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渦管の運動速度に対する高次漸近形

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1 Introduction

The motion of a thin vortex tube is a venerable problem, and since the age of Helmholtz and Kelvin, extensive study has been made on various dynamical aspects, such as formation, traveling speed, waves, instability, interactions and so on.

Concerning the steady motion of an thin axisymmetric vortex ring in an incompressible fluid of infinite extent, the traveling speed U is known, for a specific vorticity distribution in proportion to the distance from the axis of symmetry, to third (virtually fourth) order in a small parameter $\epsilon=\sigma/R_0$, the ratio of core radius σ to the ring radius R_0, as

$$U=\frac{\Gamma}{4\pi R_0}\left\{\log\left(\frac{8}{\epsilon}\right)-\frac{1}{4}-\frac{3\epsilon^2}{8}\left[\log\left(\frac{8}{\epsilon}\right)-\frac{5}{4}\right]+O(\epsilon^4\log\epsilon)\right\}, \tag{1.1}$$

where Γ is the circulation carried by the ring (Dyson 1893). The first two terms are Kelvin's formula which are considered as the first order. Dyson achieved an extension to third order, by taking account of an elliptical deformation of the cross-section of the core caused by the self-induced straining field.

The influence of viscosity ν upon traveling speed of an axisymmetric vortex ring was calculated to first order in $\varepsilon\equiv(\nu/\Gamma)^{1/2}$, a measure of the ratio of core- to ring-radii, by Saffman (1970). Fukumoto & Moffatt (1999) succeeded in constructing a formula for the third-order correction to the traveling speed.

In contrast, it is not easy to render the motion of a curved vortex filament amenable to a systematic analysis. The simplest asymptotic theory is the so called '*localized induction approximation (LIA)*' (Da Rios 1906); the induced velocity at each point of the filament is dominated by the contribution from the neighboring segment of length $2L$. In addition, introducing a short cut-off σ, we are led to the following evolution equation for the filament curve $\boldsymbol{X}=\boldsymbol{X}(s,t)$, expressed as functions of the arclength s and the time t:

$$\boldsymbol{X}_t=\tilde{A}\kappa\boldsymbol{b}\,;\quad \tilde{A}=\frac{\Gamma}{4\pi}\log\left(\frac{2L}{\sigma}\right), \tag{1.2}$$

where κ is the curvature, $\boldsymbol{b}$ is the binormal vector, and a subscript denotes a differentiation with respect to the indicated variable. In this treatment, both L and σ remain undetermined. The distinguishing feature is that, supposing that $\tilde{A}$ is a constant, (1.2) becomes a completely integrable evolution equation equivalent to a cubic nonlinear Schrödinger equation as shown by Hasimoto (1972). Langer & Perline (1991) unveiled the bi-Hamiltonian structure of (1.2) behind this integrability, and manipulated a recursion operator to generate successively an infinite sequence of commuting vector fields $\boldsymbol{V}^{(n)}$ $(n = 1, 2, \cdots)$ starting from (1.2). This sequence is referred to as the '*localized induction hierarchy (LIH)*'. A first few of them are provided, in terms of curvature κ, torsion τ and the Frenet-Serret vectors $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b})$, as follows:

$$\boldsymbol{V}^{(1)} = \kappa \boldsymbol{b}\,, \tag{1.3}$$

$$\boldsymbol{V}^{(2)} = \frac{1}{2}\kappa^2 \boldsymbol{t} + \kappa_s \boldsymbol{n} + \kappa\tau \boldsymbol{b}\,, \tag{1.4}$$

$$\boldsymbol{V}^{(3)} = \kappa^2\tau \boldsymbol{t} + (2\kappa_s\tau + \kappa\tau_s)\boldsymbol{n} + (\kappa\tau^2 - \kappa_{ss} - \frac{1}{2}\kappa^3)\boldsymbol{b}\,, \tag{1.5}$$

$$\cdots .$$

Observe that, for a circle, a superposition of (1.3) and (1.5) is no other than (1.1).

