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INDUCED REPRESENTATIONS OF RANK TWO QUASI-SPLIT UNITARY GROUPS OVER A p-ADIC FIELD

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ABSTRACT. We classify the irreducible non-supercuspidal representations of rank two quasi-split unitary groups attached to a quadratic extension E/F of p-adic fields. This extends Shahidi's classification for rank two split groups to the quasi-split groups of the same rank other than certain forms of type D_4.

1. INTRODUCTION

Let G be a connected reductive group over a non-archimedean local field F of characteristic zero. One hopes to classify the isomorphism classes of irreducible admissible representations of G. The problem divides into the following two steps: to describe the isomorphism classes of the irreducible supercuspidal representations of its Levi subgroups and to study the representations parabolically induced from them. Both steps are hard due to the rich structure of p-adic groups.

In this note, we report our results on the latter problem. More precisely, let $I_P(\rho)$ be a parabolically induced representation, where ρ is a supercuspidal representation of a Levi component $M(F)$ of a parabolic subgroup $P(F)$. The problem is to have a criterion of reducibility for $I_P(\rho)$. Such a criterion was available for $GL(n)_F$ thanks to the work of Bernstein-Zelevinskii [BZ] [Z], who utilized Gelfand-Kazhdan theory of "derivations" of representations [GK]. Unfortunately, similar theory does not exist for $G \neq GL(n)_F$. On the other hand, if the inducing representation ρ is generic, the theory of Eisenstein series relates the reducibility of $I_P(\rho)$ to the analytic behavior of certain L-factor of ρ [Sh]. If G is of rank 2 and split, ρ is always generic, and the analytic properties of the relevant L-factors are known by [Sh2] [GJ] [JL]. Consequently, the classification in this case was established by Shahidi [Sh]. Once the reducibility is determined, then the irreducible constituents of $I_P(\rho)$ at each reducible point can be calculated by their Jacquet modules. In the above case, this was given by Sally-Tadić [ST] for $G = GSp(2)$, $Sp(2)$, and Muić [Mu] for G of type G_2.

Our result extends Shahidi's result to the rank two quasi-split unitary groups. Let E be a quadratic extension of F, G_n and G'_n be the quasisplit unitary groups of $2n$ and $2n+1$ variables associated to E/F, respectively. Any proper parabolic subgroup of G_2 (resp. G'_2) is isomorphic to one of $P_i = M_iU_i$ (resp. $P'_i = M'_iU'_i$) ($i = 0, 1, 2$), whose Levi subgroups are given by $M_0 = \mathbf{T} \simeq (\mathrm{Res}_{E/F}\mathbb{G}_m)^2$, $M_1 \simeq \mathrm{Res}_{E/F}GL(2)$ and $M_2 \simeq \mathrm{Res}_{E/F}\mathbb{G}_m \times G_1$ (resp. $M'_0 = \mathbf{T}' \simeq (\mathrm{Res}_{E/F}\mathbb{G}_m)^2 \times G'_0$, $M'_1 \simeq \mathrm{Res}_{E/F}GL(2) \times G'_0$ and $M'_2 \simeq \mathrm{Res}_{E/F}\mathbb{G}_m \times G'_1$). Each $I_{P_i}(\rho)$ ($i = 0, 1, 2$) has different aspects. The (possible) reducible points of $I_{P_0}(\rho)$ are easily calculated. Those of $I_{P_1}(\rho)$ were obtained by Goldberg [G]. But for $I_{P_2}(\rho)$, we have to use the base change lift for G_1 or G'_1 [R] to derive the standard L-factor of G_1 and G'_1 from those of $GL(2)_E$ and $GL(3)_E$. Here, the key is the uniqueness result for Shahidi's γ-factor [Sh, Th.3.5]. Our method seems to apply to more general twisted endoscopic liftings of $GL(n)$ to derive the product L-factors of $G \times GL(m)$

from the Rankin product L-factor of $GL(n) \times GL(m)$, where G is a classical group. Some related results were obtained by Zhang [Zh] assuming certain conjectures.

The organization of this note is as follows. In Section 2, we describe the reducible points of $I_{P_0}(\rho)$ and its irreducible constituents at those reducible points. Section 3 begins with a review on the base change problems for unitary groups. We adopt new general set-up of twisted endoscopy [KS] for this. We review the result of D.Goldberg [G] on $I_{P_1}(\rho)$ in this framework. In Section 4, an argument on Poincaré series due to Henniart [H] and Vignéras[V] enables us to apply Shahidi's uniqueness result to calculate precise L-factors. Then the reducible points of $I_{P_2}(\rho)$ turn out to be described in terms of endoscopic liftings of G'_1 [R], and we determine its irreducible constituents at each points.

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Notation We write σ for the generator of the Galois group $\Gamma_{E/F}$ of E/F. Fix an algebraic closure $\overline{F}$ of F containing E. $W_F = W_{\overline{F}/F}$ and Γ_F denote the absolute Weil and Galois group of F, respectively. Write $|\ |$ and q for the absolute value and the cardinal of the residue field of F, respectively. We also use similar notations W_E and $|\ |_E$ for E.

Let $G = G_n$ or G'_n. Fix the usual F-splittings $\mathbf{spl}_{G_n} = (\mathbf{B}, \mathbf{T}, \{X_\alpha\})$ and $\mathbf{spl}_{G'_n} = (\mathbf{B}', \mathbf{T}', \{X'_\alpha\})$ of G_n and G'_n, respectively. In particular, $(\mathbf{B}, \mathbf{T})$ and $(\mathbf{B}', \mathbf{T}')$ are upper triangular and diagonal Borel pairs. Write $\Sigma_0 = \Sigma(\mathbf{B}, A_0)$ (resp. $\Sigma(\mathbf{B}', A_0)$) for the set of $\mathbf{B}$-positive (resp. $\mathbf{B}'$-positive) relative roots. Here A_0 is the split component of $\mathbf{T}$ or $\mathbf{T}'$. $\Delta_0 = \Delta_0^G$ and $\Delta_0^\vee = \Delta_0^{G,\vee}$ denote the set of simple roots and simple coroots of A_0 in $\mathbf{B}$ or $\mathbf{B}'$.

Put $H_n = \mathrm{Res}_{E/F} GL(n)$. The standard parabolic subgroups of G_n and G'_n are classified by the partitions $\mathbf{n} = (n_1, \dots, n_r; n_0)$ of n with a distinguished component $n_0 \geq 0$. That is, $P_{\mathbf{n}} = M_{\mathbf{n}} U_{\mathbf{n}}$ (resp. $P'_{\mathbf{n}} = M'_{\mathbf{n}} U'_{\mathbf{n}}$) is the standard parabolic subgroup, whose Levi component $M_{\mathbf{n}}$ is isomorphic to $H_{n_1} \times \cdots \times H_{n_r} \times G_{n_0}$ (resp. $H_{n_1} \times \cdots \times H_{n_r} \times G'_{n_0}$). The above P_i $(i = 0, 1, 2)$ in the introduction are $P_{(1,1;0)}$, $P_{(2;0)}$, $P_{(1;1)}$, respectively.

Let $\Pi(H(F))$ (resp. $\Pi_{\mathrm{unit}}(H(F))$, $\Pi_{\mathrm{temp}}(H(F))$, $\Pi_2(H(F))$ and $\Pi_0(H(F))$) be the set of isomorphism classes of irreducible admissible (resp. unitarizable, tempered, square integrable and supercuspidal) representations of a reductive p-adic group $H(F)$. Set $\mathfrak{a}_M := \mathrm{Hom}(X^*(M), \mathbb{R})$ and $\mathfrak{a}_M^* := X^*(M) \otimes \mathbb{R}$, where $X^*(M)$ is the group of F-rational characters of M. Recall the map $H_M : M(F) \longrightarrow \mathfrak{a}_M$ [Sh]. By this map, we identify $\nu \in \mathfrak{a}^*_{M,\mathbb{C}} = \mathfrak{a}^*_M \otimes \mathbb{C}$ with the quasi-character $M(F) \ni m \longmapsto q^{\langle \nu, H_M(m) \rangle} \in \mathbb{C}^\times$. Write $I_P^G(\pi; \nu) := \mathrm{ind}_{P(F)}^{G(F)} [\pi[\nu] \otimes \mathbf{1}_{U(F)}]$ with $\pi[\nu] := \pi \otimes \nu$, $\pi \in \Pi(M(F))$, $\nu \in \mathfrak{a}^*_M$.

