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On the characters of Wenzl's $(3,l)$-diagram representations of the Iwahori-Hecke algebras at $\sqrt[l]{1}$

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§ 1. Wenzl's (k,l)-diagram representations

(1.1) *Throughout this note, we assume q to be a non-zero complex number.* Let $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$, $\mathbf{C}$ be the set of natural numbers, the additive group of integers, the field of real numbers, the field of complex numbers, respectively, For $n\in\mathbf{N}$, define $\mathbf{H}_n(q)$ to be the $\mathbf{C}$-algebra (with 1) by the generators T_i, $1\leq i\leq n-1$, and the relations:

$$(T_i-q)(T_i+1)=0,\ T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1},\ T_iT_j=T_jT_i\ (|i-j|\geq 2).$$

The algebra $\mathbf{H}_n(q)$ is called the *Iwahori-Hecke algebra* af type A_n. Let $\mathbf{C}^\times=\mathbf{C}\setminus\{0\}$ and $\mathbf{N}'=\mathbf{N}\setminus\{1\}$. Let $\mathbf{l}:\mathbf{C}^\times\to\mathbf{N}'\cup\{\infty\}$ be the map such that

$$\mathbf{l}(q)=\begin{cases}\min\{a\in\mathbf{N}'\,|\,q^a=1\} & \text{if } \prod_{r=2}^{s}(\sum_{t=0}^{r-1}q^t)=0 \text{ for some } s\in\mathbf{N}',\\ +\infty & \text{otherwise.}\end{cases}$$

Let $\mathbf{Z}_+=\{0\}\cup\mathbf{N}$. Put $\mathbf{Z}_+^\infty=\{(x_1,x_2,\ldots)\in\mathbf{Z}^\infty\,|\,x_i\in\mathbf{Z}_+\,(i\in\mathbf{N})\}$. For $i\in\mathbf{N}$, let $p_i:\mathbf{Z}_+^\infty\to\mathbf{Z}_+$ be the map such that $p_i(x_1x_2,\ldots)=x_i$. An element λ of $\mathbf{Z}_+^\infty$ is called a *partition* if $p_i(\lambda)\geq p_{i+1}(\lambda)$ for any $i\in\mathbf{N}$ and there exists $j\in\mathbf{N}$ such that $p_j(\lambda)=0$. Let Λ be the set of partitions. Let $\mathbf{k}:\Lambda\to\mathbf{Z}_+$ be the map such that $\mathbf{k}(\lambda)=\min\{j\in\mathbf{N}\,|\,p_j(\lambda)=0\}-1$. Let $\mathbf{n}:\Lambda\to\mathbf{Z}_+$ be the map such that $\mathbf{n}(\lambda)=\sum_{i=1}^{+\infty}p_i(\lambda)$. For $n\in\mathbf{Z}_+$, let $\Lambda_n=\{\lambda\in\Lambda|\mathbf{n}(\lambda)=n\}$. For $k\in\mathbf{Z}_+$, let $\Lambda^k=\{\lambda\in\Lambda|\mathbf{k}(\lambda)=k\}$.

(1.2) Let $\phi=(0,0,\ldots)\in\Lambda$, and let $\Lambda'=\Lambda\setminus\{\phi\}$. For $l\in\mathbf{N}'\cup\{+\infty\}$, let

$$\Lambda^{(l)}=\{\phi\}\cup\{\lambda\in\Lambda'\,|\,\mathbf{k}(\lambda)\leq l-1,\ p_1(\lambda)-p_{\mathbf{k}(\lambda)}(\lambda)\leq l-\mathbf{k}(\lambda)\}.$$

For k, $1\leq k\leq l-1$, let $\Lambda^{(k,l)}=\cup_{a=1}^k(\Lambda^a\cap\Lambda^{(l+a-k)})$. Note that $\Lambda^{(l)}=\cup_{1\leq k\leq l-1}\Lambda^{(k,l)}$ and $\Lambda^{(+\infty)}=\Lambda$. Let $\Lambda_n^{(l)}=\Lambda^{(l)}\cap\Lambda_n$ and $\Lambda_n^{(k,l)}=\Lambda^{(k,l)}\cap\Lambda_n$. The element $\lambda\in\Lambda$ will also be denoted by

$$[1^{|\{i|p_1(\lambda)=1\}|}2^{|\{i|p_1(\lambda)=2\}|}\ldots]$$

if $\lambda \in \Lambda'$, and [0] if $\lambda = \phi$. For example, we have $\Lambda_0^{(l)} = \{[0]\}$, $\Lambda_1^{(l)} = \{[1]\}$ and $\Lambda_2^{(l)} = \{[2], [1^2]\}$.