This unexpected coincidence inspires us to pursue the higher-order velocity of a vortex filament in three dimensions. Fukumoto & Miyazaki (1991) showed that a vortex filament with axial velocity in the core obeys, in the LIA, an evolution equation comprising a summation of (1.3) and (1.4). In the present investigation, we rule out axial flow at leading order, and make an attempt at a further extension of matched asymptotic expansions to $O(\epsilon^3)$. It will be clarified that axial flow is induced at $O(\epsilon^2)$ by axial pressure gradient stemming from torsion and variation of curvature along the filament.

In §2, we devise a technique to derive a new asymptotic development of the Biot-Savart law. The resulting expression serves as the inner limit of the outer expansion. In §3, we give a concise description of the procedure of the inner expansion. The inner solution and the velocity of a vortex filament are obtained to $O(\epsilon^3)$.

2 Asymptotic development of the Biot-Savart law

Let us consider kinematics of vorticity in a three-dimensional space of infinite extent, filled with an incompressible fluid. Once that the vorticity $\boldsymbol{\omega}(\boldsymbol{x})$ is specified at each point, the velocity $\boldsymbol{v}(\boldsymbol{x})$ of the fluid at a position $\boldsymbol{x}$ is uniquely determined by the Biot-Savart law:

$$\boldsymbol{v} = \nabla \times \boldsymbol{A}\,; \quad \boldsymbol{A}(\boldsymbol{x}) = \frac{1}{4\pi}\iiint \frac{\boldsymbol{\omega}(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}\,dV'\,. \tag{2.1}$$

In order to evaluate (2.1) at points near the core, it is expedient to introduce local coordinates $(\tilde{x}, \tilde{y}, \xi)$ moving with the filament. Here ξ parameterizes the central curve of the filament. Given a point $\boldsymbol{x}$ sufficiently close to the core, there corresponds uniquely the nearest point $\boldsymbol{X}(\xi, t)$ on the centerline of filament. Then $\boldsymbol{x}$ is expressed as

$$\boldsymbol{x} = \boldsymbol{X}(\xi, t) + \tilde{x}\boldsymbol{n}(\xi, t) + \tilde{y}\boldsymbol{b}(\xi, t)\,, \tag{2.2}$$

$$= \boldsymbol{X} + r\cos\phi\,\boldsymbol{n} + r\sin\phi\,\boldsymbol{b}\,, \tag{2.3}$$

where (r,ϕ) are cylindrical coordinates in the plane made up from bases $(\boldsymbol{n},\boldsymbol{b})$. Note that (r,ϕ,ξ) do not constitute orthogonal coordinates. They are converted into orthogonal ones by replacing ϕ with θ defined by

$$\theta(s,t)=\phi-\int_{s_0}^{s}\tau(s',t)\,ds'\,. \tag{2.4}$$

We introduce the relative velocity $\boldsymbol{V}=(u(r,\theta,\xi,t),v(r,\theta,\xi,t),w(r,\theta,\xi,t))$:

$$\boldsymbol{v}=\dot{\boldsymbol{X}}(\xi,t)+u\boldsymbol{e}_r+v\boldsymbol{e}_\theta+w\boldsymbol{t}\,, \tag{2.5}$$

where a dot stands for a derivative in t with fixing ξ, and $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ are the unit vectors in the radial and azimuthal directions respectively. The vorticity is then represented by

$$\boldsymbol{\omega} = \omega_r\boldsymbol{e}_r+\omega_\theta\boldsymbol{e}_\theta+\zeta\boldsymbol{t}\,, \tag{2.6}$$

