the left of $\overrightarrow{Q_1Q_2}$.

3. Set $i \leftarrow 2$ and $j \leftarrow 2$.
4. Let L_i be the line that is perpendicular to $\overrightarrow{PP_{\iota(i)}}$ and that contains Q_i.
5. If $L_i \cap V(P, P_{\iota(i)})$ is empty, go to 7.
6. Set $j \leftarrow j+1$. Let Q_j be the other endpoint of the line segment $L_i \cap V(P, P_{\iota(i)})$.
7. Set $i \leftarrow i+1$.
8. If $i \leq k$, go to 4. Otherwise output $Q_1, \ldots, Q_{j-1}$ and terminate.

We denote the obtained polygon by $C(t)$, where the parameter is determined in the following manner. For $i = 1, \ldots, k$, let u_i denote the distance of the line segments $V(P, P_{\iota(i)}) \cap C(t)$, if any, from the Voronoi edge $E(P, P_{\iota(i)})$. Then, t is set to $(h_1 - u_1)/h_1$. It follows from the definition that $0 < t < 1$. Note that $C(t)$ tends to the boundary of $V(P)$ as $t \to 1$. In fact, the above procedure can be seen as the incremental algorithm for constructing Voronoi diagrams.

By setting $\partial V = C(t)$ in Lemma 1, we get the following identity:

$$\sum_{i=1}^{n} s_i^0(t; P)\boldsymbol{x} = \sum_{i=1}^{n} s_i^0(t; P)\boldsymbol{x}_i, \tag{4}$$

where

$$s_i^0(t; P) = l_i(t; P)/d_i(P),$$
$$l_i(t; P) = (\text{the length of } C(t) \cap V(P_i)).$$

We call $s_i^0(t; P)$'s the *weak Minkowski coordinates with parameter t*. Note that $s_i^0(t; P) \to s_i^0(P)$ as $t \to 1$. Hence we define $s_i^0(1; P) = s_i^0(P)$ and interpret $s_i^0(P)$ as the special case of the weak Minkowski coordinates.

For the purpose of proving Sibson's identity, we require another fact. Let $e_i(t)$ denote the line segment $V(P_i) \cap C(t)$.

Lemma 2 *Two polygons $C(t)$ and $C(t + \mathrm{d}t)$ given, assume that the edges $e_i(t)$, $e_i(t+\mathrm{d}t)$, $e_j(t)$, and $e_j(t + \mathrm{d}t)$ exist. Let $\mathrm{d}u_i$ be the width between $e_i(t)$ and $e_i(t + \mathrm{d}t)$, and let $\mathrm{d}u_j$ be the width between $e_j(t)$ and $e_j(t + \mathrm{d}t)$. Then,*

$$d_i \mathrm{d}u_i = d_j \mathrm{d}u_j.$$

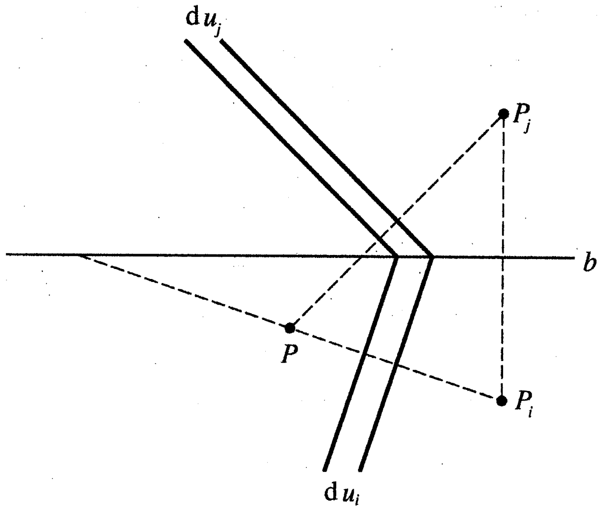


Figure 5. Consecutive edges of $C(t)$ and $C(t + \mathrm{d}t)$.