(1.3) For $\mu, \lambda \in \Lambda$, we write $\mu \subset^{(k,l)} \lambda$ if $\mu, \lambda \in \Lambda^{(k,l)}$ and $p_i(\mu) \leq p_i(\lambda)$ for any i, $1 \leq i \leq k$. We write $\mu \subset_{+1}^{(k,l)} \lambda$ if $\mu \subset^{(k,l)} \lambda$ and $\mathbf{n}(\mu) = \mathbf{n}(\lambda) - 1$. For $\mu \subset_{+1}^{(k,l)} \lambda$, put

$$\mathbf{r}(\mu, \lambda) = \sum_{i=0}^{+\infty} i \cdot (p_i(\lambda) - p_i(\mu)), \quad \text{and} \quad \mathbf{c}(\mu, \lambda) = p_{\mathbf{r}(\mu \subset_{+1}^{(k,l)} \lambda)}(\lambda).$$

(1.4) For $\mu \subset^{(k,l)} \lambda$, we set $\mathbf{n}(\lambda/\mu) = \mathbf{n}(\lambda) - \mathbf{n}(\mu)$, and call an element $(\lambda_0, \lambda_1, \ldots, \lambda_{\mathbf{n}(\lambda/\mu)})$ of $\Lambda_{\mathbf{m}(\lambda)} \times \Lambda_{\mathbf{m}(\lambda)+1} \times \cdots \times \Lambda_{\mathbf{n}(\lambda)}$ a *(k, l)-standard tabuleau* of λ/μ if $\lambda_0 = \mu$, $\lambda_{\mathbf{n}(\lambda/\mu)} = \lambda$ and $\lambda_i \subset_{+1}^{(k,l)} \lambda_{i+1}$, $1 \leq i \leq \mathbf{n}(\lambda/\mu) - 1$. Let $\mathrm{STab}^{(k,l)}(\lambda)$ denote the set of (k, l)-standard tableaux of λ/μ.

(1.5) A standard tabuleau $(\lambda_0, \lambda_1, \ldots, \lambda_{\mathbf{n}(\lambda/\mu)}) \in \mathrm{STab}^{(k,l)}(\lambda/\mu)$ will also be denoted by the table such that the i-th Arabic figure is put on $(\mathbf{r}(\lambda_{i-1}, \lambda_i), \mathbf{c}(\lambda_{i-1}, \lambda_i))$-position.

(1.6) Example.

$$\mathrm{STab}^{(2,l)}([2^1 3^1]/[1]) = \begin{cases} \left\{ \begin{smallmatrix} 12 & 13 & 23 & 14 & 24 \\ 34 & 24 & 14 & 23 & 13 \end{smallmatrix} \right\} & \text{if } l \geq 5, \\ \left\{ \begin{smallmatrix} 13 & 23 & 14 & 24 \\ 24 & 14 & 23 & 13 \end{smallmatrix} \right\} & \text{if } l = 4, \\ \left\{ \begin{smallmatrix} 24 \\ 13 \end{smallmatrix} \right\} & \text{if } l = 3 \end{cases}$$

(1.7) For $\mu \subset^{(k,l)} \lambda$, and for i, $0 \leq i \leq \mathbf{n}(\lambda/\mu)$, let $h_i : \mathrm{STab}^{(k,l)}(\lambda/\mu) \to \Lambda_{\mathbf{n}(\mu)+i}^{(k,l)}$ be the map such that $h_i(x)$ is the i-th component of x, and let $f_i : \mathrm{STab}^{(k,l)}(\lambda/\mu) \to \mathrm{STab}^{(k,l)}(\lambda/\mu)$ be the map such that $f_i(\mathbf{t}) = (h_0(\mathbf{t}), \ldots, h_{i-1}(\mathbf{t}), \nu, h_{i+1}(\mathbf{t}), \ldots, h_{\mathbf{n}(\lambda/\mu)})$ if there exists an element ν of $\Lambda^{(k,l)}$ with $h_{i-1}(\mathbf{t}) \subset_{+1}^{(k,l)} \nu \subset_{+1}^{(k,l)} h_{i+1}(\mathbf{t})$ and $\nu \neq h_i(\mathbf{t})$, and $f_i(\mathbf{t}) = \mathbf{t}$ otherwise. For $\alpha \subset_{+1}^{(k,l)} \beta \subset_{+1}^{(k,l)} \gamma$, put

$$d(\alpha, \beta, \gamma) = \mathbf{c}(\alpha, \beta) - \mathbf{c}(\beta, \gamma) + \mathbf{r}(\beta, \gamma) - \mathbf{r}(\alpha, \beta).$$

For $\mathbf{t} \in \mathrm{STab}^{(k,l)}(\lambda/\mu)$ and i, $1 \leq i \leq \mathbf{n}(\lambda/\mu) - 1$, put $\mathbf{d}(\mathbf{t}; i) = d(h_{i-1}(\mathbf{t}), h_i(\mathbf{t}), h_{i+1}(\mathbf{t}))$.