$$\begin{aligned} &= \left\{\frac{1}{r}\frac{\partial w}{\partial\theta}-\frac{1}{h_3}\frac{\partial v}{\partial\xi}+\frac{\eta\kappa}{h_3}w\sin\phi-\frac{1}{h_3}\frac{\partial\dot{\boldsymbol{X}}}{\partial\xi}\cdot\boldsymbol{e}_\theta\right\}\boldsymbol{e}_r \\ &+\left\{-\frac{\partial w}{\partial r}+\frac{1}{h_3}\frac{\partial u}{\partial\xi}+\frac{\eta\kappa}{h_3}w\cos\phi+\frac{1}{h_3}\frac{\partial\dot{\boldsymbol{X}}}{\partial\xi}\cdot\boldsymbol{e}_r\right\}\boldsymbol{e}_\theta+\left\{\frac{1}{r}\frac{\partial}{\partial r}(rv)-\frac{1}{r}\frac{\partial u}{\partial\theta}\right\}\boldsymbol{t}, \end{aligned} \tag{2.7}$$

where

$$\eta=\left|\frac{\partial\boldsymbol{X}}{\partial\xi}\right|, \quad h_3=\eta(1-\kappa r\cos\phi)\,. \tag{2.8}$$

Though incomplete, we ignore r- and θ- components of $\boldsymbol{\omega}$ and make the following ansatz:

$$\boldsymbol{\omega}=\zeta(\tilde{x},\tilde{y},t)\boldsymbol{t}(\xi,t)\,. \tag{2.9}$$

We require that $|\zeta|$ decays sufficiently rapidly to zero with the distance r from the vortex centerline. Using the shift-operator technique, the vector potential $\boldsymbol{A}$ in (2.1), with the vorticity being substituted from (2.9), is rewritten as

$$\begin{aligned} \boldsymbol{A}(\boldsymbol{x}) &= \frac{1}{4\pi}\iiint\zeta(\tilde{x},\tilde{y})\frac{\boldsymbol{t}(s)}{|\boldsymbol{x}-\boldsymbol{X}-\tilde{x}\boldsymbol{n}-\tilde{y}\boldsymbol{b}|}(1-\kappa\tilde{x})\,ds\,d\tilde{x}\,d\tilde{y} \\ &= \frac{1}{4\pi}\int ds\left\{\iint\zeta(\tilde{x},\tilde{y})(1-\kappa\tilde{x})\,e^{-\tilde{x}(\boldsymbol{n}\cdot\nabla)-\tilde{y}(\boldsymbol{b}\cdot\nabla)}\right\}\frac{\boldsymbol{t}(s)}{|\boldsymbol{x}-\boldsymbol{X}(s)|}\,. \end{aligned} \tag{2.10}$$

This expression is legitimate only when the Jacobian $(1-\kappa\tilde{x})$ of coordinate transformation is everywhere positive: $1-\kappa\tilde{x}>0$.

We are now ready to manipulate the inner limit of the outer expansion. The exponential function is formally expanded in powers of $\tilde{x}$ and $\tilde{y}$ as

$$\begin{aligned} \boldsymbol{A}(\boldsymbol{x}) &= \frac{1}{4\pi}\int ds\Big\{\iint d\tilde{x}d\tilde{y}\zeta(\tilde{x},\tilde{y})\big(1-\kappa\tilde{x}-\tilde{x}(\boldsymbol{n}\cdot\nabla)-\tilde{y}(\boldsymbol{b}\cdot\nabla) \\ &+\frac{1}{2}\left[\tilde{x}^2(\boldsymbol{n}\cdot\nabla)^2+2\tilde{x}\tilde{y}(\boldsymbol{n}\cdot\nabla)(\boldsymbol{b}\cdot\nabla)+\tilde{y}^2(\boldsymbol{b}\cdot\nabla)^2\right]+\kappa\tilde{x}^2(\boldsymbol{n}\cdot\nabla) \\ &+\kappa\tilde{x}\tilde{y}(\boldsymbol{b}\cdot\nabla)+\cdots\big)\Big\}\frac{\boldsymbol{t}(s)}{|\boldsymbol{x}-\boldsymbol{X}(s)|}\,. \end{aligned} \tag{2.11}$$

