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HIGH ENERGY RESOLVENT ESTIMATES FOR ACOUSTIC PROPAGATORS IN A STRATIFIED MEDIA

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§1 Introduction.

Let $n \geqq 2$ and $x=(y,z)\in \mathbf{R}^{n-1}\times \mathbf{R}$. In this report we study the following operator :

$$L_0 = -a_0(z)^2\triangle, \tag{1.1}$$

where

$$a_0(z)=\begin{cases} c_+ & (z\geqq h)\\ c_h & (0<z<h)\\ c_- & (z\leqq 0),\end{cases}$$

and $c_{\pm}, c_h$ and h are positive constants and

$$\triangle=\sum_{j=1}^{n-1}\frac{\partial^2}{\partial y_j^2}+\frac{\partial^2}{\partial z^2}.$$

We consider only the case $c_h < \min(c_+,c_-)$ because we can find the guided waves (cf. Wilcox [9] or Weder [6]). It seems that there are no works dealing with high energy resolvent estimates for acoustic propagators in stratified media. Here we shall prove high energy resolvent estimates for the case $c_h < c_+ = c_-$.

Kikuchi-Tamura [3] and Kadowaki [2] have proved low energy resolvent estimates for the case $c_h<\min(c_+,c_-)$ and $c_+\neq c_-$ and the case $c_h<c_+=c_-$ respectively. Both works were used Mourre's commutator method (cf. Mourre[4]). But the conjugate operator in Kadowaki [2] is different from Kikuchi-Tamura [3]. Kikuchi-Tamura [3] took the generator of dailation in $\mathbf{R}^3$ as the conjugate operator. They dealt with only media of $\mathbf{R}^3$ but their result can be extended for media of $\mathbf{R}^n (n\geqq 3)$ (cf. Kadowaki [2]). Kadowaki [2] has constructed the conjugate operator by using the generator of dailation in $\mathbf{R}^n$ and $\mathbf{R}^{n-1} (n\geqq 3)$ togerther with the generalized Fourier transform of a related operator (cf. Weder [6]). The generator of dailation in $\mathbf{R}^{n-1}$ has been used to estimate the guided wave (see §2). In this report, we also use Mourre's method. Our conjugate operator is similar to Kadowaki [2](see §2).

Let $\mathcal{H}_0 = L^2(\mathbf{R}^n; a_0^{-2}(z)dx)$ be Hilbert space with inner products

$$< u,v >_0=\int_{\mathbf{R}^n} u(x)\overline{v(x)}a_0^{-2}(z)dx.$$

In particular $L^2(\mathbf{R}_x^n)$ is the usual L^2 space defined on $\mathbf{R}_x^n$ with inner products

$$< u, v >_{L^2(\mathbf{R}_x^n)} = \int_{\mathbf{R}_x^n} u(x)\overline{v(x)}dx$$

and the corresponding norms $|\cdot|_{L^2(\mathbf{R}_x^n)}$.

L_0 is admits a unique self-adjoint realizations in $\mathcal{H}_0$. Then L_0 is a non-negative operator (zero is not an eigenvalue) and the $D(L_0)$ is given by $H^2(\mathbf{R}_x^n)$, $H^s(\mathbf{R}_x^n)$ being Sobolev space of order s over $\mathbf{R}_x^n$. We also denoted by $R(z; L_0)$ the resolvent $(L_0 - z)^{-1}$ of L for $Imz \neq 0$.

A is considered as an operator from $L^2(\mathbf{R}_x^n)$ into itself, then its norm is denoted by the notation $\|A\|$.

We remark that Weder [6] has showed the absence of eigenvalues and the limiting absorption principle for L_0. Our result is :

Theorem1.1. *Let $\alpha > 1/2$. Assume that $c_h < c_+ = c_-$. Then, we have*

$$\|X_\alpha R(\lambda \pm i\kappa; L_0)X_\alpha\| = O(\lambda^{-\frac{1}{2}}) \quad (\lambda \to \infty),$$

uniformly in $\kappa > 0$, where $X_\alpha = (1 + |x|^2)^{-\alpha/2}$.