(1.8) For $q \in \mathbf{C}^\times$, and for $d \in \mathbf{Z}$, $1 \leq |d| \leq \mathbf{l}(q) - 1$, put

$$b_d(q) = -\lim_{z \to q} \frac{1 - z}{1 - z^d},$$

and let $c_d(q) \in \{w \in \mathbf{C} | \mathrm{Im}\, w > 0\} \cup \{x \in \mathbf{R} | x \geq 0\}$ be such that

$$c_d(q)^2 = \lim_{z \to q} \frac{z(1 - z^{d-1})(1 - z^{d+1})}{(1 - z^d)^2}.$$

Note that $b_1(q) = -1$, $b_{-1}(q) = q$.

Theorem (1.9) ([Wenzl]). (i) *Let* $q \in \mathbf{C}^\times$, *and let* $k \in \mathbf{N}$ *be such that* $k \leq \mathbf{l}(q) - 1$. *Let* $\mu, \lambda \in \Lambda^{(k,\mathbf{l}(q))}$ *be such that* $\mu \subset^{(k,\mathbf{l}(q))} \lambda$. *Let* $V_{\lambda/\mu}^{(k,\mathbf{l}(q))}$ *be a* $\mathbf{C}$*-vector space with a basis* $\{ v_{\mathbf{t}} \,|\, \mathbf{t} \in \mathrm{STab}^{(k,\mathbf{l}(q))}(\lambda/\mu)\}$. *Then there exists a representation* $\pi_{\lambda/\mu}^{(k,\mathbf{l}(q))} : \mathbf{H}_{\mathbf{n}(\lambda/\mu)}(q) \to \mathbf{End}(V_{\lambda/\mu}^{(k,\mathbf{l}(q))})$ *of* $\mathbf{H}_{\mathbf{n}(\lambda/\mu)}(q)$ *such that*

$$\pi_{\lambda/\mu}^{(k,\mathbf{l}(q))}(T_i)v_{\mathbf{t}} = \begin{cases} b_{\mathbf{d}(\mathbf{t};i)}(q)v_{\mathbf{t}} + c_{\mathbf{d}(\mathbf{t};i)}(q)v_{f_i(\mathbf{t})} & \text{if } f_i(\mathbf{t}) \neq \mathbf{t}, \\ b_{\mathbf{d}(\mathbf{t};i)}(q)v_{\mathbf{t}} & \text{otherwise} \end{cases}$$

$(1 \leq i \leq \mathbf{n}(\lambda/\mu) - 1)$.

(ii) *For* $\lambda \in \Lambda^{(\mathbf{l}(q))}$, *let* $\pi_\lambda^{(\mathbf{l}(q))} = \pi_{\lambda/\phi}^{(\mathbf{k}(\lambda),\mathbf{l}(q))}$. *Then* $\pi_\lambda^{(\mathbf{l}(q))}$ *is irreducible. For* $\mu, \lambda \in \Lambda_n^{(\mathbf{l}(q))}$, $\mu \neq \lambda$, $\pi_\mu^{(\mathbf{l}(q))}$ *and* $\pi_\lambda^{(\mathbf{l}(q))}$ *are not equivalent.*

(iii) *Let* $\mathbf{l}(q) > n$. *Then* $\mathbf{H}_n(q)$ *is semisimple,* $\Lambda_n^{(\mathbf{l}(q))} = \Lambda_n$, *and* $\{ \pi_\lambda^{(\mathbf{l}(q))} \,|\, \lambda \in \Lambda_n\}$ *is a complete set of irreducible representations of* $\mathbf{H}_n(q)$.

§2. Note on irreducible characters

(2.1) Denote by $\mathbf{H}_n(q)^*$ the $\mathbf{C}$-vector space of $\mathbf{C}$-linear maps of $\mathbf{H}_n(q)$ into $\mathbf{C}$. If an element $f \in \mathbf{H}_n(q)^*$ satisfies the condition such that

$$\forall X, \forall Y \in \mathbf{H}_n(q)\ f(XY - YX) = 0,$$

we call f a *class function* of $\mathbf{H}_n(q)$. Denote by $\mathbf{CF}(\mathbf{H}_n(q))$ the set of the class functions of $\mathbf{H}_n(q)$. For $\lambda \in \Lambda_n^{(\mathbf{l}(q))}$, denote by $\chi_\lambda^{(\mathbf{l}(q))}$ the character $\mathrm{Trace} \circ \pi_\lambda^{(\mathbf{l}(q))}$. It is clear that $\chi_\lambda^{(\mathbf{l}(q))} \in \mathbf{CF}(\mathbf{H}_n(q))$.