We shall know from the inner expansion that the axial component ζ of vorticity has the following dependence on the local azimuthal coordinate ϕ:

$$\zeta(\tilde{x},\tilde{y}) = \zeta_0(r) + \zeta_{11}(r,\xi,t)\cos\phi + \zeta_{12}(r,\xi,t)\sin\phi + \zeta_{21}(r,\xi,t)\cos 2\phi + \cdots\,, \qquad (2.12)$$

where

$$\begin{aligned} \zeta_0 &= \zeta^{(0)}(r) + \kappa^2\hat{\zeta}_0^{(2)}(r) + \cdots\,, \quad \zeta_{11} = \kappa\hat{\zeta}_{11}^{(1)}(r) + \cdots\,, & (2.13)\\ \zeta_{12} &= \kappa^3\hat{\zeta}_{12}^{(3)}(r) + \cdots\,, \quad \zeta_{21} = \kappa^2\hat{\zeta}_{21}^{(2)}(r) + \cdots\,. & (2.14) \end{aligned}$$

Substituting (2.12)–(2.14) into (2.11), we get an expression of $\boldsymbol{A}$, valid to first order in κ:

$$\boldsymbol{A}(\boldsymbol{x}) = \boldsymbol{A}_m(\boldsymbol{x}) + \boldsymbol{A}_d(\boldsymbol{x}) + \cdots\,, \qquad (2.15)$$

where

$$\boldsymbol{A}_m(\boldsymbol{x}) = \frac{\Gamma}{4\pi}\int\frac{\boldsymbol{t}(s)}{|\boldsymbol{x}-\boldsymbol{X}(s)|}ds\,; \quad \Gamma = 2\pi\int_0^\infty r\zeta^{(0)}(r)dr\,, \qquad (2.16)$$

and

$$\begin{aligned} \boldsymbol{A}_d(\boldsymbol{x}) &= -\frac{1}{4\pi}[\pi\int_0^\infty r^2\hat{\zeta}_{11}^{(1)}dr]\frac{\kappa_s\boldsymbol{n}+\kappa\tau\boldsymbol{b}}{|\boldsymbol{x}-\boldsymbol{X}(s)|}ds \\ &+\frac{1}{4\pi}\left\{\frac{1}{4}[2\pi\int_0^\infty r^3\zeta^{(0)}dr] - [\pi\int_0^\infty r^2\hat{\zeta}_{11}^{(1)}dr]\right\}\int\frac{\kappa_s\boldsymbol{b}\times(\boldsymbol{x}-\boldsymbol{X}(s))}{|\boldsymbol{x}-\boldsymbol{X}(s)|^3}ds\,. \qquad (2.17) \end{aligned}$$

The first term $\boldsymbol{A}_m$ pertains to a flow field induced by a curved vortex line, and is called the 'monopole field'. The second term $\boldsymbol{A}_d$ corresponds to the flow field induced by a line of dipoles, arranged on the vortex centerline, with their axes oriented in the binormal direction. The origin of this dipole field is the curvature effect; by bending the vortex tube, the vortex lines of the outer side are stretched, while those of the inner side are contracted, producing effectively a vortex pair.

Curl of (2.17) yields the velocity field $\boldsymbol{v}_d$ of the dipoles:

$$\boldsymbol{v}_d(\boldsymbol{x}) = \int\left\{-\frac{\boldsymbol{d}(s)}{|\boldsymbol{x}-\boldsymbol{X}(s)|^3} + \frac{3\boldsymbol{d}(s)\cdot[\boldsymbol{x}-\boldsymbol{X}(s)]}{|\boldsymbol{x}-\boldsymbol{X}(s)|^5}[\boldsymbol{x}-\boldsymbol{X}(s)]\right\}ds\,, \qquad (2.18)$$

where

$$\boldsymbol{d} = D\kappa\boldsymbol{b}\,, \qquad (2.19)$$

and the strength D of the dipole is related with distribution of vorticity through (2.17).