We define the self-adjoint operator $L_0(\lambda)$ on $L^2(\mathbf{R}_x^n)$,

$$\begin{cases} L_0(\lambda) = -\Delta - \lambda(a_0^{-2}(z) - c_+^{-2}) \\ D(L_0(\lambda)) = H^2(\mathbf{R}_x^n). \end{cases}$$

This operator has been introduced by Weder [7]. Theorem 1.1 is obtained as an immediate consequence of the following proposition

Proposition 1.2. *Assume that $c_h < c_+ = c_-$. Then we have*

$$\|X_\alpha G_\kappa(0; \lambda)X_\alpha\| = O(\lambda^{-\frac{1}{2}}) \quad (\lambda \to \infty),$$

uniformly in $\kappa > 0$, where

$$G_\kappa(0; \lambda) = (L_0(\lambda) - \lambda c_+^{-2} - i\kappa a_0^{-2}(z))^{-1}$$

for $\kappa > 0$

In §§2,3 we shall give the proof of above proposition.

We give a comment for the assumption of Theorem 1.1. This follows from our method. Applying Mourre's method to the original operator, L_0, we do not get the Mourre's estimates on the neighborhood of threshods of L_0 (cf. Wilcox [9] or Weder [6]). The conjugate operator for L_0 is contructed by using generator of dailation in $\mathbf{R}^n$ and exterior domains of ball in $\mathbf{R}^{n-1}$ together with the generalized Fourier transform for L_0 (cf. Kadowaki [1]). While, since $L_0(\lambda)$ dose not have threshods on $[0, \infty)$ (see Weder [7]) , we can obtain Mourre's estimates. But, to prove only Lemma 3.6 in §3, we need the assumption $c_h < c_+ = c_-$. In brief we deal with only $c_h < c_+ \leqq c_-$.

As an application of our theorem, we can consider scattering problem for wave equations with disspative terms in stratified media. This is due to Mochizuki [4]. He has proved existence of scattering states for wave equations with disspative terms in the case $c_h = c_+ = c_- = 1$. His idea is due to Kato's smooth pertabation theory together with low and high energy resolvent estimates for Laplacian in $\mathbf{R}^n (n \neq 2)$. To consider scattering problem for stratified media, we need low energy estimates which is requred in Mochizuki [4]. Kikuchi-Tamura [3] and Kadowaki [2] have proved low energy etimates in perturbed stratified media. But the 3-demensional case in Kadowaki [3] and Kikuchi-Tamura [2] do not satisfy Mochizuki's condition (for detail see Mochizuki [4]). For Kikuchi-Tamura's result, we can remake it to satisfy Mochizuki's condition (see Kadowaki [3]). We will give low energy etimates for stratified media of $\mathbf{R}^n (n \geqq 2)$ elsewhere and consider scattering problem.

§2 Conjugate operator and Mourre's estimates.

In this section we conctruct the conjugate operators and show Mourre's estimates (2.1). First we define conjugate operator, $D(\lambda)$, as follows :

$$\begin{aligned} D(\lambda) =& F_0(\lambda)^*(-D_n)F_0(\lambda) + F_1(\lambda)^*(-D_{n-1})F_1(\lambda) \\ &+ \sum_{j=1}^{Q(\lambda)} G_j(\lambda)^*(-D_{n-1})G_j(\lambda), \end{aligned}$$

where $k = (\overline{k}, k_0) \in \mathbf{R}^{n-1} \times \mathbf{R}$, $F_0(\lambda), F_1(\lambda)$ and $G_j(\lambda)$ are partially isometric operators for $L_0(\lambda)$ (see Appendix) and

$$D_n = \frac{1}{2i}(k \cdot \nabla_k + \nabla_k \cdot k), \quad D_{n-1} = \frac{1}{2i}(\overline{k} \cdot \nabla_{\overline{k}} + \nabla_{\overline{k}} \cdot \overline{k}).$$