For $k, m \in \mathbf{N}$, $1 \leq m < k \leq n$, put

$$\begin{aligned} S_{m,k} &= (T_{k-m} \cdots T_1)(T_{k-m+1} \cdots T_2) \cdots (T_{k-1} \cdots T_m) \\ &= (T_{k-m} \cdots T_{k-1})(T_{k-m-1} \cdots T_{k-2}) \cdots (T_1 \cdots T_m). \end{aligned}$$

Then we have $S_{m,k}T_j = T_{j-m}S_{m,k}$ $(m+1 \leq j \leq k-1)$ and $S_{m,k}T_i = T_{k-m+i}S_{m,k}$ $(1 \leq i \leq m-1)$.

For $a, n \in \mathbf{N}$, $1 < a < n$, denote by $\iota_{a,n}$ the monomorphism of $\mathbf{H}_a(q)$ into $\mathbf{H}_n(q)$ such that $\iota_{a,n}(T_i) = T_i$ $(1 \leq i \leq a-1)$. For $X \in \mathbf{H}_a(q)$, $\iota_{a,n}(X)$ will also be denoted by X. For $k, m \in \mathbf{N}$, $1 \leq m < k \leq n$, and for $X \in \mathbf{H}_a(q)$, $Y \in \mathbf{H}_{k-m}(q)$, put

$$X \sharp Y = X S_{k-m,k} Y S_{k-m,k}^{-1} \in \mathbf{H}_n(q).$$

For $f \in \mathbf{CF}(\mathbf{H}_n(q))$, we have

$$f(X \sharp Y) = f(Y S_{k-m,k}^{-1} X S_{k-m,k}) = f(Y S_{m,k} X S_{m,k}^{-1}) = f(Y \sharp X).$$

We also have $f((X \sharp Y) \sharp Z) = f(X \sharp (Y \sharp Z))$.

Let $\chi_{\lambda/\mu}^{(\mathbf{k}(\lambda),\mathbf{l}(q))} = \mathrm{Trace} \circ \pi_{\lambda/\mu}^{(\mathbf{k}(\lambda),\mathbf{l}(q))}$. The following lemma is easily obtained by the definition of $\pi^{(l)}$.

Lemma (2.2) Let k, $m \in \mathbf{N}$, $1 < m < n$, and $X \in \mathbf{H}_m(q)$, $Y \in \mathbf{H}_{k-n}(q)$. Let $\lambda \in \Lambda^{(\mathbf{l}(q))}$. Then

$$\pi_\lambda^{(\mathbf{l}(q))}(X\sharp Y) \cong \bigoplus_{\mu \in \Lambda_m^{(\mathbf{k}(\lambda),\mathbf{l}(q))}} \pi_\mu^{(\mathbf{l}(q))}(X) \otimes \pi_{\lambda/\mu}^{(\mathbf{k}(\lambda),\mathbf{l}(q))}(Y).$$

In particular,

$$\chi_\lambda^{(\mathbf{l}(q))}(X\sharp Y) = \sum_{\mu \in \Lambda_m^{(\mathbf{k}(\lambda),\mathbf{l}(q))}} \chi_\mu^{(\mathbf{l}(q))}(X)\chi_{\lambda/\mu}^{(\mathbf{k}(\lambda),\mathbf{l}(q))}(Y).$$