Denote by $\omega_{E/F}$ the non-trivial character of $F^\times / N_{E/F}(E^\times)$. We reserve the scripts μ and η for unitary characters of $E^\times$ such that $\mu|_{F^\times} = \omega_{E/F}$ and $\eta|_{F^\times} = \mathbf{1}$, respectively. Another such characters are denoted by μ', η', etc. η being as such, let η_u be the unitary character of $G'_0 = U(1, F)_{E/F}$ given by $\eta_u(x\sigma(x^{-1})) = \eta(x)$.

2. Irreducible representations supported on P_0

We begin with $G = G_n$ or G'_n. Each irreducible admissible representation of $\mathbf{T}(F)$ (resp. $\mathbf{T}'(F)$) is of the form of $\underline{\chi}[\nu]$, where $\underline{\chi} = \bigotimes_{i=1}^n \chi_i$ (resp. $\bigotimes_{i=1}^n \chi_i \otimes \eta_u$) $(\chi_i \in \Pi_{\mathrm{unit}}(E^\times))$ and $\nu \in \mathfrak{a}_0^* := \mathfrak{a}^*_{M_0}$.

Since $\nu = \nu_0$ is a reducible point of $I(\underline{\chi}; \nu) := I_{\mathbf{B}}^G(\underline{\chi}; \nu)$ (resp. $I_{\mathbf{B}'}^G(\underline{\chi}; \nu)$) if and only if so is $w(\nu_0)$ for $I(w(\underline{\chi}); \nu)$ [BZ, 2.9], it suffices to study $I(\underline{\chi}; \nu)$ with ν in some closed positive chamber:

$$\mathfrak{c}_{P_{\mathbf{n}}} := \{\lambda \in \mathfrak{a}^*_{M_{\mathbf{n}}} \,|\, \alpha^\vee(\lambda) > 0 \quad (\forall \alpha \in \Delta_0 \setminus \Delta_0^{M_{\mathbf{n}}}), \quad \alpha^\vee(\lambda) = 0 \quad (\forall \alpha \in \Delta_0^{M_{\mathbf{n}}})\}.$$

Putting $m_i := \sum_{j=1}^{i} n_j$, we write $\underline{\chi}_i^{\mathbf{n}} = \bigotimes_{j=m_{i-1}+1}^{m_i} \chi_j$, $(1 \le i \le r)$ and $\underline{\chi}_0^{\mathbf{n}} = \bigotimes_{j=m_r+1}^{n} \chi_j$. Then we have $I(\underline{\chi};\nu) = I_{P_{\mathbf{n}}}^G(I^{M_{\mathbf{n}}}(\underline{\chi});\nu)$ where

$$I^{M_{\mathbf{n}}}(\underline{\chi}) = \begin{cases} \bigotimes_{i=1}^r I^{H_{n_i}}(\underline{\chi}_i^{\mathbf{n}}) \otimes I^{G_{n_0}}(\underline{\chi}_0^{\mathbf{n}}) & \text{if } G = G_n \\ \bigotimes_{i=1}^r I^{H_{n_i}}(\underline{\chi}_i^{\mathbf{n}}) \otimes I^{G'_{n_0}}(\underline{\chi}^{\mathbf{n}}{}_0, \eta_u) & \text{if } G = G'_n. \end{cases}$$

Since the R-group of $H_n(F)$ is trivial, $I^{H_{n_i}}(\underline{\chi}_i^{\mathbf{n}})$ are all irreducible and tempered. Suppose that s denotes the number of different χ_i $(m_r + 1 \le i \le n)$ such that $\chi_i|_{F^\times} = \omega_{E/F}$ (resp. $\chi_i|_{F^\times}$ is trivial but $\chi_i \ne \eta$). Since the R-group of G_n (resp. G'_n) is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^s$ [Ke, Th.3.6][Ke2, Th.8], $I^{G_{n_0}}(\underline{\chi}_0^{\mathbf{n}})$ and $I^{G'_{n_0}}(\underline{\chi}_0^{\mathbf{n}} \otimes \eta_u)$ are direct sums of 2^s different irreducible tempered representations:

$$I^{G_{n_0}}(\underline{\chi}_0^{\mathbf{n}}) \simeq \bigoplus_{i=1}^{2^s} \tau_i^{G_{n_0}}(\underline{\chi}_0^{\mathbf{n}}), \quad I^{G'_{n_0}}(\underline{\chi}_0^{\mathbf{n}} \otimes \eta_u) \simeq \bigoplus_{i=1}^{2^s} \tau_i^{G'_{n_0}}(\underline{\chi}_0^{\mathbf{n}}, \eta_u).$$

Thus we are reduced to study the reducibility of $I_{P_{\mathbf{n}}}^G(\tau_i(\underline{\chi});\nu)$ and $I_{P'_{\mathbf{n}}}^G(\tau_i(\underline{\chi},\eta_u);\nu)$ with $\tau_i(\underline{\chi}) := \bigotimes_{j=1}^r I^{H_{n_i}}(\underline{\chi}_i^{\mathbf{n}}) \otimes \tau_i^{G_{n_0}}(\underline{\chi}_0^{\mathbf{n}})$ and $\tau_i(\underline{\chi},\eta_u) = \bigotimes_{j=1}^r I^{H_{n_i}}(\underline{\chi}_i^{\mathbf{n}}) \otimes \tau_i^{G'_{n_0}}(\underline{\chi}_0^{\mathbf{n}}, \eta_u)$.

For each standard Levi subgroup M, write W_M for the set of $w \in W$ of minimal length in the coset wW^M such that $w(M)$ is again a standard Levi subgroup. For $w \in W_M$, $P_w = M_w U_w$ denotes the standard parabolic subgroup with the Levi component $M_w = w(M)$. For a standard parabolic P, let $\Sigma_P := \{(\alpha|_{\mathfrak{a}_M}) \,|\, \alpha \in \Sigma_0 \setminus \Sigma_0^M\}$ and write Σ_P^r for the set of reduced elements in it. Define

$$\mathrm{inv}_P(w) := \{\alpha \in \Sigma_P^r \,|\, w(\alpha) \notin \Sigma_{P_w}\}.$$

For $\pi[\nu] \in \Pi(M(F))$ the integral

$$[M(w,\pi[\nu])\phi](g) := \int_{(U_w \cap w(U))(F)\setminus U_w(F)} \phi(w^{-1}ug)\,du, \quad \phi \in I_P^G(\pi[\nu])$$

converges absolutely if $\alpha^\vee(\nu) \gg 0$ for every $\alpha \in \mathrm{inv}_P(w)$. It extends to a meromorphic function of ν on all $\mathfrak{a}^*_{M,\mathbb{C}}$ (cf. [Sh3], [Si]). Outside its poles it defines an intertwining operator $M(w,\pi[\nu]) : I_P^G(\pi[\nu]) \to I_{P_w}^G(w(\pi[\nu]))$. It follows from the properties of the intertwining operator that:

Lemma 2.1. *The set of zeros of $M(w_{\mathbf{n}}^-,\underline{\chi}[\lambda])$ in the region $\lambda \in \mathfrak{a}^*_{M_{\mathbf{n}},\mathbb{C}}$, $\mathrm{Re}(\lambda) \in \mathfrak{c}_{P_{\mathbf{n}}}$ is the union of those of $M(r_\alpha,\underline{\chi}[\lambda])$, $\alpha \in \Sigma_0^r \setminus \Sigma_0^{M_{\mathbf{n}}}$.*

$M(r_\alpha,\underline{\chi}[\lambda])$, $\alpha \in \Sigma_0^r \setminus \Sigma_0^{M_{\mathbf{n}}}$ are essentially intertwining operators for rank one subgroups G_α. More precisely, we have

Lemma 2.2. *Let $\alpha \in \Sigma_0^r \setminus \Sigma_0^{M_{\mathbf{n}}}$ and take $w \in W$ such that $w(\alpha) \in \Delta_0$. Write $P_{w(\alpha)} = M_{w(\alpha)}U_{w(\alpha)}$ for the standard parabolic subgroup satisfying $\Delta_0^{M_{w(\alpha)}} = \{w(\alpha)\}$. Then the set of zeros of $M(r_\alpha,\underline{\chi}[\nu])$ coincides with that of $M^{M_{w(\alpha)}}(r_{w(\alpha)}, w(\underline{\chi}[\nu]))$.*

In our case, G_α is isomorphic to either H_2, G_1 or G'_1. The zeros of intertwining operator of those are given by the following. In any case α denotes the unique simple relative root. Write δ^H for the Steinberg representation of a reductive p-adic group $H(F)$.