We consider the commutator $i[L_0(\lambda), D(\lambda)]$ as a form on $H^2(\mathbf{R}^n_x) \cap D(D(\lambda))$ as follows :

$$\begin{aligned} &< i[L_0(\lambda), D(\lambda)]u, u >_{L^2(\mathbf{R}^n_x)} \\ &\quad = i(< D(\lambda)u, L_0(\lambda)u >_{L^2(\mathbf{R}^n_x)} - < L_0(\lambda)u, D(\lambda)u >_{L^2(\mathbf{R}^n_x)}) \end{aligned}$$

for $u \in H^2(\mathbf{R}^n) \cap D(D(\lambda))$. Then Lemma A of the Appendix implies that

$$\begin{aligned} &< i[L_0(\lambda), D(\lambda)]u, u >_{L^2(\mathbf{R}^n_x)} \\ &= i\{< |k|^2 F_0(\lambda)u, D_n F_0(\lambda)u >_{L^2(\mathbf{R}^n_k)} - < D_n F_0(\lambda)u, |k|^2 F_0(\lambda)u >_{L^2(\mathbf{R}^n_k)} \\ &+ < |\overline{k}|^2 F_1(\lambda)u, D_{n-1} F_1(\lambda)u >_{L^2(\Omega_0)} - < D_{n-1} F_1(\lambda)u, |\overline{k}|^2 F_1(\lambda)u >_{L^2(\Omega_0)} \\ &+ \sum_{j=1}^{Q(\lambda)} (< |\overline{k}|^2 G_j(\lambda)u, D_{n-1} G_j(\lambda)u >_{L^2(\mathbf{R}^{n-1}_{\overline{k}})} \\ &- < D_{n-1} G_j(\lambda)u, |\overline{k}|^2 G_j(\lambda)u >_{L^2(\mathbf{R}^{n-1}_{\overline{k}})})\}. \end{aligned}$$

Thus we have by integral by parts

$$\begin{aligned} &< i[L_0(\lambda), D(\lambda)]u, u >_{L^2(\mathbf{R}^n_x)} \\ &=< 2(F_0(\lambda)^*|k|^2 F_0(\lambda) + F_1(\lambda)^*|\overline{k}|^2 F_1(\lambda) + \sum_{j=1}^{Q(\lambda)} G_j(\lambda)^*|\overline{k}|^2 G_j(\lambda))u, u >_{L^2(\mathbf{R}^n_x)} \end{aligned}$$

for $u \in H^2(\mathbf{R}^n_x) \cap D(D(\lambda))$. Thus the form $i[L_0(\lambda), D(\lambda)]$ can be extended to a bounded operator from $H^1(\mathbf{R}^n_x)$ to $H^{-1}(\mathbf{R}^n_x)$ which is denoted by $i[L_0(\lambda), D(\lambda)]^0$. Let $\lambda > 1$, take $f_\lambda(r) \in C_0^\infty(\mathbf{R}), 0 \leqq f_\lambda \leqq 1$ such that f_λ has support in $((c_+^{-2} - c_-^{-2}/2)\lambda, 2c_+^{-2}\lambda)$ and $f_\lambda = 1$ on $[(c_+^{-2} - c_-^{-2}/4)\lambda, 3c_+^{-2}\lambda/2]$. Noting that

$$\begin{aligned} &f_\lambda(L_0(\lambda))i[L_0(\lambda), D(\lambda)]^0 f_\lambda(L_0(\lambda)) \\ &= 2(F_0(\lambda)^*|k|^2 f_\lambda(|k|^2 + q_-(\lambda))^2 F_0(\lambda) + F_1(\lambda)^*|\overline{k}|^2 f_\lambda(|\overline{k}|^2 - k_0^2 + q_-(\lambda))^2 F_1(\lambda) \\ &+ \sum_{j=1}^{Q(\lambda)} G_j(\lambda)^*|\overline{k}|^2 f_\lambda(|\overline{k}|^2 - \omega_j^2(\lambda))^2 G_j(\lambda). \end{aligned}$$

Then there exists a positive constant C which is independent of λ such that

$$f_\lambda(L_0(\lambda))i[L_0(\lambda), D(\lambda)]^0 f_\lambda(L_0(\lambda)) \geqq C\lambda f_\lambda(L_0(\lambda))^2 \tag{2.1}$$

in the form sense.