Proof. Let b denote the perpendicular bisector of the line segment $\overline{P_iP_j}$. Then, the Voronoi edge $E(P_i, P_j)$ is a part of b. Let θ_i denote the angle generated by the lines b and $\overline{PP_i}$, and let θ_j denote the angle generated by the lines b and $\overline{PP_j}$. See Figure 5.

It is clear that

$$\frac{\mathrm{d}u_i}{\cos\theta_i} = \frac{\mathrm{d}u_j}{\cos\theta_j}.$$

Since $b \perp \overline{P_iP_j}$, the following equations hold:

$$\angle PP_iP_j = 90^\circ - \theta_i, \quad \angle PP_jP_i = 90^\circ - \theta_j.$$

On the other hand, the sine theorem on the triangle PP_iP_j yields the equation

$$\frac{d_j}{\sin\angle PP_iP_j} = \frac{d_i}{\sin\angle PP_jP_i}.$$

From the above equations, we get

$$d_i \mathrm{d}u_i = d_j \mathrm{d}u_j.$$

□

Now we give another proof of Sibson's identity. Minkowski's theorem guarantees (4) for any $0 < t < 1$. Operating $d_1 h_1 \int_0^1 \mathrm{d}t$ on (4), we get

$$d_1 h_1 \sum_{i=1}^{n} \int_0^1 s_i^0(t;P)\mathrm{d}t\boldsymbol{x}$$
$$= d_1 h_1 \sum_{i=1}^{n} \int_0^1 s_i^0(t;P)\mathrm{d}t\boldsymbol{x}_i.$$

Note that Lemma 2 implies that

$$\mathrm{d}t = -\frac{1}{h_1}\mathrm{d}u_1 = -\frac{d_i}{d_1 h_1}\mathrm{d}u_i$$

for $i = 1, \dots, k$. Hence we get

$$\sum_{i=1}^{k} \int_0^{h_i} l_{\iota(i)}(t(u_i);P)\mathrm{d}u_i\boldsymbol{x}$$
$$= \sum_{i=1}^{k} \int_0^{h_i} l_{\iota(i)}(t(u_i);P)\mathrm{d}u_i\boldsymbol{x}_{\iota(i)}.$$

Since

$$s^1_{\iota(i)}(t;P) = \int_0^{h_i} l_{\iota(i)}(t(u_i);P)\mathrm{d}u_i,$$

the above equation completes the proof.

4 Generalization of Coordinate Systems

Observing the proof given in Section 3.2 carefully, we notice that Sibson's coordinate system is not the only one that we can obtain; other coordinate systems can be obtained by modifying the expression slightly. The following are typical modifications:

1. Any subinterval (a, b) of $(0, 1)$ is available as the integration interval.
2. Furthermore, any interval (a, b) with $0 < a < b$ is available when we extend $C(t)$ naturally for $t > 1$. $C(t)$ with $t > 1$ lies outside the Voronoi region $V(P)$.
3. In the integration step, the weight function $w(t)$ can be multiplied.
4. Multiple integration. We can construct other coordinate systems by integrating previously obtained coordinates repeatedly.

4.1 Standard interpolants

Although a lot of coordinate systems can be constructed by combining the above modifications, we guess the following subclass of $s_i^k(P)$ is especially important. Each $k = 0, 1, \dots$, the coordinate $s_i^k(P)$ is calculated in the following recursion:

$$s_i^k(P) = s_i^k(1;P),$$

where

$$s_i^k(t;P) = d_1 h_1 \int_0^t s_i^{k-1}(s;P)\mathrm{d}s \quad \text{for } 0 < t < 1.$$

We call $s_i^k(P)$ the *order-k standard coordinates.* Minkowski's coordinate system and Sibson's coordinate system are the first two coordinate systems in this subclass. The corresponding interpolant

$$\tilde{\varphi}^k(P) = \sum_{i=1}^{n} y_i s_i^k(P) \tag{5}$$

is called the *order-k standard interpolant.*

5 Computational Experiment

In this section, we present some computational results about the order-k standard interpolant.

