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A Short Survey on Coarse Topology

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1. Introduction

The purpose of this article is to present a short introduction to the coarse (or asymptotic) topology (as a part of the coarse geometry) for persons who are interested in this field. The exposition is a short summary of [1] and related papers, and readers are recommended to refer directly to the referenced papers.

The main object of coarse geometry is proper metric spaces. Typical examples in mind are (i) open Riemannian manifolds (for instance, the universal coverings of compact Riemannian manifolds) and (ii) finitely generated groups with word-length metrics. Here the view point is that we study a proper metric space X in larger and larger scales. We look at X in a scale R so that every part of scale less than R is ignored. Next we make the scale R larger and larger. In this process some global geometric properties of X keep remaining. These properties are the main objects to be studied in the coarse geometry. Hence global sturctures and end structures have important roles, while most local structures are ignored.

To develop coarse geometry, first we have to clarify the notion of coarse equivalence of proper metric spaces. This is accomplished by formulating appropriate categories: Coarse category [3, 4] and Asymptotic category [1] (§.2.1). In these categories we have the basic notions of subobjects, asymptotic neighborhoods, products, cones, etc. and homotopy (equivalence) (§.2.2). After this frame work, the theory can be developed in the similar way as in the usual topological category (of compact metric spaces): (1) Coarse (or Asymptotic) general topology (§.3): coarse dimension theory, coarse ANE theory, polyhedral approximations and Higson compactification, (2) Coarse algebraic topology (§.4): coarse Čech, Steenrod (co)homology, coarse (Top, C^*-alg) K-theory etc.

The term "Asymptotic topology" is used in [1] to include these topics. More complehensive survey on the coarse geometry should include further topics: Geometric group theory, Index theory for open Riemannian manifolds, Riemannian metrics of positive scalar curvature, etc. For these topics see the references in the cited papers [1, 2, 3, 4, 5].

These theories are intended to have some applications to topological or geometric problems on compact manifolds M related to $\pi_1(M)$. The main problem is the Novikov conjecture:

(1) **Novikov conjecture on the higher signatures**: Suppose M^{4k} is a closed oriented $4k$-manifold. The Hirzebruch signature formula represents the signature of M as $\sigma(M) = \langle L, [M] \rangle$. For $\Gamma = \pi_1(M)$ let $B\Gamma$ denote the classifying space of Γ and $f : M \to B\Gamma$ be the classifying map. Let $\alpha \in H^*(B\Gamma, \mathbb{Q})$. The higher signature of M associated to α is defined by $\langle L \cup f^*(\alpha), [M] \rangle$. The Novikov conjecture asserts that this higher signature is an oriented homotopy invariant of M.

(2) **Gromov-Lawson-Rosenberg conjecture**: If M is a compact aspherical manifold, then M cannot admit any Riemannian metric of positive scalor curvature.

(3) **Gromov's zero-in-the spectrum conjecture**: Suppose M is a Riemannian manifold which is uniformly contractible and has bounded geometry. Then the spectrum of Laplacian on space of L^2-forms of M contains 0.

These problems have their coarse versions and Higson compactification can be used to translate the coarse problems into some topological problems in the usual category of compact Hausdorff spaces. The descent principle asserts that the coarse version implies the original conjecture.

For these problems, G. Yu [5] has obtained the following results:

Theorem. (1) Suppose Γ is a finitely generated group with a word-length metric and such that $B\Gamma$ is a finite complex. If Γ has finite asymptotic dimension, then the Novikov conjecture and Gromov-Lawson-Rosenberg conjecture hold for Γ.
(2) Suppose M is a Riemannian manifold which is uniformly contractible and has finite asymptotic dimension, then (i) M cannot have uniform positive scalor curvature and (ii) the Gromov's zero-in-the spectrum conjecture holds for M.

2. Coarse category

Below all spaces are proper metric spaces.