§3 Proof of Proposition 1.2.

Proposition 1.2 follows from lemmas in this section. But we omit the proof of lemmas and give only a comment of the proof.

We can prove the following lemmas in the same way as in the proof of Lemma 2.5 of Weder [7].

Lemma 3.1. *Let $f \in C_0^\infty(\mathbf{R})$. Then*
(i) $f(L_0(\lambda))$ sends $D(D(\lambda))$ into $D(D(\lambda))$.
(ii)$[f(L_0(\lambda)), D(\lambda)]$ defined as operator on $D(D(\lambda))$ is extended to a bounded operator on $L^2(\mathbf{R}^n_x)$ which is denoted by $[f(L_0(\lambda)), D(\lambda)]^0$.

It follows from (2.1) that $M_0(\lambda)$ is non-negative and hence we define an operator, $G_\kappa(\epsilon;\lambda)$, on $L^2(\mathbf{R}^n_x)$ by

$$G_\kappa(\epsilon;\lambda) = (L_0(\lambda) - \lambda c_+^{-2} - i\kappa a_0^{-2}(z) - i\epsilon M_0(\lambda))^{-1} \tag{3.1}$$

for $\kappa > 0$ and $\epsilon > 0$. Using (2.1), we can prove the following lemma (for detail, see that of Lemma 5.3 of Kikuchi-Tamura [3]).

Lemma 3.2. *For $\epsilon > 0$, as $\lambda \to \infty$, one has*

$$\|G_\kappa(\epsilon;\lambda)\| = \epsilon^{-1}O(\lambda^{-1}), \quad (\lambda \to \infty)$$

uniformly in $\kappa > 0$.

We write

$$F_\kappa(\epsilon;\lambda) = \lambda^{\frac{1}{2}} Z_\alpha(\epsilon,\lambda) G_\kappa(\lambda^{-\frac{1}{2}}\epsilon;\lambda) Z_\alpha(\epsilon,\lambda),$$

where $Z_\alpha(\epsilon,\lambda) = (\lambda^{\frac{1}{2}} + |D(\lambda)|)^{-\alpha}(\lambda^{\frac{1}{2}} + \epsilon|D(\lambda)|)^{\alpha-1}$.

This is due to Yafaev [8]. But we do not use the scaling argument for λ (cf. (3.1)).

Let $g_\lambda(p) = 1 - f_\lambda(p)$. We write in brief f_λ and g_λ for $f_\lambda(L_0(\lambda))$ and $g_\lambda(L_0(\lambda))$ respectively.

Noting that

$$G_\kappa(\epsilon;\lambda)D(D(\lambda)) \subset D(D(\lambda)) \cap H^2(\mathbf{R}^n)$$

(cf. Kadowaki [2]), we decompose $(d/d\epsilon)F_\kappa(\epsilon;\lambda)$ as a form on $L^2(\mathbf{R}^n_x)$

$$(d/d\epsilon)F_\kappa(\epsilon;\lambda) = \sum_{j=1}^{8} Y^j_\kappa(\epsilon;\lambda), \tag{3.2}$$