Assume that the data $y_j = \delta_{ij}$ for some $i = 1, \dots, n$ are given. Then we obtain the function $\tilde{\varphi}^k(P) = \hat{s}_i^k(P)$ as the result. Therefore, we give $\hat{s}_i^k$ the alias "hat function". The hat functions are important because the standard interpolant is a linear combination of the hat functions.

Figure 6 describes examples of hat functions of $\tilde{\varphi}^0$, $\tilde{\varphi}^1$ and $\tilde{\varphi}^2$ for the data sites drawn in Figure 3. Although our eyes can hardly see the difference among the figures, the following discussion clarifies their difference.

Figure 7 describes the cross sections of hat functions cut by a vertical plane. If we see the cross section of $\tilde{\varphi}^0$, we notice that some non-smooth points, as explained in Section 2. The slopes of these cross sections are drawn in Figure 8. The differentiation was done by the numerical manner.

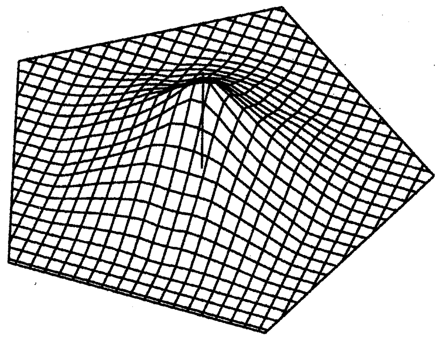
Figure 8 tells that $\tilde{\varphi}^1$ loses C^2-continuity, whereas $\tilde{\varphi}^2$ still has C^2-continuity. Therefore these computational experiments suggest that the obtained surface becomes smoother as k increases. We conjecture that $\tilde{\varphi}^k$ has C^k-continuity except at the data sites.

6 Concluding remarks

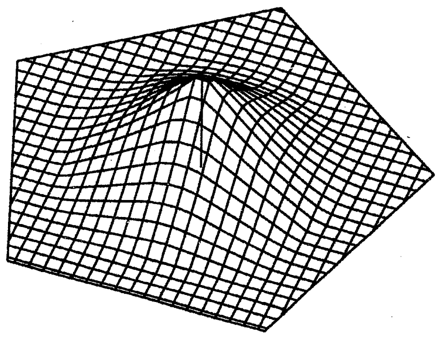
This paper gave another proof of Sibson's identity from Minkowski's theorem. Based on the underlying idea, this paper generalized Sibson's coordinate system and Minkowski's coordinate system. This generalization implies direction of the research of the Voronoi diagram approach. Indeed the generalization contains both Sibson's and Minkowski's interpolant, and it also contains infinitely many new interpolants.

A lot of works must be done, including the following:

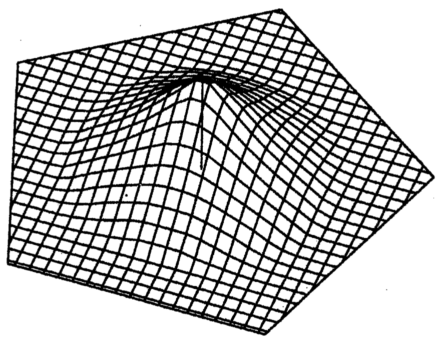
- To research the smoothness and other properties of the standard interpolants.
- To select better interpolants, if any, than the standard interpolants.
- To develop the applications of the Voronoi diagram approach. Spatial surface construction is one of potential applications.



(a) A hat function of $\tilde{\varphi}^0$.

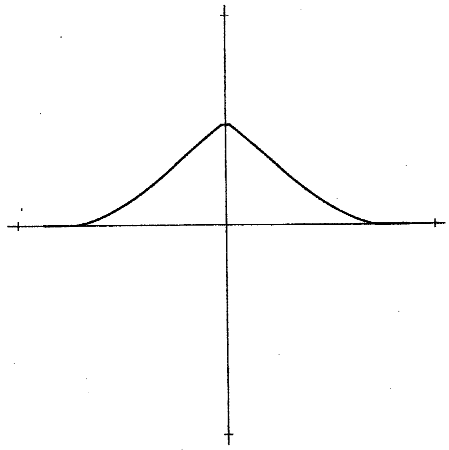


(b) A hat function of $\tilde{\varphi}^1$.