2.1. The coarse category.

Definition. (1) A map $f : X \to Y$ is said to be a coarse map if it is (i) coarsely proper ($f^{-1}(B)$ is a bounded set in X for each bounded subset B of Y) and (ii) coarsely uniform (for each $R > 0$ there is $S > 0$ such that if $x, x' \in X$ and $d_X(x, x') \leq R$, then $d_Y(f(x), f(x')) \leq S$).
(2) Two coarse maps $f_0, f_1 : X \to Y$ are said to be coarsely equivalent and written as $f_0 \sim f_1$ if there is a constant $K > 0$ such that $d_Y(f_0(x), f_1(x)) \leq K$ for any $x \in X$. A coarse map $f : X \to Y$ is a coarse equivalence if there exists a coarse map $g : Y \to X$

such that $f \circ g \sim 1_Y$ and $g \circ f \sim 1_X$.
(3) The coarse category $\mathcal{C}$ is the category consisting of proper metric spaces and coarse maps (up to coarse equivalence).

Example. (1) If the Gromov–Hausdorff distance of X and Y is finite, then X and Y are coarsely equivalent.
(2) Suppose M is a compact Riemannian manifold, $\tilde{M}$ is the universal cover of M with the metric induced from M and $\Gamma = \pi_1(X)$ is equipped with the word-length metric. Then for any fixed point $p \in \widetilde{M}$, the map $\Gamma \to \widetilde{M} : \gamma \mapsto \gamma p$ is a coarse equivalence.
(3) For a compact metric space X, we denote its open cone by $\mathcal{O}X$. The coarse geometry of $\mathcal{O}X$ reflects the geometry of X.

2.2. **The asymptotic category.** To develop a reasonable coarse ANE/Dimension theory, it is necessary to restrict ourselves to a subclass of coarse maps which satisfy some sorts of Lipschitz condition. A. N. Dranishnikov introduced two subcategories $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ of $\mathcal{C}$.

Definition. (1) A map $f : X \to Y$ is asymptotically Lipschitz if there exist $\lambda, \mu > 0$ such that $d_Y(f(x), f(x')) \leq \lambda\, d_X(x, x') + \mu$.
(2) The norm $\|f\|$ of a map $f : X \to Y$ with respect to base points $x_0 \in X$ and $y_0 \in Y$ is defined by $\|f\| = \lim\limits_{x\to\infty} \frac{d(f(x),y_0)}{d(x,x_0)}$.
(3) The subcategory $\mathcal{A}$ ($\widetilde{\mathcal{A}}$) consists of proper metric spaces and proper asymptotically Lipschitz maps (with nonzero norms).

When X is a geodesic metric space, every coarse map $f : X \to Y$ is asymptotically Lipschitz.

2.3. **Basic Notions in the asymptotic category.**

Our next task is to formulate the coarse versions of various basic notions in the usual topological category: (1) subobject, asymptotic closed subset, asymptotic neighborhood, etc, (2) product, wedge, quotient, cone, join, etc.

First of all, it is important to note a role of open cones of compact metric spaces. The open cone construction induces a correspondence between the geometry of a compact metric space X and the coarse geometry of its open cone $\mathcal{O}X$, and we have the following table of correspondence between several fundametal objects in the topological category and the asymptotic category. The correct theory should reflect these correspondence:

X	Geometry	1 point	2 point	$[0,1]$	S^n	B^n
$\mathcal{O}X$	Coarse Geoemtry	$\mathbb{R}_+$	$\mathbb{R}$	$\mathbb{R}^2_+$	$\mathbb{R}^{n+1}$	$\mathbb{R}^{n+1}_+$

$\mathbb{R}^n_+$: the closed upper half-space of $\mathbb{R}^n$

As examples, we consider products and cones in $\mathcal{A}$. For convenience we work in the pointed category (slight modifications are necessary in the unpointed case):

The products in $\mathcal{A}$ is defined as follows: for pointed proper metric spaces $((X, x_0), d_X)$, $((Y, y_0), d_Y)$, first consider the Cartesian product $(X \times Y, d)$, where $d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$. In order that the projections $X \times Y$ to X and Y are $\mathcal{A}$-morphisms, it is necessary to restrict the full product to the subspace $X \tilde{\times} Y$ defined as the pull back of the diagram:

$$\begin{array}{ccc} X\tilde{\times}Y & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle d_Y(\ ,y_0)} \\ X & \xrightarrow[d_X(\ ,x_0)]{} & \mathbb{R}_+ \end{array}$$

When X and Y are geodesic metric spaces, $X \tilde{\times} Y$ does not depend on the choice of base points up to coarse equivalence. It follows that $X \tilde{\times} \mathbb{R}_+ \cong X$ and the triple $X \tilde{\times} (\mathbb{I}, 0, 1) \equiv X \tilde{\times} (\mathbb{R}^2_+, \mathbb{R}_- \times \{0\}, \mathbb{R}_+ \times \{0\})$ can be identified with (Y, Y_0, Y_1): $Y = \{(x, t) \in X \times \mathbb{R}_+) \mid 0 \leq t \leq \|x\|\}$, $Y_0 = \{(x, 0) \mid x \in X\}$, $Y_1 = \{(x, \|x\|) \mid x \in X\}$, where $\|x\| = d_X(x, x_0)$ $(x \in X)$.