(2.3) For $\lambda \in \Lambda_n^{(\mathbf{l}(q))}$, denote by $e_\lambda^{(\mathbf{l}(q))}$ a primitive idempotent of $\mathbf{H}_n(q)$ corresponding to the irreducible representation $\pi_\lambda^{(\mathbf{l}(q))}$. For k, m, $n \in \mathbf{N}$, $1 < m < n$, and for $\mu \in \Lambda_m^{(k,\mathbf{l}(q))}$, $\nu \in \Lambda_{n-m}^{(k,\mathbf{l}(q))}$, $\lambda \in \Lambda_n^{(k,\mathbf{l}(q))}$, μ, $\nu \subset^{(k,\mathbf{l}(q))} \lambda$, define the integer $d^{(k,\mathbf{l}(q))}(\lambda;\mu,\nu)$ to be $\chi_\lambda^{(k,\mathbf{l}(q))}(e_\mu^{(\mathbf{l}(q))}\sharp e_\nu^{(\mathbf{l}(q))}) = \chi_{\lambda/\mu}^{(k,\mathbf{l}(q))}(e_\nu^{(\mathbf{l}(q))})$. Denote by $\mathbf{S}(\mathbf{N})$ the group of the bijective maps of $\mathbf{N}$ onto itsself. Let $s_i \in \mathbf{S}(\mathbf{N})$ $(i \in \mathbf{N})$ be the map such that $p_i(s_i(\alpha)) = p_{i+1}(\alpha)$, $p_{i+1}(s_i(\alpha)) = p_i(\alpha)$ and $p_j(s_i(\alpha)) = p_j(\alpha)$ $(j \neq i,\ i+1)$. Let $s_0^{(k,l)} \in \mathbf{S}(\mathbf{N})$ $(i \in \mathbf{N})$ be the map such that $p_1(s_0^{(k,l)}(\alpha)) = p_k(\alpha) + l$, $p_k(s_0^{(k,l)}(\alpha)) = p_1(\alpha) - l$ and $p_j(s_0^{(k,l)}(\alpha)) = p_j(\alpha)$ $(j \neq 1,\ k)$. Let $W^{(k,l)}$ be the subgroup of $\mathbf{S}(\mathbf{N})$ generated by $s_0^{(k,l)}$ and s_i $(1 \le i \le k)$. Define $\delta(k) \in \lambda$ to be $(k-1, k-2, \ldots, 1, 0, \ldots)$. Goodman-Wenzl [GW] proved:

Theorem (2.4)([GW])

$$d^{(k,l)}(\lambda;\mu,\nu) = \sum_{w \in W^{(k,l)}} \mathrm{sgn}(w) d^{(k,l)}(w(\lambda + \delta(k)) - \delta(k);\mu,\nu).$$

Since this identity is coincides with the Kac-Walton one, $d^{(k,l)}(\lambda;\mu,\nu)$ is the same as the SU(k)-fusion coefficient with level $l-k$ (see [GW]).

(2.5) An element α of $\mathbf{Z}_+^\infty$ is called a *composition* if there exists $j \in \mathbf{N}$ such that $p_i(\alpha) > 0$ for $i < j$ and $p_k(\alpha) = 0$ for $k \ge j$. Denote by Ω the set of the compositions. Let $\omega : \Omega \to \Lambda$ be the map such that $\omega(\alpha) = \sigma(\alpha)$ for some $\sigma \in \mathbf{S}(\mathbf{N})$. Let $\Omega_n = \omega^{-1}(\Lambda_n)$ and $\Omega' = \omega^{-1}(\Lambda')$.

(2.6) The maps $\mathbf{k} \circ \omega$, $\mathbf{n} \circ \omega$ of Ω into $\mathbf{Z}_+$ will also be denoted by $\mathbf{k}$, $\mathbf{n}$ respectively. For $\alpha \in \Omega'$ and i, $1 \le i \le \mathbf{k}(\alpha)$, let $x_i(\alpha) = \sum_{1 \le j < i} p_j(\alpha) \in \mathbf{Z}_+$, and let $X(\alpha;i)$ be the element of $\mathbf{H}_{\mathbf{n}(\alpha)}(q)$ such that $X(\alpha;i) = T_1T_2\cdots T_{p_i(\alpha)-1}$ if $p_i(\alpha) \ge 2$, and $X(\alpha;i) = 1$ if $p_i(\alpha) = 1$. Put

$$X(\alpha) = X(\alpha;1)\sharp\cdots\sharp X(\alpha;\mathbf{k}(\alpha)).$$

By the following theorem proved by Ram[Ram], we see that any class function $f : \mathbf{H}_n(q) \to \mathbf{C}$ is determined only by the values $f(X(\lambda))$ $(\lambda \in \Lambda_n)$.

Theorem (2.7)([Ram]). *Let* $n \in \mathbf{N}$. *Then*

$$^{\forall}X \in \mathbf{H}_n(q)\ ^{\exists}x_\lambda \in \mathbf{C}\ (\lambda \in \Lambda_n)\ ^{\forall}f \in \mathbf{CF}(\mathbf{H}_n(q)) \quad f(X) = \sum_{\lambda \in \Lambda_n} x_\lambda f(C(\lambda))$$

We can calculate the coefficients x_λ's via an inductive process.