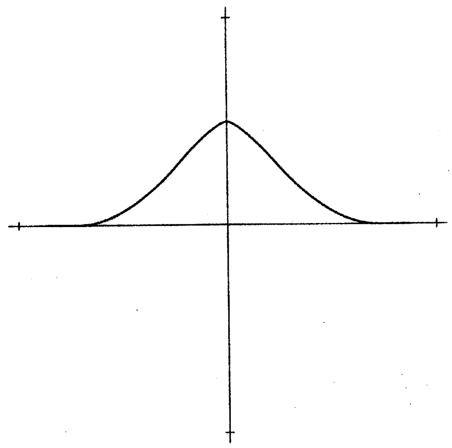


(c) A hat function of $\tilde{\varphi}^2$

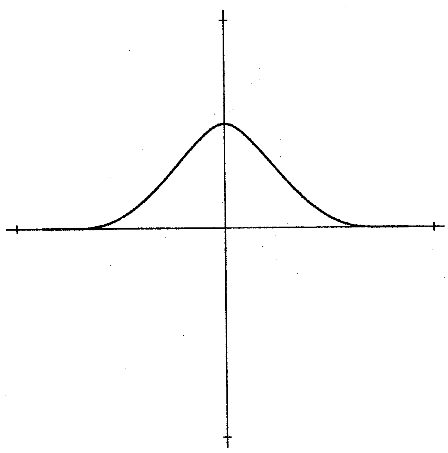
Figure 6. Hat functions of standard interpolants.



(a) A hat function of $\tilde{\varphi}^0$.

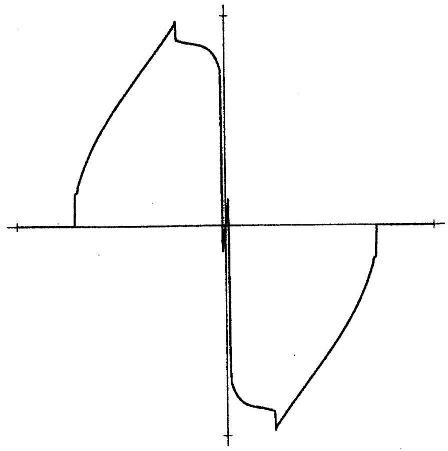


(b) A hat function of $\tilde{\varphi}^1$.

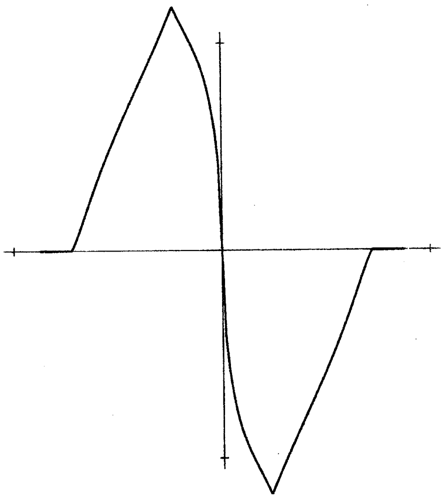


(c) A hat function of $\tilde{\varphi}^2$

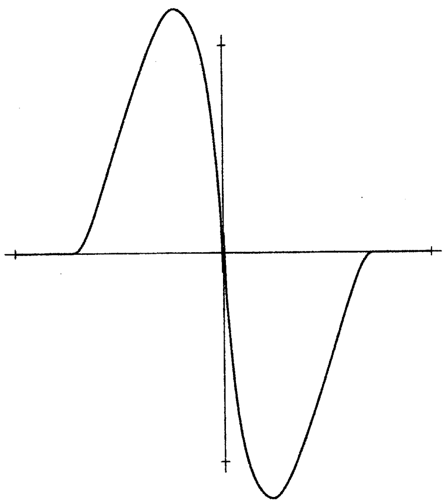
Figure 7. Cross sections of hat functions.



(a) Derivative function of the cross section.



(b) Derivative function of the cross section.



(c) Derivative function of the cross section.

Figure 8. Derivative functions of the cross sections.

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