The cone Cone(X) over X is defined as the quotient metric space of Y obtained by collapsing each compact subset $Y_{1,t} = \{(x, \|x\|) \in Y_1 \mid \|x\| = t\}$ of Y_1 $(t \geq 0)$ to a point. It follows that Cone$(\mathbb{R}^n) \cong \mathbb{R}^{n+1}_+$ in $\widetilde{\mathcal{A}}$.

Homotopy in $\mathcal{A}$ is defined using the product with $\mathbb{I} \equiv \mathbb{R}^2_+$:

Definition. A (pointed) homotopy $H : f \simeq g : X \to Y$ is an $\mathcal{A}$-morphism $H : X \tilde{\times} \mathbb{I} \to Y$ with $H|_{X\tilde{\times}0} = f$, $H|_{X\tilde{\times}1} = g$. Homotopy equivalences are defined in the usual way.

For the n-dimensional hyperbolic space $\mathbb{H}^n$, it is known that $\mathbb{H}^n \simeq \mathbb{R}^n$ in $\mathcal{A}$, but $\mathbb{R}^n \not\simeq \mathbb{H}^n$ in $\widetilde{\mathcal{A}}$.

Various concepts in the topological category also have their coarse versions. The followings are the coarse versions of compactness and local contractibility:

Definition. (1) X has bounded geometry (BG) if for every $L > 0$ there is a uniformly bounded cover $\mathcal{U}$ of X with Lubesgue number $> L$ and of finite multiplicity

(2) X is uniformly contractible (UC) if for every $R > 0$ there exists $S > 0$ such that $B_R(x)$ (the R-ball with the center x) is contractible in $B_S(x)$ for every $x \in X$.

3. Asymptotic General Topology

3.1. Higson compactification.

Definition. Suppose $f : X \to \mathbb{R}$ is a function and $R > 0$. The R-variation of f is defined by

$$\mathrm{Var}_R f : X \to [0, \infty), \quad (\mathrm{Var}_R f)(x) = \sup_{y \in B_R(x)} |f(x) - f(y)|$$

Let $C_h(X)$ denote the algebra of bounded continuous functions $f : X \to \mathbb{R}$ with $\lim_{x \to \infty} \mathrm{Var}_R f(x) = 0$ for any $R > 0$. There is a natural embedding $F = \prod_{f \in C_h(X)} f : X \to \mathbb{R}^{C_h(X)}$.

Definition. The closure $\overline{X} \equiv \overline{F(X)} \subset \mathbb{R}^{C_h(X)}$ is a compactification of X. This compactification $\overline{X}$ and the reminder $\nu(X) = \overline{X} \setminus X$ are called the Higson compactification and the Higson corona of X respectively.

The Higson corona ν is a functor from the coarse category $\mathcal{C}$ to the category of (non-metrizable) compact Hausdorff spaces and continuous maps. However, $\check{H}^*(\nu(X))$ is not a coarse homotopy invariant, since $\mathbb{R}^n \simeq \mathbb{H}^n$ in $\mathcal{A}$ but $\check{H}^n(\nu(\mathbb{R}^n)) \not\cong \check{H}^n(\nu(\mathbb{H}^n))$. The functor ν transforms various problems in the (equivariant) coarse category to ones in the (equivariant) topological category (see §.5).

3.2. **Asymptotic ANE Theory.**

The notion of absolute neighborhood extensors (ANE) in the asymptotic category is defined as follows:

Definition. Suppose $Y \in Ob(\mathcal{A})$.
(1) $Y \in ANE(\mathcal{A})$ if for any $X \in Ob(\mathcal{A})$, any subobject $A \subset X$ and any $\mathcal{A}$-morphism $f : A \to Y$ there exist a closed asymptotic neighborhood W of A in X and an $\mathcal{A}$-morphism extension $\overline{f} : W \to Y$ of f. $Y \in AE(\mathcal{A})$ if we can always take as $W = X$.
(2) $Y \in ANE_0(\mathcal{A})$ if $Y \times \mathbb{R}_+ \in AE(\mathcal{A})$.