(1) $H_2(F) = GL(2,E)$ [JL]. Let $\chi \in \Pi_{\mathrm{unit}}(E^\times)$. $\mathfrak{a}_0^*$ is identified with $\mathbb{R}^2$ so that $\nu = (\nu_1, \nu_2) \in \mathbb{R}^2$ corresponds to

$$\mathbf{T}^{H_2}(F) \ni \mathrm{diag}(t_1, t_2) \mapsto |t_1|_E^{\nu_1/2}|t_2|_E^{\nu_2/2} \in \mathbb{C}^\times.$$

Then $M(r_\alpha; \underline{\chi}[\nu])$ has a zero in the region $\alpha^\vee(\nu) \geq 0$ if and only if $\chi_1 = \chi_2$. In this case, the only zero occurs at $\alpha^\vee(\nu) = 2$. If we write such ν as $(\lambda+1, \lambda-1)$,

$$(2.1) \qquad 0 \longrightarrow \chi(\det)\delta^{H_2}[\nu] \longrightarrow I(\chi[\nu+1] \otimes \chi[\nu-1]) \longrightarrow \chi(\det)[\nu] \longrightarrow 0.$$

(2) $G_1(F) = U(1,1)_{E/F}(F)$ [LL]. $\underline{\chi} = \chi$ for $\chi \in \Pi_{\mathrm{unit}}(E^\times)$. Note that $\nu \in \mathbb{R}$ is identified with

$$\mathbf{T}(F) \ni \mathrm{diag}(t, \sigma(t)^{-1}) \mapsto |t|_E^{\nu/2} \in \mathbb{C}^\times.$$

$M(r_\alpha; \underline{\chi}[\nu])$ has a zero in the region $\alpha^\vee(\nu) \geq 0$ if and only if $\chi|_{F^\times} = \mathbf{1}$. In this case, the only zero located at $\alpha^\vee(\nu) = 1$ and we have

$$(2.2) \qquad 0 \longrightarrow \eta_u(\det)\delta^{G_1} \longrightarrow I(\eta; 1) \longrightarrow \eta_u(\det) \longrightarrow 0.$$

(3) $G_1'(F) = U(2,1)_{E/F}(F)$ [KeS]. Let $\underline{\chi} = \chi \in \Pi_{\mathrm{unit}}(E^\times)$. Again, $\mathfrak{a}_0^*$ is identified with $\mathbb{R}$ in such a way that $\nu \in \mathbb{R}$ corresponds to

$$\mathbf{T}'(F) \ni \mathrm{diag}(t, z, \sigma(t)^{-1}) \mapsto |t|_E^{\nu/2} \in \mathbb{C}^\times.$$

$M(r_\alpha; \underline{\chi} \otimes \eta_u)[\nu])$ has a zero in the positive region if and only if either $\chi = \eta$ or $\chi|_{F^\times} = \omega_{E/F}$.

(i) If $\chi = \eta$, the zero occurs at $\alpha^\vee(\nu) = 4$ and we have

$$(2.3) \qquad 0 \longrightarrow \eta_u(\det)\delta^{G_1'} \longrightarrow I(\eta[2] \otimes \eta_u) \longrightarrow \eta_u(\det) \longrightarrow 0.$$

(ii) If $\chi|_{F^\times} = \omega_{E/F}$, the zero occurs at $\alpha^\vee(\nu) = 2$ and we have

$$(2.4) \qquad 0 \longrightarrow \delta^1(\mu, \eta) \longrightarrow I(\mu[1] \otimes \eta_u) \longrightarrow \pi^1_{\mathrm{nt}}(\mu, \eta) \longrightarrow 0.$$

Here $\delta^1(\mu, \eta) \in \Pi_2(G'(F))$ and $\pi^1_{\mathrm{nt}}(\mu, \eta)$ is the non-tempered representation.

The above implies

Proposition 2.3. *(i) Suppose $G = G_n$. The set of the reducible points of $I^G_{P_n}(\tau_i(\underline{\chi}); \nu)$ is given by*

$$\mathfrak{r} := \left\{ \underline{\chi}[\nu] \;\middle|\; \begin{array}{ll} \chi_i\chi_j^{-1} = |\;|_E^{\pm}, & 1 \leq i < j \leq n \\ \chi_i|_{F^\times} = |\;|_F^{\pm}, & 1 \leq i \leq n \\ \chi_i\chi_j = |\;|_E^{\pm}, & 1 \leq i < j \leq n \end{array} \right\}.$$

(ii) Suppose $G = G_n'$. The set of the reducible points of $I^G_{P_n}(\tau_i(\underline{\chi}, \eta_u); \nu)$ is given by

$$\mathfrak{r}' := \left\{ \underline{\chi}[\nu] \;\middle|\; \begin{array}{ll} \chi_i\chi_j^{-1} = |\;|_E^{\pm}, & 1 \leq i < j \leq n \\ \chi_i = \eta|\;|_E^{\pm}, & 1 \leq i \leq n \\ \chi_i|_{F^\times} = \omega_{E/F}|\;|_F^{\pm}, & 1 \leq i \leq n \\ \chi_i\chi_j = |\;|_E^{\pm}, & 1 \leq i < j \leq n \end{array} \right\},$$

Now we restrict ourselves to the cases $G = G_2$ and $G' = G_2'$. By [BZ, 2.9], it suffices to consider the case of

$$\mathfrak{r}_{\alpha_1} := \{\underline{\chi}[\nu] \,|\, \chi_1\chi_2^{-1} = |\;|_E\}, \quad \mathfrak{r}_{\alpha_2} := \{\underline{\chi}[\nu] \,|\, \chi_2|_{F^\times} = |\;|_F\}$$

for G and

$$\mathfrak{r}_{\alpha_1} := \{\underline{\chi}[\nu] \,|\, \chi_1\chi_2^{-1} = |\;|_E\}, \quad \mathfrak{r}_{\alpha_2} := \{\underline{\chi}[\nu] \,|\, \chi_2 = \eta|\;|_E\}, \quad \mathfrak{r}_{2\alpha_2} := \{\underline{\chi}[\nu] \,|\, \chi_2|_{F^\times} = \omega_{E/F}|\;|_F\}$$

for G'.