In the spirit of the LIA, (2.18) simplifies to

$$\begin{aligned} \boldsymbol{v}_d &= D\left\{\frac{2\kappa}{r^2}[\sin\phi\boldsymbol{e}_r - \cos\phi\boldsymbol{e}_\theta] - \frac{\kappa^2}{r}\cos 2\phi\boldsymbol{e}_\theta\right. \\ &\left.+\log\left(\frac{2L}{r}\right)\left[(2\kappa_s\tau+\kappa\tau_s)\boldsymbol{n} + (\kappa\tau^2 - \kappa_{ss} - \frac{1}{4}\kappa^3)\boldsymbol{b}\right]\right\} + \cdots\,. \qquad (2.20) \end{aligned}$$

The first two terms imply that the dipoles are distributed along the line of $r = 0$.

Complying with the LIA, we focus on the logarithmic terms. Intriguingly, these terms are almost identical with the third vector fields (1.5) of the LIH. The only difference lies in the coefficient of κ^3. We are reminded of the fact that, for the speed of a vortex ring, the logarithmic terms at $O(\epsilon^3)$ arise also from the inner solution. The same will be true for a curved vortex filament in general. The above remarkable coincidence invites a further investigation of the inner solution to higher orders.

3 Inner solution

The inner solution is addressed by solving the Euler equations in the moving coordinates. We introduce the following dimensionless variables endowed with star:

$$\left.\begin{array}{c} r = \sigma r^*\,, \quad \boldsymbol{X} = R_0 \boldsymbol{X}^*\,, \quad \xi = R_0 \xi^*\,, \quad \kappa = \kappa^*/R_0\,, \quad t = (R_0^2/\Gamma) t^*\,, \\ (u, v, w) = \frac{\Gamma}{\sigma}(u^*, v^*, w^*)\,, \quad \dot{\boldsymbol{X}} = \frac{\Gamma}{R_0}\dot{\boldsymbol{X}}^*\,, \quad \frac{p}{\rho} = \left(\frac{\Gamma}{\sigma}\right)^2 \frac{p^*}{\rho^*}\,, \end{array}\right\} \tag{3.1}$$

where R_0 signifies a measure of the curvature radius, and ρ is tentatively used for density with abuse of notation. In order to eliminate the pressure, it is advantageous to handle the vorticity equation rather than the Euler equations. Dropping the stars, the vorticity equation in the axial direction takes the following form:

$$\begin{aligned} &\epsilon^2 \left[\dot{\zeta} + \omega_r(\dot{\boldsymbol{e}}_r \cdot \boldsymbol{t}) + \omega_\theta(\dot{\boldsymbol{e}}_\theta \cdot \boldsymbol{t})\right] \\ &\quad + \epsilon\left[w - \epsilon^2 r(\dot{\boldsymbol{e}}_r \cdot \boldsymbol{t})\right]\left[\frac{1}{h_3}\frac{\partial \zeta}{\partial \xi} + \frac{\eta\kappa}{h_3}(-\omega_r \cos\phi + \omega_\theta \sin\phi)\right] \\ &\quad - \epsilon^2(\dot{\boldsymbol{e}}_r \cdot \boldsymbol{e}_\theta)\frac{\partial \zeta}{\partial \theta} - \epsilon^3 r(\dot{\boldsymbol{e}}_r \cdot \boldsymbol{t})\frac{1}{h_3}\frac{\partial \zeta}{\partial \xi} + u\frac{\partial \zeta}{\partial r} + \frac{v}{r}\frac{\partial \zeta}{\partial \theta} \\ &= \epsilon^2 \frac{\zeta}{h_3}\frac{\partial \dot{\boldsymbol{X}}}{\partial \xi} \cdot \boldsymbol{t} + \epsilon\eta\kappa(-u\cos\phi + v\sin\phi)\frac{\zeta}{h_3} + \omega_r \frac{\partial w}{\partial r} + \frac{\omega_\theta}{r}\frac{\partial w}{\partial \theta} + \epsilon\frac{\zeta}{h_3}\frac{\partial w}{\partial \xi}\,. \end{aligned} \tag{3.2}$$