$ANE(\widetilde{\mathcal{A}})$ and $ANE_0(\widetilde{\mathcal{A}})$ are defined similarly. By the open cone correspondence in §.2.3, one expects that $\mathbb{R}^{n+1}_+$ and $\mathbb{R}^{n+1}$ play the roles of the closed n-disk and the n-sphere. However, $\mathbb{R}_+ \notin AE(\mathcal{C})$. Next statement shows that the required properties are obtained in the category $\mathcal{A}$:

Theorem. (1) $ANE_0(\mathcal{A}) \subset ANE(\mathcal{A})$, $ANE_0(\widetilde{\mathcal{A}}) \subset ANE(\widetilde{\mathcal{A}})$ (the converse implication does not hold in $\mathcal{A}$).
(2) $\mathbb{R}^n_+ \in AE(\mathcal{A}) \cap AE(\widetilde{\mathcal{A}})$ and $\mathbb{R}^n \in ANE_0(\mathcal{A}) \cap ANE_0(\widetilde{\mathcal{A}})$.

Homotopy Extension Theorem. Suppose $Y \in ANE(\widetilde{\mathcal{A}})$ and A is an asymptotically closed subset of X. Then any $\widetilde{\mathcal{A}}$-morphism $f : X \tilde{\times} 0 \cup A \tilde{\times} \mathbb{I} \to Y$ extends to an $\widetilde{\mathcal{A}}$-morphism $\overline{f} : X \tilde{\times} \mathbb{I} \to Y$.

3.3. Anti-Čech approximation by polyhedra.

In the homotopy theory, any space can be replaced by a polyhedron up to weak homotopy equivalence. The coarse version of this notion is the coarsening: a coarsening of a (proper metric) space X is a coarse equivalence $\pi : X \to E$ to a metric simplicial complex E of UC & BG. When X has coarsenings, they are unique up to proper homotopy equivalence. However, unlike in the topological case, every space do not necessarily admit a coarsening. On the other hand, in the shape theory (the Čech-type homotopy theory) one replaces any space by a Čech-system (a system of finer and finer open covers or their nerves). Its coarse version is the notion of anti-Čech-systems (a system of open covers with larger and larger Lubesgue numbers or their nerves).

First we recall some notions on metric polyhedra:

Definition. (1) A uniform (PE) polyhedra is a simplicial complex K with a geodesic metric such that all simplicies are isometric to the standard simplex of size L. We call L the mesh of K.
(2) An asymptotic (PE) polyhedron is a locally finite simplicial complex with a geodesic metric such that each simplex has a metric induced from a (same) Banach space such that the widths of simplices $\to \infty$ as simplices $\to \infty$.

Suppose X has BG.

Definition. An anti-Čech system of X is a sequence $\mathcal{U}_k$ $(k \geq 1)$ of locally finite and uniformly bounded open covers of X such that there exists $R_k \to \infty$ such that (i) $\operatorname{mesh} \mathcal{U}_k \leq R_k$ and (ii) the Lubesgue number of $\mathcal{U}_{k+1} \geq R_k$.

The induced nerve sequence $\{N(\mathcal{U}_k), i_{k\ell}\}$ satisfies the following conditions:

Definition. An anti-Čech approximation of X is a direct sequence $\{K_i, g^i_{i+1} : K_i \to K_{i+1}\}$ together with short maps $f_i : X \to K_i$ such that
(i) K_i is a uniform simplicial complex with mesh L_i and lim $L_i = \infty$.
(ii) g^i_{i+1} is a simplicial projections and f_{i+1} is proper homotopic to $g^i_{i+1} \circ f_i$

If X is UC, then the maps f_i admit left proper homotopy inverse $p_i : K_i \to X$ such that f^i_{i+1} and $f_{i+1} \circ p_i$ are proper homotopic. The above definitions are naturally extended to the case of pairs (X, A), where A is an asymptotically closed subset of X.