Proposition 2.4. *Suppose $\chi \in \Pi_{\mathrm{unit}}(E^\times)$ and $\lambda \in \mathbb{R}_{\geq 0}$. We have the following* (1) *and* (2) *for G, and* (3), (4), (5) *for G'.*
(1) *Both $I_{P_1}^G(\chi[\lambda]\delta^{H_2})$ and $I_1^G(\chi(\det)[\lambda])$ are irreducible outside the points W_{M_1}-conjugate to one of the following:*

$$
\begin{aligned}
&\mathfrak{r}_{\alpha_1,0}(\chi) := \chi[1] \otimes \chi[-1], && \chi = \mu \text{ or } \eta,\\
&\mathfrak{r}_{\alpha_1,1}(\chi) := \chi[2] \otimes \chi, && \chi = \mu \text{ or } \eta,\\
&\mathfrak{r}_{\alpha_1,2}(\eta) := \eta[3] \otimes \eta[1].
\end{aligned}
$$

(2) *Both $I_2^G(\chi[\lambda] \otimes \eta_u\delta^{G_1})$ and $I_2^G(\chi[\lambda] \otimes \eta_u(\det))$ are irreducible outside the points W_{M_2}-conjugate to one of the following:*

$$
\begin{aligned}
&\mathfrak{r}_{\alpha_2,0}(\chi,\eta) = \chi \otimes \eta[1], && \chi = \mu \text{or } \eta,\\
&\mathfrak{r}_{\alpha_2,1}(\eta',\eta) = \eta'[1] \otimes \eta[1], && \eta' \text{ may be } \eta,\\
&\mathfrak{r}_{\alpha_2,3}(\eta) = \mathfrak{r}_{\alpha_1,2}(\eta).
\end{aligned}
$$

(3) *Both $I_1^{G'}(\chi[\lambda]\delta^{H_2} \otimes \eta_u)$ and $I_1^{G'}(\chi(\det)[\lambda] \otimes \eta_u)$ are irreducible outside the points W_{M_1}-conjugate to one of the following:*

$$
\begin{aligned}
&\mathfrak{r}_{\alpha_1,0}(\chi) = \chi[1] \otimes \chi[-1] \otimes \eta_u, && \chi = \mu \text{ or } \eta',\\
&\mathfrak{r}_{\alpha_1,1}(\chi) = \chi[2] \otimes \chi \otimes \eta_u, && \chi = \mu \text{ or } \eta,\\
&\mathfrak{r}_{\alpha_1,2}(\mu) = \mu[3] \otimes \mu[1] \otimes \eta_u,\\
&\mathfrak{r}_{\alpha_1,3} = \eta[4] \otimes \eta[2] \otimes \eta_u.
\end{aligned}
$$

(4) *Both $I_2^{G'}(\chi[\lambda] \otimes \eta_u\delta^{G_1'})$ and $I_2^{G'}(\chi[\lambda] \otimes \eta_u(\det))$ are irreducible outside the points W_{M_2}-conjugate to one of the following:*

$$
\begin{aligned}
&\mathfrak{r}_{\alpha_2,0}(\chi) = \chi \otimes \eta[2] \otimes \eta_u, && \chi = \mu \text{ or } \eta',\\
&\mathfrak{r}_{\alpha_2,1}(\mu) = \mu[1] \otimes \eta[2] \otimes \eta_u,\\
&\mathfrak{r}_{\alpha_2,2} = \eta[2] \otimes \eta[2] \otimes \eta_u,\\
&\mathfrak{r}_{\alpha_2,4} = \mathfrak{r}_{\alpha_1,3}.
\end{aligned}
$$

(5) *Both $I_2^{G'}(\chi[\lambda] \otimes \delta(\mu,\eta))$ and $I_2^{G'}(\chi[\lambda] \otimes \pi_{\mathrm{nt}}^1(\mu,\eta))$ are irreducible outside the points W_{M_2}-conjugate to one of the following:*

$$
\begin{aligned}
&\mathfrak{r}_{2\alpha_2,0}(\chi,\mu) = \chi \otimes \mu[1] \otimes \eta_u, && \chi = \mu' \text{ or } \eta',\\
&\mathfrak{r}_{2\alpha_2,1}(\mu',\mu) = \mu'[1] \otimes \mu[1] \otimes \eta_u, && \mu' \text{ may be } \mu,\\
&\mathfrak{r}_{2\alpha_2,2}(\mu) = \eta[2] \otimes \mu[1] \otimes \eta_u,\\
&\mathfrak{r}_{2\alpha_2,3}(\mu) = \mathfrak{r}_{\alpha_1,2}(\mu).
\end{aligned}
$$

The real part of these reducible points are illustrated as follows.

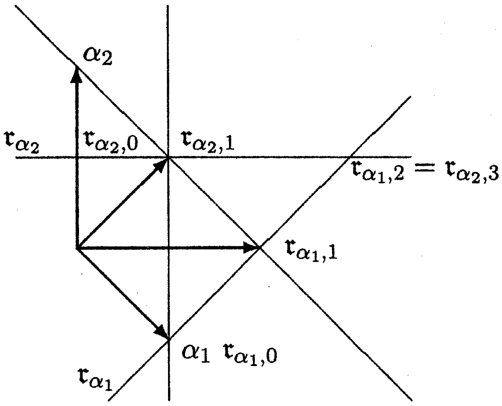

Fig.1 $\mathrm{Re}(\mathfrak{t}_\alpha)$'s for G_2

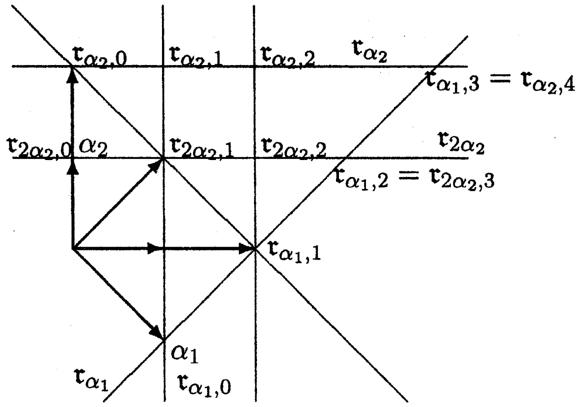

Fig.2 $\mathrm{Re}(\mathfrak{t}_\alpha)$'s for G_2'

A formula for Jacquet modules [T] at each reducible points combined with Langlands classification enables us to calculate the irreducible constituents of $I_P(\underline{\chi};\nu)$ (resp. $I_{P'}(\underline{\chi}\otimes \eta_u;\nu)$).

Write $J_i^G(\pi)$ for the Langlands quotient of $I_i^G(\pi)$. $i_j^G(\pi)$ denotes the image of $I_j^G(\pi)$ in the Grothendieck group $K\Pi(G(F))$.

Theorem 2.5. *Suppose that $I_0^G(\pi;s)$ with $\pi \in \Pi_{\mathrm{unit}}(M_0(F))$, $s \in \mathbb{R}_{\geq 0}$ has more than two irreducible constituents. Then its irreducible constituents are given by the following.*

(A) *First we consider the reducible points which is regular, that is, $\mathfrak{t}_{\alpha_1,2}(\eta)$ for G, and $\mathfrak{t}_{\alpha_1,3}$, $\mathfrak{t}_{\alpha_1,2}(\mu)$ and $\mathfrak{t}_{\alpha_2,1}(\mu)$ for G'.*

$$i_0^G(\eta[3]\otimes\eta[1]) = \eta_u(\det)\delta^{G_2} + J_1^G(\eta(\det)\delta^{H_2}[2]) + J_2^G(\eta[3]\otimes\eta_u(\det)\delta^{G_1}) + \eta_u(\det),$$

$$\begin{aligned} i_0^{G'}(\eta[4]\otimes\eta[2]\otimes\eta_u) &= \eta_u(\det)\delta^{G_2'} + J_1^{G'}(\eta(\det)\delta^{H_2}[3]\otimes\eta_u) \\ &\quad + J_2^{G'}(\eta[4]\otimes\eta_u(\det)\delta^{G_1'}) + \eta_u(\det), \end{aligned}$$

$$\begin{aligned} i_0^{G'}(\mu[3]\otimes\mu[1]\otimes\eta_u) &= \eta_u\delta_0^{G_1'}(\mu) + J_1^{G'}(\mu(\det)\delta^{H_2}[1]\otimes\eta_u) \\ &\quad + J_2^{G'}(\mu[3]\otimes\delta^1(\mu,\eta)) + J_0^{G'}(\mu[3]\otimes\mu[1]\otimes\eta_u), \end{aligned}$$

$$\begin{aligned} i_0^{G'}(\eta[2]\otimes\mu[1]\otimes\eta_u) &= \delta_0^{G'}(\mu,\eta) + J_2^{G'}(\eta[2]\otimes\delta^1(\mu,\eta)) \\ &\quad + J_2^{G'}(\mu[1]\otimes\eta_u\delta^{G_1'}) + J_0^{G'}(\eta[2]\otimes\mu[1]\otimes\eta_u). \end{aligned}$$

Here $\delta_0^{G'}(\mu)$ is the unique square integrable constituent of $i_0^{G'}(\mu[3]\otimes\mu[1]\otimes\mathbf{1})$ and $\delta_0^{G'}(\mu,\eta)$ denotes the unique square integrable constituent of $i_0^{G'}(\mu[1]\otimes\eta[2]\otimes\eta_u)$.