(2.8) For $\mu, \lambda \in \Lambda^{(k,\mathbf{l}(q))}$, $\mu \subset^{(k,\mathbf{l}(q))} \lambda$, put $\Delta^{(k,\mathbf{l}(q))}(\lambda/\mu) = \chi^{(k,\mathbf{l}(q))}(\lambda/\mu)(X([\mathbf{n}(\lambda/\mu)^1]))$. It turns out that

$$\Delta^{(k,\mathbf{l}(q))}(\lambda/\mu) = \sum_{\mathbf{t} \in \mathbf{STab}^{(k,\mathbf{l}(q))}(\lambda/\mu)} \prod_{1 \le i \le \mathbf{n}(\lambda/\mu)} b_{\mathbf{d}(\mathbf{t};i)}(q).$$

As an immediate consequence of Lemma (2.2), we have:

Theorem (2.9) *Let* $\lambda \in \Lambda_n^{(k,l)}$. *Let* $\alpha \in \Omega_n$. *Then*

$$\chi_\lambda^{(k,l)}(X(\alpha)) = \sum_{\substack{\phi=\mu_0 \subset^{(k,\mathbf{l}(q))} \cdots \subset^{(k,\mathbf{l}(q))} \mu_r = \lambda \\ p_i(\alpha) = \mathbf{n}(\mu_i/\mu_{i-1})}} \prod_{i=1}^{r} \Delta^{(k,\mathbf{l}(q))}(\mu_i/\mu_{i-1}).$$

Ram[Ram] gave an explicit formula of $\Delta^{(k,\mathbf{l}(q))}(\mu_i/\mu_{i-1})$ for $\mathbf{l}(q) > \mathbf{n}(\lambda)$.

§3. Main result

(3.1) Let $l = \mathbf{l}(q)$. In §3, $a \equiv b$ means $a - b \in l\mathbf{Z}$. For $\lambda \in \Lambda^{(k,\mathbf{l}(q))}$, $\Delta(\lambda/\phi)$ will also be denoted by $\{p_1(\lambda), \ldots, p_{\mathbf{k}(\lambda)}(\lambda)\}$. If $\mathbf{k}(\lambda) = 2$, we have:

$$\{r, r-a\} = \begin{cases} -q^{-a-3} & r \equiv -1 \\ q^{-a-2} & a \ge 1,\ r \equiv 0 \\ q^{-a-2} + q^{a-1} & a = 0,\ r \equiv a \\ -q^{a} & r \equiv a+1 \\ 0 & \text{otherwise} \end{cases}$$

(3.2) If $\mathbf{k}(\lambda) = 3$, we have:

$\{r,r,r\}$	$l \geq 5$	$q^{-4}\,(r \equiv -1)$, $q^{-3}+q^{-2}+q^{-1}\,(r \equiv 0)$, $1\,(r \equiv 1)$, $0\,(\text{oth.})$
—	$l = 4$	$1\,(r \equiv 1,\,3)$, $-1\,(r \equiv 0,\,2)$
$\{r,r,r-1\}$	$l \geq 6$	$q^{-5}\,(r \equiv -1)$, $q^{-4}\,(r \equiv 0)$, $-1\,(r \equiv 1)$, $0\,(\text{oth.})$
—	$l = 5$	$1\,(r \equiv -1)$, $q\,(r \equiv 0)$, $-1\,(r \equiv 1)$, $0\,(r \equiv 2)$, $-q\,(r \equiv -2)$
—	$l = 4$	$1\,(r \equiv 0,\,2)$, $-1\,(r \equiv 1,\,3)$
$\{r,r,r-a\}$ $(a \geq 2)$	$l \geq a+5$	$q^{-(a+4)}\,(r \equiv -1)$, $q^{-(a+3)}\,(r \equiv 0)$, $0\,(\text{oth.})$
—	$l = a+4$	$q\,(r \equiv 0)$, $-q^{a}\,(r \equiv -2)$, $1\,(r \equiv -1)$, $0\,(\text{oth.})$
—	$l = a+3$	$1\,(r \equiv 0)$, $q^{a-1}\,(r \equiv -2)$, $q^{a}+q^{a+1}+q^{a+2}\,(r \equiv -1)$, $0\,(\text{oth.})$
$\{r,r-1,r-1\}$	$l \geq 6$	$-q^{-4}\,(r \equiv 0)$, $1\,(r \equiv 1)$, $q\,(r \equiv 2)$, $0\,(\text{oth.})$
—	$l = 5$	$-q\,(r \equiv 0)$, $1\,(r \equiv 1)$, $q\,(r \equiv 2)$, $-1\,(r \equiv 3)$, $0\,(r \equiv 4)$
—	$l = 4$	$1\,(r \equiv 1,\,3)$, $-1\,(r \equiv 0,\,2)$
$\{r,r-a,r-a\}$ $(a \geq 2)$	$l \geq a+5$	$q^{a-1}\,(r \equiv a)$, $q^{a}\,(r \equiv a+1)$, $0\,(\text{oth.})$
—	$l = a+4$	$q^{a-1}\,(r \equiv a)$, $q^{a}\,(r \equiv a+1)$, $-1\,(r \equiv a+2)$, $0\,(\text{oth.})$
—	$l = a+3$	$q^{a-1}\,(r \equiv a)$, $q^{a}+q^{a+1}+q^{a+2}\,(r \equiv a+1)$, $1\,(r \equiv a+2)$, $0\,(\text{oth.})$
$\{r,r-1,r-2\}$	$l \geq 7$	$-q^{-5}\,(r \equiv 0)$, $-q\,(r \equiv 2)$, $0\,(\text{oth.})$
—	$l = 6$	$-q\,(r \equiv 0,\,2,\,4)$, $0\,(\text{oth.})$
—	$l = 5$	$-1\,(r \equiv 0)$, $0\,(r \equiv 1)$, $-q\,(r \equiv 2)$, $1\,(r \equiv -2)$, $q\,(r \equiv -1)$
$\{r,r-1,r-a\}$ $(a \geq 3)$	$l \geq a+5$	$-q^{-(a+3)}\,(r \equiv 0)$, $0\,(\text{oth.})$
—	$l = a+4$	$-q\,(r \equiv 0)$, $-q^{a-1}\,(r \equiv -2)$, $0\,(\text{oth.})$
—	$l = a+3$	$-1\,(r \equiv 0)$, $q^{a-2}\,(r \equiv -2)$, $q^{a-1}\,(r \equiv -1)$, $0\,(\text{oth.})$
$\{r,r-a+1,r-a\}$ $(a \geq 3)$	$l \geq a+5$	$-q^{a-1}\,(r \equiv a)$, $0\,(\text{oth.})$
—	$l = a+4$	$-q^{a-1}\,(r \equiv a)$, $-q\,(r \equiv a+2)$, $0\,(\text{oth.})$
—	$l = a+3$	$-q^{a-1}\,(r \equiv a)$, $1\,(r \equiv a+1)$, $q\,(r \equiv a+2)$, $0\,(\text{oth.})$
$\{r,r-b,r-a\}$ $(a-2 \geq b \geq 2)$	$l \geq a+5$	0
—	$l = a+4$	$-q^{a-b}\,(r \equiv -2)$, $0\,(\text{oth.})$
—	$l = a+3$	$q^{a-b-1}\,(r \equiv -2)$, $q^{a-b}\,(r \equiv -1)$, $0\,(\text{oth.})$

§4. On Littlewood-Richardson rule

(4.1) In this section, we use terminology in [Macdonald]. Let $\mu \subset \lambda$. Denote by $\mathbf{LP}(\lambda/\mu)$ the set of the lattice permutations of shape λ/μ. For $\mathbf{z} \in \mathbf{LP}(\lambda/\mu)$ and $\nu \in \Lambda_{\mathbf{n}(\lambda/\mu)}$, let $\mathbf{LP}(\lambda/\mu;\nu)$ be the set of the lattice permutations of shape λ/μ and weight ν. It is well-known that the cardinarity of $\mathbf{LP}(\lambda/\mu;\nu)$ is equal to $d^{(k,\mathbf{l}(q))}(\lambda;\mu,\nu)$ for $\mathbf{l}(q) > \mathbf{n}(\lambda)$. See [Macdonald].

(4.2) The set $\mathbf{LP}([2^1 3^1 4^1]/[2])$ consists of the lattice permutations

$$\begin{array}{cccc} & & 1 & 1 \\ 1 & 1 & 2 & \\ 2 & 2 & & \end{array} , \quad \begin{array}{cccc} & & 1 & 1 \\ 1 & 1 & 2 & \\ 2 & 3 & & \end{array} , \quad \begin{array}{cccc} & & 1 & 1 \\ 1 & 2 & 2 & \\ 2 & 3 & & \end{array} , \quad \begin{array}{cccc} & & 1 & 1 \\ 1 & 2 & 2 & \\ 3 & 3 & & \end{array} .$$

The weights of the first, the second, the third, and the fourth lattice permutations are $[3^1 4^1]$, $[1^1 2^1 4^1]$, $[1^1 3^2]$, $[2^2 3^1]$, respectively.