3.4. Asymptotic Dimension Theory.

A main subject in the asymptotic topology is the dimension theory. In the topological dimension theory, there are several approaches to describe the concepts of dimension of spaces: (1) Topological dimension: (i) covering dimension, polyhedral displacement, extension dimension, (ii) infinite dimensions, and (2) Cohomological dimension. Alexandroff problem asked whether $\dim X = \dim_{\mathbb{Z}} X$ for any compact metric space X, and counterexamples were found by A. N. Dranishnikov et al.

Asymptotic dimension theory is the coarse version of the topological dimension theory, and we have the corresponding approaches: (1) Asymptotic topological dimension: (i) $as\dim X$, $as\dim_* X$, $\dim^c X$, (ii) slow dimension growth, (2) Asymptotic cohomological dimension.

[1] Asymptotic topological dimensions

(1) $as\dim X$:

Definition. $as\dim X \leq n$ if for any $L > 0$ there is a uniformly bounded open cover of X of multiplicity $\leq n+1$ with Lebesgue number L.

Proposition. The following conditions are equivalent:
(i) $as\dim X \leq n$
(ii) for any $L > 0$ there is a uniformly bounded open cover $\mathcal{U} = \cup_{i=0}^{n} \mathcal{U}_i$ of X such that each $\mathcal{U}_i$ is L-disjoint.
(iii) for any $D > 0$ there is a uniformly cobounded short proper map $f : X \to K$ to an n-dimensional PE complex of mesh D.
(iv) X admits an anti-Čech approximation by n-dimensional simplicial complexes.

(2) $as\dim_* X$:

Definition. $as\dim_* X \leq n$ if for any proper function $f : X \to \mathbb{R}_+$ there is a short map $\varphi : X \to K$ to an n-dimensional asymptotic polyhedron such that $\|\varphi\|_{AG} < f$ ($\|\varphi\|_U < f$).

Here, the Alexandrof norm $\|\psi\|_{AG}$ and the Uryshon norm $\|\psi\|_U$ of a map $\psi : X \to Y$ are defined as follows:
(a) $\|\psi\|_{AG} < f$ if $d_Z(x, \psi(x)) < f(x)$ for some proper metric space (Z, d_Z) which contains X and Y as metric subspaces.
(b) $\|\psi\|_U < f$ if for any $R > 0$ there exists a compact subset C of X such that $\operatorname{diam} \psi^{-1}(B_R(\psi(x))) < f(x)$ for any $x \in X \setminus C$.

Proposition. $as\dim_* X \leq n$ iff for a given proper map $f : X \to \mathbb{R}_+$ there is a uniformly bounded cover $\mathcal{U}$ of order $\leq n+1$ and such that $\lim_{x\to\infty} L_{\mathcal{U}}(x) = \infty$ and $mesh_{\mathcal{U}}(x) < f(x)$ ($x \in X \setminus C$) for some compact set C.

(3) $\dim^c X$:

Definition. $\dim^c X \leq n$ if for any asymptotically closed subset A of X and any $\mathcal{A}$-morphism $f : A \to \mathbb{R}^{n+1}$ there is an $\mathcal{A}$-extension $\overline{f} : X \to \mathbb{R}^{n+1}$.

(4) Comparison of $as\dim X$, $as\dim_* X$, $\dim^c X$ and $\dim \nu(X)$:

Theorem. (1) $\dim \nu(X) = \dim^c X \leq as\dim_* X \leq as\dim X$
(2) (a) If $as\dim X < \infty$, then $as\dim X = \dim \nu(X)$. (b) If $as\dim_* X < \infty$, then $as\dim_* X = \dim \nu(X)$.

(5) Slow dimension growth:

Even if $as\dim X = \infty$, we can consider a rate $d(L)$ of dimension growth:

$$d(L) = \min\{n \geq 0 \mid \text{There is a uniformly bounded cover of } X \text{ with Lubesgue number } L \text{ and of order } n+1\}$$

Note that $as\dim X = \sup\{d(L) \mid L > 0\}$.

Definition. X has a slow dimension growth if $\lim\limits_{L\to\infty} \frac{d(L)}{L} = 0$.

[2] Asymptotic cohomological dimension

Let G be an abelian group. (See §.4.1 for the definition of $H^n_{R,c}(X, A; G)$.)