There two other types of reducible points where the generalized principal series contains a square integrable constituent.

(B) *The first case occurs only for G at $\mathfrak{t}_{\alpha_1,1}(\mu)$.*

$$i_0^G(\mu[2]\otimes\mu) = i_2^G(\mu[2]\otimes\tau^1(\mu)_+) \oplus i_2^G(\mu[2]\otimes\tau^1(\mu)_-)$$
$$i_2^G(\mu[2]\otimes\tau^1(\mu)_\pm) = \delta_0^G(\mu)_\pm + J_1^G(\mu(\det)\delta^{H_2}[1]) + J_2^G(\mu[2]\otimes\tau^1(\mu)_\pm),$$

where $\delta_0^G(\mu)_\pm$ are the square-integrable constituents. Note that $J_1^G(\mu(\det)\delta^{H_2}[1])$ has the multiplicity two in $i_0^G(\mu[2]\otimes\mu)$.

(C) *The second case, they consist of* $\mathfrak{r}_{\alpha_2,1}(\eta',\eta)$ *for* G *and* $\mathfrak{r}_{2\alpha_2,1}(\mu',\mu)$ *for* G', *where* $\eta'\neq\eta$ *and* $\mu'\neq\mu$.

$$i_0^G(\eta'[1]\otimes\eta[1]) = \delta_0(\eta',\eta) + J_2^G(\eta'[1]\otimes\eta_u(\det)\delta^{G_1}) + J_2^G(\eta[1]\otimes\eta'_u(\det)\delta^{G_1}) + J_1^G(I_0^{H_2}(\eta\otimes\eta')[1]),$$

$$i_0^{G'}(\mu'[1]\otimes\mu[1]\otimes\eta_u) = \delta_0(\mu,\mu';\eta) + J_2^{G'}(\mu'[1]\otimes\delta^1(\mu,\eta)) + J_2^{G'}(\mu[1]\otimes\delta^1(\mu',\eta)) + J_1^{G'}(i_0^{H_2}(\mu'\otimes\mu)[1]\otimes\eta_u),$$

where $\delta_0(\eta',\eta)=\delta_0(\eta,\eta')$ *and* $\delta_0(\mu,\mu';\eta)=\delta_0(\mu',\mu;\eta)$ *are the unique square integrable constituent, respectively.*

Next we treat the rest reducible where the generalized principal series contains some tempered constituents. These fall into two patterns.

(D) *First we consider* $\mathfrak{r}_{\alpha_2,0}(\mu,\eta)$ *for* G *and* $\mathfrak{r}_{\alpha_2,0}(\eta')$ *and* $\mathfrak{r}_{2\alpha_2,0}(\eta')$ *for* G'.

$$i_0^G(\mu\otimes\eta[1]) = (\tau_0(\mu,\eta)_+\oplus\tau_0(\mu,\eta)_-) + J_2^G(\eta[1]\otimes\tau^1(\mu)_+) + J_2^G(\eta[1]\otimes\tau^1(\mu)_-),$$

$$i_0^{G'}(\eta'\otimes\eta[2]\otimes\eta_u) = \tau_0(\eta',\eta)_+\oplus\tau_0(\eta',\eta)_- + J_2^{G'}(\eta[2]\otimes\tau_0^1(\eta';\eta)_+) + J_2^{G'}(\eta[2]\otimes\tau_0^1(\eta';\eta)_-),$$

$$i_0^{G'}(\eta'\otimes\mu[1]\otimes\eta_u) = (\tau(\eta',\delta^1(\mu,\eta))_+\oplus\tau(\eta',\delta^1(\mu,\eta))_-) + J_2^{G'}(\mu[1]\otimes\tau^1(\eta';\eta)_+) + J_2^{G'}(\mu[1]\otimes\tau^1(\eta';\eta)_-),$$

where $\tau_0(\mu,\eta)_\pm\in\Pi_{\mathrm{temp}}(G(F))$ *and* $\tau_0(\eta',\eta)_\pm$, $\tau(\eta',\delta^1(\mu,\eta))_\pm\in\Pi_{\mathrm{temp}}(G'(F))$.

(E) *At* $\mathfrak{r}_{\alpha_2,1}(\eta)$ *for* G *and* $\mathfrak{r}_{\alpha_1,0}$ *and* $\mathfrak{r}_{\alpha_1,1}$ *for* G'.

$$i_0^G(\eta[1]\otimes\eta[1]) = \eta_u\tau(\delta^{H_2}) + J_2^G(\eta[1]\otimes\eta_u(\det)\delta^{G_1}) + J_1^G(i_0^{H_2}(\eta\otimes\eta)[1]) + \eta_u\tau(\mathbf{1}_{H_2}),$$

$$i_0^{G'}(\mu[1]\otimes\mu[1]\otimes\eta_u) = \tau(\delta^1(\mu,\eta))\oplus\tau(\pi_{\mathrm{nt}}^1(\mu,\eta)) + J_2^{G'}(\mu[1]\otimes\delta^1(\mu,\eta)) + J_1^{G'}(i_0^{H_2}(\mu\otimes\mu)[1]\otimes\eta_u),$$

$$i_0^G(\eta[2]\otimes\eta\otimes\eta_u) = \eta_u(\tau'(\delta^{H_2}))\oplus\eta_u(\tau'(\mathbf{1}_{H_2})) + J_1^{G'}(\eta\delta^{H_2}[1]\otimes\eta_u) + J_2^{G'}(\eta[2]\otimes i_0^{G'_1}(\eta\otimes\eta_u)),$$

where $\eta_u\tau(\delta^{H_2})$ *and* $\eta_u\tau(\mathbf{1}_{H_2})\in\Pi_{\mathrm{temp}}(G(F))$, *and* $\tau(\delta^1(\mu,\eta))$, $\tau(\pi_{\mathrm{nt}}^1(\mu,\eta))$, $\eta_u(\tau'(\delta^{H_2}))$ *and* $\eta_u(\tau'(\mathbf{1}_{H_2}))\in\Pi_{\mathrm{temp}}(G'(F))$.

3. Base change problems and the result of Goldberg

3.1. **Base change problems for unitary groups.** We first review some definitions from [KS]. We always work over a fixed non-archimedean local field F of characteristic zero. A twisted endoscopy problem is considered for a triple $(G,\theta,\mathbf{a})$ where G is a connected reductive group defined over F, θ is a quasi-semisimple F-automorphism of G (i.e. its restriction to Lie $(G)_{\mathrm{der}}$ is semisimple) and $\mathbf{a}$ is a class in $H^1(W_F,Z(\widehat{G}))$. For convenience we fix a splitting $\mathbf{spl}_G:=(\mathbf{B},\mathbf{T},\{X_\alpha\})$ of G and an L-group datum $(\widehat{G},\rho_G,\eta_G)$ where $\widehat{G}$ is the dual group of G, ρ_G is an L-action of Γ on $\widehat{G}$ and η_G is a Γ-bijection between canonical based root data. We fix a splitting $\mathbf{spl}_{\widehat{G}}:=(\mathcal{B},\mathcal{T},\{\mathcal{X}\})$ of $\widehat{G}$ which is fixed by the Γ_F-action ρ_G. The dual of the inner class of θ determines an automorphism of the based root datum of $\widehat{G}$. This lifts to an automorphism $\widehat{\theta}$ of $\widehat{G}$ which preserves $\mathbf{spl}_{\widehat{G}}$.