The equation of continuity is

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\epsilon}{h_3}\frac{\partial w}{\partial \xi} + \epsilon\frac{\eta\kappa}{h_3}(-u\cos\phi + v\sin\phi) + \frac{\epsilon^2}{h_3}\dot{\boldsymbol{X}}_\xi \cdot \boldsymbol{t} = 0\,. \tag{3.3}$$

Suppose that the leading-order flow consists only of the azimuthal component $v^{(0)}$ possessing both rotational and translational symmetry about the local central axis $\boldsymbol{t}$:

$$v^{(0)} = v^{(0)}(r, t)\,, \tag{3.4}$$

which is compatible with the Euler equations. Going into higher orders, we will be led to the following form of the inner expansions:

$$u \;=\; \epsilon u^{(1)} + \epsilon^2 u^{(2)} + \epsilon^3 u^{(3)} + \cdots\,, \tag{3.5}$$

$$v = v^{(0)}(r,t) + \epsilon v^{(1)} + \epsilon^2 v^{(2)} + \epsilon^3 v^{(3)} + \cdots, \tag{3.6}$$
$$w = \epsilon^2 w^{(2)} + \cdots, \tag{3.7}$$
$$\zeta = \zeta^{(0)}(r,t) + \epsilon\zeta^{(1)} + \epsilon^2\zeta^{(2)} + \epsilon^3\zeta^{(3)} + \cdots. \tag{3.8}$$

Consistently with (3.3), we can conveniently introduce the streamfunction ψ for the local flow (u, v) in the plane transversal to the $\boldsymbol{t}$-direction,

$$\psi = \psi^{(0)}(r,t) + \epsilon\psi^{(1)} + \epsilon^2\psi^{(2)} + \epsilon^3\psi^{(3)} + \cdots. \tag{3.9}$$

As assumed above, the leading-order flow is the local circulatory flow (3.4) of axial symmetry. We note that this statement may have been proved, in the context of elliptic partial differential equations, by Caffarelli & Friedman (1980). The local stretching of vortex lines is restricted in such a way that its effect enters only through the dependence on t.

The solution at $O(\epsilon)$, meeting the condition that the relative velocity u and v are finite at $r = 0$, is written out as follows:

$$\psi^{(1)} = \left[\tilde{\psi}_{11}^{(1)} - \frac{1}{\kappa}(\dot{\boldsymbol{X}}^{(0)} \cdot \boldsymbol{b})\right]\cos\phi; \quad \tilde{\psi}_{11}^{(1)} = \Psi_{11}^{(1)} + c_{11}^{(1)} v^{(0)}, \tag{3.10}$$

where

$$\Psi_{11}^{(1)} = v^{(0)}\left\{\frac{r^2}{2} + \int_0^r \frac{dr'}{r'[v^{(0)}(r')]^2}\int_0^{r'} r''\left[v^{(0)}(r'')\right]^2 dr''\right\}, \tag{3.11}$$

and $c_{11}^{(1)}$ is a disposable parameter bearing with the freedom of choosing the local origin $r = 0$ in the ρ-direction, within an accuracy of $O(\epsilon)$, in the given moving frame (Fukumoto & Moffatt 1999). The matching condition then gives rise to

$$\dot{\boldsymbol{X}}^{(0)} = A\kappa\boldsymbol{b}, \tag{3.12}$$

where

$$A = \frac{1}{4\pi R_0}\left[\log\left(\frac{2L}{\epsilon}\right) - \frac{1}{2} + \lim_{r\to\infty}\left\{4\pi^2\int_0^r r'[v^{(0)}(r')]^2 dr' - \log r\right\}\right], \tag{3.13}$$

(Widnall, Bliss & Zalay 1971). For the present purpose, the discrepancy of (1.2) from (3.13) may be looked upon as inconsequential.