Theorem (4.3) *Let μ, $\lambda \in \Lambda^{(3,l)}$ be such that $\mu \subset^{(3,l)} \lambda$. Assume $p_1(\mu) - p_2(\mu) \leq 1$. For $\mathbf{z} \in \mathbf{LP}(\lambda/\mu)$, let $\mathbf{z}_{ij}$ be the Arabic figure at (i,j)-position, and let $\mathbf{bot}(\lambda/\mu; a)$ be the number of j's such that $\mathbf{z}_{3j} = a$. Denote by $\mathbf{Y}^{(3,\mathbf{l}(q))}(\lambda/\mu)$ the set of the lattice permutations $\mathbf{z} \in \mathbf{LP}(\lambda/\mu)$ such that*

(1) $\mathbf{bot}(\lambda/\mu; 1) + 1 + p_1(\lambda) - p_3(\lambda) \leq \mathbf{l}(q) - 3$ *if* $p_1(\mu) = p_2(\mu) + 1$, $\mathbf{z}_{2,p_1(\mu)} = 1$, $\mathbf{z}_{3,p_1(\mu)} = 2$,

(2) $\mathbf{bot}(\lambda/\mu; 1) + 1 + p_1(\lambda) - p_3(\lambda) \leq \mathbf{l}(q) - 3$ *if* $p_1(\mu) = p_2(\mu) + 1$, $\mathbf{z}_{2,p_1(\mu)} = 1$, $\mathbf{z}_{3,p_1(\mu)} \neq 2$, $\mathbf{bot}(\lambda/\mu; 2) = p_1(\lambda) - p_2(\lambda) + 1$,

(3) $\mathbf{bot}(\lambda/\mu; 1) + 1 + p_1(\lambda) - p_3(\lambda) \leq \mathbf{l}(q) - 3$ *otherwise.*

Then $d^{(3,\mathbf{l}(q))}(\lambda; \mu, \nu)$ is equal to the number of the elements of $\mathbf{Y}^{(3,\mathbf{l}(q))}(\lambda/\mu)$ whose weights are ν.

Example (10.3). *On* $Y^{(3,\mathbf{l}(q))}([7^1 10^1 12^1]/[5^2])$. We shall only write $a\,b\,c\,d\,e$ for

$$\mathbf{z} = \begin{matrix} & & & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & & & & & 2 & 2 & 2 & 2 & 2 & & \\ a & b & c & d & e & 3 & 3 & & & & & \end{matrix} .$$

(x, y, z) denotes weight. Then it consists of:

$\mathbf{l}(q) \geq 8$, $(7,7,5)$ 22333,

$\mathbf{l}(q) \geq 9$, $(8,6,5)$ 12333, $(8,7,4)$ 12233,

$\mathbf{l}(q) \geq 10$, $(9,5,5)$ 11333, $(9,6,4)$ 11233, $(9,7,3)$ 11223,

$\mathbf{l}(q) \geq 11$, $(10,5,4)$ 11133, $(10,6,3)$ 11123, $(10,7,2)$ 11122,

$\mathbf{l}(q) \geq 12$, $(11,5,3)$ 11113, $(11,6,2)$ 11112,

$\mathbf{l}(q) \geq 13$, $(12,5,2)$ 11111,

Example (10.4). *On* $Y^{(3,\mathbf{l}(q))}([6^1 9^1 11^1]/[3^1 4^1])$.

$$\mathbf{z} = \begin{matrix} & & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & & & a & 2 & 2 & 2 & 2 & 2 & & \\ b & c & d & e & 3 & 3 & & & & & \end{matrix}$$

$\mathbf{l}(q) \geq 8$, $(7,6,6)\ {}_{3333}^{2}$, $(7,7,5)\ {}_{2333}^{2}$, $(8,6,5)\ {}_{2333}^{1}$, $(8,7,4)\ {}_{2233}^{1}$,

$\mathbf{l}(q) \geq 9$, $(8,6,5)\ {}_{1333}^{2}$, $(8,7,4)\ {}_{1233}^{2}$, $(8,8,3)\ {}_{2223}^{1}$, $(9,5,5)\ {}_{1333}^{1}$, $(9,6,4)\ {}_{1233}^{1}$, $(9,7,3)\ {}_{1223}^{1}$,

$\mathbf{l}(q) \geq 10$, $(9,6,4)\ {}_{1133}^{2}$, $(9,7,3)\ {}_{1123}^{2}$, $(9,8,2)\ {}_{1222}^{1}$, $(10,5,4)\ {}_{1133}^{1}$, $(10,6,3)\ {}_{1123}^{1}$,

$\mathbf{l}(q) \geq 11$, $(10,6,3)\ {}_{1113}^{2}$, $(10,7,2)\ {}_{1122}^{1}$, $(11,5,3)\ {}_{1113}^{1}$,

$\mathbf{l}(q) \geq 12$, $(11,6,2)\ {}_{1112}^{1}$

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