Recall that a quadruple $(H,\mathcal{H},s,\xi)$ is an *endoscopic datum* for $(G,\theta,\mathbf{a})$ if

(1) H is a quasi-split group over F. We fix an L-group datum $(\widehat{H},\rho_H,\eta_H)$ for H.

(2) $\mathcal{H}$ is a split extension

$$1 \to \widehat{H} \to \mathcal{H} \xrightarrow{\pi} W_F \to 1$$

Thus we have a splitting, that is, an injective homomorphism $\iota : W_F \hookrightarrow \mathcal{H}$ satisfying $\pi \circ \iota = \mathrm{id}_{W_F}$. We impose that the inner class of $\mathrm{Ad}(\iota(w))|_{\widehat{H}}$ coincides with that of $\rho_H(w)$ for any $w \in W_F$.

(3) s is a $\widehat{\theta}$-semisimple element in $\widehat{G}$. That is, $\mathrm{Ad}(s) \circ \widehat{\theta}$ is a quasi-semisimple automorphism of $\widehat{G}$.

(4) $\xi : \mathcal{H} \hookrightarrow {}^L G$ is an L-embedding satisfying
  (4a) $\mathrm{Ad}(s) \circ \widehat{\theta} \circ \xi = a' \cdot \xi$, for some $a' \in \mathbf{a}$.
  (4b) $\xi(\widehat{H}) = (\widehat{G}^{\mathrm{Ad}(s)\widehat{\theta}})^0$.

An endoscopic datum $(H, \mathcal{H}, s, \xi)$ is *elliptic* if $\xi(Z(\widehat{H})^{\Gamma_F})^0 \subset Z(\widehat{G})$. Two elliptic data $(H, \mathcal{H}, s, \xi)$ and $(H', \mathcal{H}', s', \xi')$ are isomorphic if there exists $g \in \widehat{G}$ such that

$$\mathrm{Ad}(g)\xi(\mathcal{H}) = \xi'(\mathcal{H}') \tag{3.1}$$

$$s' \in \mathrm{Ad}_{\widehat{\theta}}(g)s \cdot Z(\widehat{G}). \tag{3.2}$$

We now classify the base change problems for unitary groups. Let G be the quasi-split unitary group in n variables associated to E/F. Put $\widetilde{G} := \mathrm{Res}_{E/F}G$. $\widetilde{\sigma}$ denotes the F-automorphism of $\widetilde{G}$ associated to σ by the F-structure of G. Then $\widetilde{G}^\wedge \simeq \widehat{G}^2$ and we may choose $\widetilde{\sigma}^\wedge$ to be $\widetilde{\sigma}^\wedge(x, y) = (y, x)$. Since $Z(\widetilde{G}^\wedge) \rtimes_{\rho_{\widetilde{G}}} W_F \simeq {}^L(\mathrm{Res}_{E/F}G'_0)$, we may identify each class $\mathbf{a} \in H^1(W_F, Z(\widetilde{G}^\wedge))$ with the $(\mathrm{Res}_{E/F}G'_0)^\wedge$-conjugacy class of Langlands parameters attached to some $\chi \in \Pi(E^\times)$ by the Langlands correspondence for tori. We write this as $\mathbf{a} = \mathbf{a}_\chi$.

Suppose that $(H, \mathcal{H}, 1, \xi)$ is an endoscopic datum for $(\widetilde{G}, \widetilde{\sigma}, \mathbf{a}_\chi)$.

First note that the quasi-split group H must be G itself. We identify $\widehat{H}$ and $\mathcal{H}$ with its image under $\xi : \mathcal{H} \hookrightarrow {}^L\widetilde{G}$. The definition of the endoscopy (4*b*) implies $\widehat{H} = (\widetilde{G}^\wedge)^{\widetilde{\sigma}^\wedge} = \{(g, g) \,|\, g \in \widehat{G}\} = \widehat{G}$. ι in (2) can be written as

$$\iota(w) = a_\iota(w) \rtimes_{\rho_{\widetilde{G}}} w, \qquad w \in W_F$$

for some $\widetilde{G}^\wedge$-valued 1-cocycle $\{a_\iota(w)\}$ satisfying $\mathrm{Ad}(a_\iota(w)) \circ \rho_{\widetilde{G}}(w)\widehat{H} = \widehat{H}$. Since our $\widehat{H}$ is preserved by $\rho_{\widetilde{G}}(w)$, we have $a_\iota(w) \in \mathrm{Norm}(\widehat{H}, \widetilde{G}^\wedge) = \widehat{H}Z(\widetilde{G}^\wedge)$ and the inner class $\rho_H(w)$ of $\mathrm{Ad}(\iota(w))$ coincides with $\rho_{\widetilde{G}}(w)|_{\widehat{H}}$. Hence $H = G$.

Next analize $\mathcal{H} = \widehat{H}\iota(W_F)$. Fix once for all $w_\sigma \in W_F \setminus W_E$. For $\chi \in \Pi(E^\times)$, the character of W_E corresponding to it by Langlands' version of classfield theory is denoted by the same symbol χ. Then the 1-cocycle a_χ given by

$$a_\chi(w) = \begin{cases} (\chi(w), \sigma(\chi)^{-1}(w)) & \text{if } w \in W_E \\ (\chi(w_\sigma^2), 1) & \text{if } w = w_\sigma \end{cases}$$

belongs to $\mathbf{a}_\chi$. Since a' in (4*a*) belongs to the same class $\mathbf{a}$, it must be of the form

$$a'(w) = (x, y)a_\chi(w)\rho_{\widetilde{G}}(w)(x^{-1}, y^{-1}) = \begin{cases} (\chi(w), \sigma(\chi)^{-1}(w)) & \text{if } w \in W_E \\ (xy\chi(w_\sigma^2), yx) & \text{if } w = w_\sigma \end{cases}$$

for some $(x, y) \in Z(\widetilde{G}^\wedge)$. That is,

$$a'(w) = \begin{cases} (\chi(w), \sigma(\chi)^{-1}(w)) & \text{if } w \in W_E \\ (z'\chi(w_\sigma^2), z') & \text{if } w = w_\sigma, \end{cases}$$

for some $z' \in Z(\widehat{G})$. Writing $a_\iota(w) \in \widehat{G}Z(\widetilde{G}^\wedge)$ as $a_\iota(w) = (x(w), x(w)z(w))$, $(x(w) \in \widehat{G}, z(w) \in Z(\widehat{G}))$, the condition $(4a)$ becomes

$$(x(w)z(w), x(w)) = \begin{cases} (\chi(w)x(w), \sigma(\chi)^{-1}(w)x(w)z(w)) & \text{if } w \in W_E \\ (z'\chi(w_\sigma^2)x(w_\sigma), z'x(w_\sigma)z(w_\sigma)) & \text{if } w = w_\sigma, \end{cases}$$

or equivalently,

$$\chi(w) = \sigma(\chi)(w) = z(w), \quad \forall w \in W_E,$$
$$z(w_\sigma) = z'\chi(w_\sigma^2), \quad z'z(w_\sigma) = 1.$$

In particular, we have no endoscopic data with $s = 1$ unless $\chi(w) = \sigma(\chi)(w)$, $w \in W_E$. If $\chi(w) = \sigma(\chi)(w)$, $(4a)$ is equivalent to

(3.3) $\qquad z|_{W_E} = \chi \quad \text{for } \forall w \in W_E, \quad z(w_\sigma)^2 = \chi(w_\sigma^2), \quad z' = z(w_\sigma)^{-1}$

Since a_ι is a 1-cocycle, $x|_{W_E}$ and $z|_{W_E}$ are homomorphisms and

$$a_\iota(w_\sigma^2) = a_\iota(w_\sigma)\rho_{\widetilde{G}}(w_\sigma)(a_\iota(w_\sigma))$$

gives $z(w_\sigma^2) = z(w_\sigma)^2$. Regarding this, (3.3) is equivalent to

(i) $x|_{W_E}$ and $z|_{W_E} = \chi$ are homomorphism,
(ii) $z(w_\sigma^2) = z(w_\sigma)^2 = \chi(w_\sigma^2)$, $z' = z(w_\sigma)^{-1}$.