Fortunately $p^{(1)}$ is straightforwardly tractable in the form:

$$p^{(1)} = \kappa\left\{v^{(0)}\frac{\partial\tilde{\psi}_{11}^{(1)}}{\partial r} - \zeta^{(0)}\tilde{\psi}_{11}^{(1)} - r[v^{(0)}(r)]^2\right\}\cos\phi. \tag{3.14}$$

The gradient of $p^{(1)}$, in turn, drives axial flow at $O(\epsilon^2)$. Discarding the irrelevant terms from the Euler equation, we are left with

$$v^{(0)}(\dot{e}_\theta^{(0)} \cdot \boldsymbol{t}) + \frac{v^{(0)}}{r}\frac{\partial w^{(2)}}{\partial\theta} = -\frac{1}{\eta}\frac{\partial p^{(1)}}{\partial\xi}. \tag{3.15}$$

In the LIA, (3.15) admits a compact form of the solution for $w^{(2)}$ as

$$w^{(2)} = \hat{w}(-\kappa\tau\cos\phi + \kappa_s \sin\phi)\,, \tag{3.16}$$

where

$$\hat{w} = Ar - r\frac{\partial \tilde{\psi}_{11}^{(1)}}{\partial r} + \frac{r\zeta^{(0)}}{v^{(0)}}\tilde{\psi}_{11}^{(1)} + r^2 v^{(0)}\,. \tag{3.17}$$

In this way, we have clarified that, for a curved vortex filament, the axial flow shows up at $O(\epsilon^2)$. In view of (3.16), torsion or arcwise variation of curvature is vital for the presence of pressure gradient and thus of axial velocity.

The streamfunction $\psi^{(2)}$ at $O(\epsilon^2)$ for flow in the transversal plane is built in parallel with the case of a circular vortex ring. The detail is relegated to a full paper.

We are now in a position to make headway to deduce the third-order velocity. At $O(\epsilon^3)$, the vorticity equation in the axial direction is reducible to

$$\begin{aligned}
&\dot{\zeta}^{(1)} + A\left(\tau^2 - \frac{\kappa_{ss}}{\kappa}\right)\frac{\partial \zeta^{(1)}}{\partial\theta} + \dot{T}\frac{\partial \zeta^{(1)}}{\partial\theta} + \frac{v^{(0)}}{r}\frac{\partial \zeta^{(3)}}{\partial\theta} + u^{(3)}\frac{\partial \zeta^{(0)}}{\partial r} \\
&+\frac{v^{(1)}}{r}\frac{\partial \zeta^{(2)}}{\partial\theta} + u^{(2)}\frac{\partial \zeta^{(1)}}{\partial r} + \frac{v^{(2)}}{r}\frac{\partial \zeta^{(1)}}{\partial\theta} + u^{(1)}\frac{\partial \zeta^{(2)}}{\partial r} = \kappa v^{(0)}\zeta^{(2)}\sin\phi \\
&-\kappa\zeta^{(1)}(u^{(1)}\cos\phi - v^{(1)}\sin\phi) - \kappa\zeta^{(0)}(u^{(2)}\cos\phi - v^{(2)}\sin\phi) + \frac{\kappa^2}{2} r v^{(0)}\zeta^{(1)}\sin 2\phi \\
&-\frac{\kappa^2}{2} r\zeta^{(0)}\left[u^{(1)}(1+\cos 2\phi) - v^{(1)}\sin 2\phi\right] + \frac{\kappa^3}{4} r^2 v^{(0)}\zeta^{(0)}(\sin\phi + \sin 3\phi) + \frac{\zeta^{(0)}}{\eta}\frac{\partial w^{(2)}}{\partial\xi}\,,
\end{aligned} \tag{3.18}$$