where

$$\begin{aligned}
Y^1_\kappa &= iZ_\alpha(\epsilon,\lambda)G_\kappa(\lambda^{-\frac{1}{2}}\epsilon;\lambda)g_\lambda[L_0(\lambda),D(\lambda)]^0 f_\lambda G_\kappa(\lambda^{-\frac{1}{2}}\epsilon;\lambda)Z_\alpha(\epsilon,\lambda),\\
Y^2_\kappa &= iZ_\alpha(\epsilon,\lambda)G_\kappa(\lambda^{-\frac{1}{2}}\epsilon;\lambda)g_\lambda[L_0(\lambda),D(\lambda)]^0 g_\lambda G_\kappa(\lambda^{-\frac{1}{2}}\epsilon;\lambda)Z_\alpha(\epsilon,\lambda),\\
Y^3_\kappa &= iZ_\alpha(\epsilon,\lambda)G_\kappa(\lambda^{-\frac{1}{2}}\epsilon;\lambda)f_\lambda[L_0(\lambda),D(\lambda)]^0 g_\lambda G_\kappa(\lambda^{-\frac{1}{2}}\epsilon;\lambda)Z_\alpha(\epsilon,\lambda),\\
Y^4_\kappa &= -iZ_\alpha(\epsilon,\lambda)\{D(\lambda)G_\kappa(\lambda^{-\frac{1}{2}}\epsilon;\lambda)+G_\kappa(\lambda^{-\frac{1}{2}}\epsilon;\lambda)D(\lambda)\}Z_\alpha(\epsilon,\lambda)\\
Y^5_\kappa &= \kappa Z_\alpha(\epsilon,\lambda)G_\kappa(\lambda^{-\frac{1}{2}}\epsilon;\lambda)[a_0(z)^{-2},D(\lambda)]G_\kappa(\lambda^{-\frac{1}{2}}\epsilon;\lambda)Z_\alpha(\epsilon,\lambda),\\
Y^6_\kappa &= \epsilon Z_\alpha(\epsilon,\lambda)G_\kappa(\lambda^{-\frac{1}{2}}\epsilon;\lambda)[M_0(\lambda),D(\lambda)]G_\kappa(\lambda^{-\frac{1}{2}}\epsilon;\lambda)Z_\alpha(\epsilon,\lambda),\\
Y^7_\kappa &= \lambda^{-\frac{1}{2}}\{\frac{d}{d\epsilon}Z_\alpha(\epsilon,\lambda)\}G_\kappa(\lambda^{-\frac{1}{2}}\epsilon;\lambda)Z_\alpha(\epsilon,\lambda),\\
Y^8_\kappa &= \lambda^{-\frac{1}{2}}Z_\alpha(\epsilon,\lambda)G_\kappa(\lambda^{-\frac{1}{2}}\epsilon;\lambda)\frac{d}{d\epsilon}Z_\alpha(\epsilon,\lambda).
\end{aligned}$$

We need the following lemmas (Lemma 3.3~Lemma 3.5) to estimate each term of the right side of (3.2).

Note that there is $c_0, c_0 > 0$ suth that $(L_0(\lambda)+c_0\lambda)^{-1}$ exists.

Lemma 3.3. *As* $\lambda \to \infty$, *one has :*
(i) $\|g_\lambda G_\kappa(\lambda^{-\frac{1}{2}}\epsilon;\lambda)\| = O(\lambda^{-1})$,
(ii) $\|(L_0(\lambda)+c_0\lambda)^{1/2} f_\lambda G_\kappa(\lambda^{-\frac{1}{2}}\epsilon;\lambda)Z_\alpha(\epsilon,\lambda)\| = \epsilon^{-1/2}\|F_\kappa\|^{1/2}O(1)$,
(iii) $\|(L_0(\lambda)+c_0\lambda)^{1/2} g_\lambda G_\kappa(\lambda^{-\frac{1}{2}}\epsilon;\lambda)Z_\alpha(\epsilon,\lambda)\| = O(\lambda^{-1})$,
(iv) $\|F_\kappa(\epsilon;\lambda)\| = \epsilon^{-1}O(\lambda^{-1})$,
uniformly in $\kappa > 0$.

For a proof of Lemma 3.5 (i), see that of Lemma 5.4 of Kikuchi- Tamura [3]. Also, for a proof of (ii) and (iii), see that of Lemma 5.5 of Kikuchi-Tamura [3]. (ii) and (iii) imply (iv).