Recall that the base change problem for G is, by definition, the twisted endoscopy problem for the triple $(\widetilde{G}, \widetilde{\sigma}, 1)$. More precisely the base change problems for G is the endoscopy problems attached to the endoscopic data of the form $(H, \mathcal{H}, 1, \xi)$ for $(\widetilde{G}, \widetilde{\sigma}, 1)$. Thus we suppose $\chi = 1$. Then (i) and (ii) become

(i)$'$ $z|_{W_E} = 1$ and $x|_{W_E}$ is homomorphism,
(ii)$'$ $z(w_\sigma^2) = z(w_\sigma)^2 = 1$, $z' = z(w_\sigma)^{-1}$.

By (ii)$'$, we have $z(w_\sigma) = 1$ or -1 and

$$a_\iota(w) = \begin{cases} (x(w), x(w)) & \text{if } w \in W_E, \\ (x(w), x(w)) & \text{if } w = w_\sigma,\ z(w_\sigma) = 1, \\ (x(w), -x(w)) & \text{if } w = w_\sigma,\ z(w_\sigma) = -1. \end{cases}$$

Now granting (3.1), we see that if $z(w_\sigma) = 1$ $(H, \mathcal{H}, s, \xi)$ is isomorphic to $(G, {}^LG, 1, \xi)$ with

$$\xi : {}^LG \ni g \rtimes_{\rho_{\widetilde{G}}} w \mapsto (g, g) \rtimes_{\rho_{\widetilde{G}}} w \in {}^L\widetilde{G},$$

and if $z(w_\sigma) = -1$ it is isomorphic to $(G, {}^LG, 1, \xi')$ with

$$\xi' : {}^LG \ni g \rtimes_{\rho_{\widetilde{G}}} w \mapsto \begin{cases} (g, g) \rtimes_{\rho_{\widetilde{G}}} w \in {}^L\widetilde{G} & \text{if } w \in W_E \\ (g, -g) \rtimes_{\rho_{\widetilde{G}}} w \in {}^L\widetilde{G} & \text{otherwise.} \end{cases}$$

Thus we recover the following.

Proposition 3.1. (*Rogawski* [R]) *Up to isomorphism, the base change problem for G is the endoscopic liftings from endoscopic data $(G, {}^LG, 1, \xi)$ and $(G, {}^LG, 1, \xi')$ for $(\widetilde{G}, \widetilde{\sigma}, 1)$ to $\widetilde{G}$. Here*

$$\xi : {}^LG \ni g \rtimes_{\rho_{\widetilde{G}}} w \mapsto (g, g) \rtimes_{\rho_{\widetilde{G}}} w \in {}^L\widetilde{G}$$

$$\xi' : {}^LG \ni g \rtimes_{\rho_{\widetilde{G}}} w \mapsto \begin{cases} (g, g) \rtimes_{\rho_{\widetilde{G}}} w \in {}^L\widetilde{G} & \text{if } w \in W_E \\ (g, -g) \rtimes_{\rho_{\widetilde{G}}} w \in {}^L\widetilde{G} & \text{otherwise.} \end{cases}$$

We call the former the standard base change and the latter the twisted base change, respectively.

3.2. **The result of Goldberg.** Now we review the result of Goldberg about the irreducible constituents of $I_{P_1}(\pi[\nu])$ $\pi \in \Pi_0(M_1(F))$, $\nu \in \mathfrak{a}^*_{M_1}$. $\widetilde{\pi}$ denotes the contragredient of π. From [G] and [Sh, Th.8.1], the result is summarized as follows.

Proposition 3.2. (*Goldberg*) *Let $G = G_2$, $G' = G'_2$ and $\pi \in \Pi_0(H_2(F))$.*

(1) *$I_1^G(\pi[\nu])$ and $I_1^{G'}(\pi[\nu] \otimes \eta_u)$ are irreducible unless $\sigma(\widetilde{\pi}) \simeq \pi$.*
(2) *If $\sigma(\widetilde{\pi}) \simeq \pi$, there are the following two cases.*
   (a) *Suppose that $\pi, \pi' \in \Pi_0(M_1(F))$ are the twisted and standard base change lifts of some irreducible supercuspidal representations of $G_1(F)$, respectively. Then $I_1^G(\pi[\nu])$ and $I_1^{G'}(\pi'[\nu] \otimes \eta_u)$ are reducible only at $\nu = \pm 1$. Each induced representation has only two irreducible constituents, a square integrable representation and the Langlands quotient.*
   (b) *Suppose that π and π' are the standard and twisted base change lifts of some irreducible supercuspidal representation of $G_1(F)$, respectively. Then $I_1^G(\pi[\nu])$ and $I_1^{G'}(\pi[\nu] \otimes \eta_u)$ are reducible only at $\nu = 0$, each of them decomposes into the direct sum of two irreducible tempered representations.*

4. Irreducible representations supported on P_2

4.1. **Product L-factor for $G \times H_m$.** Let $G := G_n$ or G'_n and $\mathbf{G} := G_{m+n}$ or G'_{m+n}, respectively. $P = MU$ denotes the standard parabolic subgroup of $\mathbf{G}$ such that $M \simeq H_m \times G$. Take $\chi \in \Pi(H_m(F))$ and $\tau \in \Pi(G(F))$ and consider the parabolically induced representation

$$I_P^{\mathbf{G}}(\pi; s) := \mathrm{ind}_{P(F)}^{\mathbf{G}(F)}[(|\det|_E^{s/2}\chi \otimes \tau) \otimes \mathbf{1}_{U(F)}], \quad \pi = \chi \otimes \tau,$$

and the intertwining operator $M(w, \pi; s) : I_P^{\mathbf{G}}(\pi; s) \to I_P^{\mathbf{G}}(w(\pi); -s)$. Here w denotes the unique non-trivial element in W_M.

Write St_n for the standard representation of $GL(n, \mathbb{C})$ and $\widetilde{\mathrm{St}}_n$ for its dual. Let $r_{m,n}$ be the representation of ${}^LM$ defined by

$$r_{m,n}|_{\widehat{M}} = [\widetilde{\mathrm{St}}_\ell \otimes (\mathrm{St}_m \otimes \mathbf{1}_{GL(m)})] \oplus [\mathrm{St}_\ell \otimes (\mathbf{1}_{GL(m)} \otimes \mathrm{St}_m)],$$

$$r_{m,n}(w)(v_1 \oplus v_2) = \begin{cases} v_1 \oplus v_2 & \text{if } w \in W_E, \\ v_2 \oplus v_1 & \text{otherwise,} \end{cases}$$

where $\ell = 2n$ or $2n + 1$ according to $G = G_{m+n}$ or G'_{m+n}. Also let r_{Asai} be the twisted tensor representation of ${}^LH_m$:

$$r_{\mathrm{Asai}}|_{\widehat{H}_m} = \mathrm{St}_m \otimes \mathrm{St}_m, \quad r_{\mathrm{Asai}}(w)(v_1 \otimes v_2) = \begin{cases} v_1 \otimes v_2 & \text{if } w \in W_E, \\ v_2 \otimes v_1 & \text{otherwise.} \end{cases}$$

We view this as a representation of ${}^L M$ trivial on $\widehat{G}_n$ or $\widehat{G}'_n$.