where

$$T(\xi, t) = \int_0^{s(\xi,t)} \tau(s', t)ds'\,. \tag{3.19}$$

Relevant to the traveling speed is the terms proportional to $\cos\phi$ and $\sin\phi$. Equation (3.18) has much in common with that for a circular vortex ring. The effect of τ and κ_s, κ_{ss}, $\cdots$, which is missing in the latter case, makes its appearance only in the first few terms $\dot{\zeta}^{(1)}$, $A(-\tau^2 + \kappa_{ss}/\kappa)\partial\zeta^{(1)}/\partial\theta$, $\dot{T}\partial\zeta^{(1)}/\partial\theta$, and in the last term $(\zeta^{(0)}/\eta)\partial w^{(2)}/\partial\xi$. The first term $\dot{\zeta}^{(1)}$ is

$$\dot{\zeta}^{(1)} = -\left(a\tilde{\psi}_{11}^{(1)} + r\zeta^{(0)}\right)\left(\dot{\kappa}\cos\phi + \kappa\dot{T}\sin\phi\right)\,; \quad a = \frac{1}{v^{(0)}}\frac{\partial \zeta^{(0)}}{\partial r}\,. \tag{3.20}$$

This is further simplified, under the LIA, by invoking the Betchov-Da Rios equation:

$$\dot{\kappa} = -A\left(2\kappa_s\tau + \kappa\tau_s\right)\,, \tag{3.21}$$

$$\dot{\tau} = A\frac{\partial}{\partial s}\left(\frac{\kappa_{ss}}{\kappa} - \tau^2 + \frac{\kappa^2}{2}\right)\,, \tag{3.22}$$

(Da Rios 1906). The matching condition is, when only the terms tied with torsion and non-constancy of curvature in the $\cos\phi$ and $\sin\phi$ components are retained, written as

$$\begin{aligned}
\kappa\psi^{(3)} \sim &\left(\frac{3}{32\pi}r^3 + \frac{D}{\Gamma R_0^2}r\right)\log\left(\frac{2L}{\epsilon r}\right)\left[(\kappa_{ss} - \kappa\tau^2)\cos\phi + (2\kappa_s\tau + \kappa\tau_s)\sin\phi\right] \\
&+\cdots \quad \text{as } r \to \infty\,.
\end{aligned} \tag{3.23}$$

We limit ourselves to a specific vorticity distribution at $O(\epsilon^0)$ of constant vorticity in the circular domain $r \leq 1$ of unit radius surrounded by an irrotational flow, which is known as the Rankine vortex. The velocity at $O(\epsilon^0)$ takes the form:

$$v^{(0)} = \begin{cases} \frac{r}{2\pi}, & (r \leq 1) \\ \frac{1}{2\pi r}. & (r > 1) \end{cases} \tag{3.24}$$

The strength D of dipole, defined by (2.19), is evaluated as $D/(\Gamma R_0^2) = 3/(32\pi)$. Imposition of the matching condition (3.23) on (3.18) gives rise to, after some manipulation, the third-order correction $\dot{\boldsymbol{X}}^{(2)}$ to the traveling speed. Combining with (1.2), we eventually arrive at the evolution equation of a vortex filament in the LIA, which is expressed, in terms of dimensional variables, as

$$\frac{\partial \boldsymbol{X}}{\partial t}(s,t) = \frac{\Gamma}{4\pi}\log\left(\frac{2L}{\sigma}\right)\kappa\boldsymbol{b} + \frac{\Gamma\sigma^2}{16\pi}\log\left(\frac{2L}{\sigma}\right)\left\{(2\kappa_s\tau + \kappa\tau_s)\boldsymbol{n} + \left(\kappa\tau^2 - \kappa_{ss} - \frac{3}{2}\kappa^3\right)\boldsymbol{b} + \kappa^2\tau\boldsymbol{t}\right\}. \tag{3.25}$$

It deserves emphasis that the LIA equation as extended to $O(\epsilon^3)$ is nearly equal to a summation of the first (1.3) and the third (1.5) belonging to the vector fields of the LIH. The only exception is the coefficient of the term $-(3/2)\kappa^3\boldsymbol{b}$ in (3.25).

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