Lemma 3.4. *Assume that* $c_h < c_+ = c_-$. *Then* $[a_0^{-2}(z), D(\lambda)]$ *defined as a form on* $D(D(\lambda))$ *is extended to a bounded operator from* $H^1(\mathbf{R}^n_x)$ *to* $H^{-1}(\mathbf{R}^n_x)$ *which is denoted by* $[a_0^{-2}(z), D(\lambda)]^0$. *Moreover we have*

$$\begin{aligned}
&i[a_0^{-2}(z), D(\lambda)]^0\\
&= (c_h^{-2}-1)((n-1)\chi_{0<z<h}(z) - (\partial_{k_0}F_0(\lambda)\chi_{0<z<h}(z))^* k_0 F_0(\lambda)\\
&\quad - (k_0F_0(\lambda))^*\partial_{k_0}F_0(\lambda)\chi_{0<z<h}(z) + F_0(\lambda)^*F_0(\lambda))
\end{aligned}$$

and

$$\|(L_0(\lambda)+c_0\lambda)^{-1/2} i[a_0^{-2}(z), D(\lambda)]^0 (L_0(\lambda)+c_0\lambda)^{-1/2}\| = O(\lambda^{-\frac{1}{2}}) \quad (\lambda \to \infty).$$

proof. Noting that the reprensention of $F_0(\lambda)$, we show this lemma by straighforward calculation (cf. Kadowaki [2]).

Using Lemma 3.1 and the representation of $i[L_0(\lambda), D(\lambda)]^0$ we show the follwing lemma.

Lemma 3.5. *As $\lambda \to \infty$, one has :*

$$\|[M_0(\lambda), D(\lambda)]^0\| = O(\lambda).$$

Using Lemma 3.2 ~ 3.5, we can evaluate the norm of $Y_\kappa^j, 1 \le j \le 8$ (see Kikuchi-Tamura [3]). Thus we obtain the following differential inequality :

$$\|(d/d\epsilon)F_\kappa(\epsilon;\lambda)\| \leqq C(\lambda^{-1}\epsilon^{\alpha-1} + \lambda^{-\frac{1}{2}}\epsilon^{\alpha-\frac{3}{2}}\|F_\kappa\|^{1/2} + \|F_\kappa\|) \tag{3.3}$$

It follows from Lemma 3.3(iv) and (3.3) that

$$\|(\lambda^{\frac{1}{2}} + |D(\lambda)|)^{-\alpha}G_\kappa(0;\lambda)(\lambda^{\frac{1}{2}} + |D(\lambda)|)^{-\alpha}\| = O(\lambda^{-\frac{1}{2}-\alpha}), \quad (\lambda \to \infty), \tag{3.4}$$

uniformly in $\kappa > 0$.

Noting Lemma 3.1 we rewrite $D(\lambda)f_\lambda X_1$ as

(3.5)
$$\begin{aligned}&\frac{1}{i}(f_\lambda \nabla_y \cdot y X_1 + \frac{n-1}{2} f_\lambda X_1)\\ &\qquad - \frac{1}{i}(f_\lambda F_0(\lambda)^* k_0 \partial_{k_0} F_0(\lambda) X_1 + \frac{1}{2} f_\lambda F_0(\lambda)^* F_0(\lambda) X_1)\\ &\qquad\qquad + [D(\lambda), f_\lambda]^0 X_1.\end{aligned}$$

We can show that

$$\|[D(\lambda), f_\lambda]^0\| = O(1), \quad (\lambda \to \infty), \tag{3.6}$$

(for proof, see that of Lemma 5.6 of Kikuchi-Tamura [3]).

By straighforward calculation we can show next lemma.