Suppose χ and τ are generic for some non-degenerate characters of $\mathbf{U}^{H_m}(F)$ and of $\mathbf{U}^G(F)$. Then Shahidi defined the automorphic L and ε-factors attached to $r_{n,m}$ and r_{Asai} [Sh, §7]:

$$\begin{aligned} L(s,\tau\times\chi) &= L(s,\pi,r_{m,n}), \quad \varepsilon(s,\tau\times\chi,\psi)=\varepsilon(s,\pi,r_{m,n},\psi) \\ L_{\mathrm{Asai}}(s,\chi) &= L(s,\pi,r_{\mathrm{Asai}}), \quad \varepsilon_{\mathrm{Asai}}(s,\chi,\psi)=\varepsilon(s,\pi,r_{\mathrm{Asai}},\psi). \end{aligned}$$

Here ψ is a fixed non-trivial character of F. Moreover setting

$$r(w,\pi;s) := \frac{L(s,\tau\times\chi)}{\varepsilon(s,\tau\times\chi,\psi)L(s+1,\tau\times\chi)}\frac{L_{\mathrm{Asai}}(2s,\chi)}{\varepsilon_{\mathrm{Asai}}(2s,\chi,\psi)L_{\mathrm{Asai}}(2s+1,\chi)},$$

he showed that the normalized intertwining operator

$$N(w,\pi;s) := r(w,\pi;s)^{-1}M(w,\pi;s)$$

is holomorphic on $\{s\in\mathbb{C}\,|\,\mathrm{Re}(s)\geq 0\}$ [Sh, Prop.7.3, Th.7.9]. Since the reducibility of $I_P^{\mathbf{G}}(\pi;s)$ is controlled by the poles of $M(w,\pi;s)$ [Sh, Th.8.1], we have to calculate the poles of $L(s,\tau\times\chi)$ and $L_{\mathrm{Asai}}(2s,\chi)$. Since $L_{\mathrm{Asai}}(2s,\chi)$ is treated in [G, §5], we concentrate on $L(s,\tau\times\chi)$.

4.2. **Application of the base change.** We now turn to the case where $m=n=1$. Then we have the standard base change lift attached to $(G,{}^L G,1,\xi_1)$. For each $\tau\in\Pi_0(G)$, we write $\xi_1(\tau)$ for the base change lift of the unique tempered L-packet containing τ. Define

$$\begin{aligned} L_{\mathrm{BC}}(s,\tau\times\chi) &:= L(s,\widetilde{\xi(\tau)}\times\chi), \\ \varepsilon_{\mathrm{BC}}(s,\tau\times\chi,\psi) &:= \lambda(E/F,\psi)^{\ell m}\varepsilon(s,\widetilde{\xi(\tau)}\times\chi,\psi\circ\mathrm{Tr}_{E/F}). \end{aligned}$$

Here the factors on the right hand side are the Rankin product factors [JPSS]. Then by some local-global argument we can prove:

Proposition 4.1. *Suppose that $\tau\in\Pi_0(G_1(F))$ or $\Pi_0(G'_1(F))$ and $\chi\in\Pi(H_1(F))$ are generic representations. Then the two product L and ε-factors defined above coincide:*

$$L(s,\tau\times\chi)=L_{\mathrm{BC}}(s,\tau\times\chi), \quad \varepsilon(s,\tau\times\chi,\psi)=\varepsilon_{\mathrm{BC}}(s,\tau\times\chi,\psi).$$

4.3. **Reducible points.** Any supercuspidal representation of $G_1(F)$ is generic, but there exists a non-generic representation τ' in $\Pi_0(G'_1(F))$. We have a tempered L-packet T which contain τ' [R]. By [FGJR], T contains a unique generic representation τ. The result of base change [R] yields that the Plancherel measures $\mu(\chi\otimes\tau,w)$ have a same value for any τ in same L-packet. Thus from Prop.4.1 we have

$$L(s,\tau'\times\chi)=L_{\mathrm{BC}}(s,\tau\times\chi), \quad \varepsilon(s,\tau'\times\chi,\psi)=\varepsilon_{\mathrm{BC}}(s,\tau\times\chi,\psi).$$

Using this we can determine the reducibility of $I_{P_2}^G(\pi;s)$. Let $\lambda_\mu : {}^L(U(1)_{E/F}\times U(1)_{E/F})\to{}^L G_1$ and $\lambda'_\mu : {}^L(G_1\times U(1)_{E/F})\to{}^L G'_1$ be the L-embeddings :

$$\lambda_\mu : (z_1,z_2)\rtimes w \longmapsto \begin{cases} \begin{pmatrix} z_1\mu(w)^{-1} & \\ & z_2\mu(w)^{-1}\end{pmatrix}\rtimes w & \text{if } w\in W_E \\ \begin{pmatrix} & -z_1 \\ z_2 & \end{pmatrix}\rtimes w & \text{otherwise} \end{cases}$$

$$\lambda'_\mu : \left(\begin{pmatrix} a & b \\ c & d\end{pmatrix},z\right)\rtimes w \longmapsto \begin{cases} \begin{pmatrix} a\mu(w) & & b\mu(w) \\ & z & \\ c\mu(w) & & d\mu(w)\end{pmatrix}\rtimes w & \text{if } w\in W_E \\ \begin{pmatrix} a & & -b \\ & z & \\ c & & -d\end{pmatrix}\rtimes w & \text{otherwise,} \end{cases}$$

respectively. The associated lifting of $\eta_u \otimes \eta'_u \in \Pi(G'_0(F)^2)$ (resp. $\pi \otimes \eta'_u \in \Pi(G_1(F) \times G'_0(F))$) to an L-packet $\lambda_\mu(\eta, \eta')$ of $G_1(F)$ (resp. $\lambda_\mu(\pi, \eta')$ of $G'_1(F)$) are constructed in [R]. From the above argument and [Sh, Th.8.1], we can deduce the following theorem.

Theorem 4.2. *Let $G = G_2$ or G'_2. $I_2^G(\chi| \, |_E^s \otimes \tau)$ with $\chi \in \Pi_0(E^\times)$, $\tau \in \Pi_0(G_1(F))$ or $\Pi(G'_1(F))$ and $s \in \mathbb{R}_{\geq 0}$ is irreducible unless the next three cases.*

(1) *Suppose that $\chi = \mu$ and $\tau \notin \lambda_\mu(\eta, \eta')$ if $G = G_2$, and $\chi = \eta$ and $\tau \notin \lambda_\mu(\pi, \eta')$ if $G = G'_2$. Then $I_2^G(\chi| \, |_E^s \otimes \tau)$ is reducible only at $s = 0$. It decomposes into the direct sum of two tempered representations.*

(2) *Suppose that $\chi = \mu\eta^{-1}$ and $\tau \in \lambda_\mu(\eta, \eta')$ if $G = G_2$, and $\chi = \eta$ and $\tau \in \lambda_\mu(\pi, \eta')$ where η may be η' if $G = G'_2$. Then $I_2^G(\chi| \, |_E^s \otimes \tau)$ is reducible only at $s = 1$.*

$$i_2^G(\mu\eta^{-1}| \, |_E \otimes \tau) = \delta_2^{G_2}(\mu^{-1}\eta, \mu^{-1}\eta') + J_2^G(\mu\eta^{-1}| \, |_E \otimes \lambda_\mu(\eta, \eta')),$$

where $\delta_2^{G_2}(\mu^{-1}\eta, \mu^{-1}\eta') \in \Pi_2(G(F))$.

$$i_2^{G'_2}(\eta| \, | \otimes \tau) = \delta_2^{G'_2}(\eta, \tau) + J_2^{G'_2}(\eta| \, | \otimes \tau).$$

Here $\delta_2^{G'_2}(\eta, \tau) \in \Pi_2(G'_2(F))$.

(3) *Suppose $\chi = \eta$ if $G = G_2$, and $\chi = \mu$ if $G = G'_2$. Then $I(\chi| \, |_E^s \otimes \tau)$ is reducible only at $s = \frac{1}{2}$. It has two irreducible constituents, its Langlands quotient and square integrable representation.*

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