Lemma 3.6. *As $\lambda \to \infty$, one has :*

$$\|f_\lambda F_0(\lambda)^* k_0 \partial_{k_0} F_0(\lambda) X_1\| = O(\lambda^{\frac{1}{2}})$$

It follows from (3.5), (3.6) and Lemma 3.8 that

$$\|D(\lambda) f_\lambda X_1\| = O(\lambda^{\frac{1}{2}}) \quad (\lambda \to \infty).$$

Thus we obtain by interpolation

(3.) $$\|(\lambda^{\frac{1}{2}} + |D(\lambda)|)^\alpha f_\lambda X_\alpha\| = O(\lambda^{\frac{\alpha}{2}}) \quad (\lambda \to \infty).$$

Note that

$$\|g_\lambda G_\kappa(0;\lambda)\| = O(\lambda^{-1}) \quad (\lambda \to \infty).$$

(3.4) and (3.7) imply that

$$\|X_\alpha G_\kappa(0;\lambda) X_\alpha\| = O(\lambda^{-\frac{1}{2}}) \quad (\lambda \to \infty).$$

Now the proof of Proposition 1.2 is complete.

Appendix.

In this Appendix we state the generalizied Fourier transform of $L_0(\lambda)$ established by Weder (cf. Weder [6]).

For $\lambda >> 1$ large enough, we consider the following operator :

$$\begin{cases} h(\lambda) = -\dfrac{d^2}{dz^2} - \lambda(a_0^{-2}(z) - c_+^{-2}), \\ D(h(\lambda)) = H^2(\mathbf{R}_z). \end{cases}$$

This is the self-adjoint operator in $L^2(\mathbf{R}_z)$.

$h(\lambda)$ has finite number $Q(\lambda) \in \mathbf{N}$, of eigenvalues, $-\omega_j^2(\lambda), 0 < \omega_j^2(\lambda) < q_h(\lambda) = \lambda(c_h^{-2} - c_+^{-2}), 1 \leqq j \leqq Q(\lambda)$, of multiplicity one. There exist $F_0(\lambda), F_1(\lambda)$ and $G_j(\lambda)(j = 1, 2, 3 \cdots Q(\lambda))$ which are partially isometric operators from $L^2(\mathbf{R}_x^n)$ onto $L^2(\mathbf{R}_k^n), L^2(\Omega_0)$ and $L^2(\mathbf{R}_{\overline{k}}^{n-1})$ respectively, where $\Omega_0 = \{k \in \mathbf{R}^n; 0 < k_0 < \sqrt{q_-(\lambda)} = \sqrt{\lambda(c_+^{-2} - c_-^{-2})}\}$. Defining the operator $F(\lambda)$ as

$$F(\lambda)u = (F_0(\lambda)u, F_1(\lambda)u, G_1(\lambda)u, G_2(\lambda)u, G_3(\lambda)u \cdots G_{Q(\lambda)u}(\lambda))$$

for $u \in L^2(\mathbf{R}_x^n)$, we have

Lemma A. *$F(\lambda)$ is unitary operator from $L^2(\mathbf{R}_x^n)$ onto*

$$\hat{\mathcal{H}} = L^2(\mathbf{R}_k^n) \bigoplus L^2(\Omega_0) \bigoplus_{j=1}^{Q(\lambda)} L^2(\mathbf{R}_{\overline{k}}^{n-1})$$

and for every $u \in D(L_0(\lambda)) = H^2(\mathbf{R}_x^n)$

$$\begin{aligned} F(\lambda)L_0(\lambda)u = (&(|k|^2 + q_-(\lambda))F_0(\lambda)u, (|\overline{k}|^2 - k_0^2 + q_-(\lambda))F_1(\lambda)u, \\ &(|\overline{k}|^2 - \omega_1^2(\lambda))G_1(\lambda)u, (|\overline{k}|^2 - \omega_2^2(\lambda))G_2(\lambda)u, \cdots, \\ &(|\overline{k}|^2 - \omega_{Q(\lambda)}^2(\lambda))G_{Q(\lambda)}(\lambda)u). \end{aligned}$$

For the proof, see Weder [6].

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