Definition. $as\dim_G X = \sup\{n \mid H^n_{R,c}(X, A; G) \neq 0 \text{ for some closed subset } A\}$.

Proposition. If X is UC, then $as\dim_G X \leq \dim_G X \leq \dim X$.

Coarse Alexandroff Problem. Does $as\dim X = as\dim_{\mathbb{Z}} X$ hold for any proper metric space X ?

The following facts are known:

Theorem. (1) There is a UC Riemannian metric on $\mathbb{R}^8$ with $as\dim = \infty$ and $as\dim_{\mathbb{Z}} < \infty$.
(2) Suppose G is a finitely generated group with a word-length metric and such that BG is a finite complex. Then $as\dim_{\mathbb{Z}} G < \infty$. (It is unkown whether $as\dim G = as\dim_{\mathbb{Z}} G$. If this holds, then the Novikov conjecture holds for G.)

4. Coarse Algebraic Topology

For compact metric spaces, the Čech homology theory and Steenrod homology theory are defined by using Čech-systems and their telescopes. Their coarse versions can be defined by using anti-Čech-approximations and their telescopes. Note that an anti-Čech system is a direct system, while a Čech-system is an inverse system.

4.1. **Coarse (co)homology theory.**

Suppose X is a proper metric space and $\{K_i, g^i_{i+1}\}$ is an anti-Čech-approximation of X. Let T denote the telescope constructed from $\{K_i, g^i_{i+1}\}$. The (co)homology theory for coarse spaces are defined as follows:

Coarse (co)homology. :

(1) $\widehat{H}_*(X) = \varinjlim \{H_*(K_i)\}$ (Coarse homology)

(2) $\widehat{H}^{lf}_*(X) = \varinjlim \{H^{lf}_*(K_i)\}$ (Roe's exotic homology)

(3) $H^*_R(X) = H^*(T)$ (Roe's cohomology)

(4) $H^*_{R,c}(X) = H^*(T^\alpha)$ (Roe's cohomology with compact support)

where T^α is the telescope of one point compactifications of αK_i.

The (co)homology groups for a pair (X, A) can be defined similarly. There are natural homomorphisms and exact sequences connecting these groups:

Exact Sequences.

(1) $0 \to \varinjlim{}^1 H^{k-1}_c(K_i) \to H^k_{R,c}(X) \to \widehat{H}^k_c(X) \to 0$

(2) (i) $\cdots \to \widehat{H}^{lf}_i(A) \to \widehat{H}^{lf}_i(X) \to \widehat{H}^{lf}_i(X, A) \to \widehat{H}^{lf}_{i-1}(A) \to \cdots$

(ii) $\cdots \leftarrow \widehat{H}^i_{R,c}(A) \leftarrow \widehat{H}^i_{R,c}(X) \leftarrow \widehat{H}^i_{R,c}(X, A) \leftarrow \widehat{H}^{i-1}_{R,c}(A) \leftarrow \cdots$

The inclusion of αX into the telescope T^α_X induces homomorphisms

$$c_* : H^{lf}_*(X) \to \widehat{H}^{lf}_*(X) \text{ and } c^* : \widehat{H}^*_{R,c}(X) \to \widehat{H}^*_c(X).$$

Proposition. Suppose X is UC. Then

(1) c_* and c^* are isomorphisms.

(2) For any asymptotically closed set A of X

(i) $\widehat{H}^{lf}_*(X, A) = \lim_{k\to} \widehat{H}^{lf}_*(X, N_k(A))$,

(ii) there exists the exact sequence

$$0 \to \lim_{\leftarrow k}{}^1 H^{*-1}_c(X, N_k(A)) \to H^*_{R,c}(X, A) \to \lim_{\leftarrow k} \widehat{H}^*_c(X, N_k(A)) \to 0$$

4.2. **Coarse K-theory.**

The above construction also can be applied to the generalized homology theory. We will be concerned with K-theories. In the following subsections we recall basic facts on (i) the C^*-algebra K-theory $K_*(C^*(X))$, (ii) the locally finite K-homology $K^{lf}_*(X)$, (iii) the index map $\mathrm{Ind} : K^{lf}_*(X) \to K_*(C^*(X))$ (iv) the coarse index map $\mathrm{Ind} : \widehat{K}^{lf}_*(X) \to K_*(C^*(X))$

4.2.1. **The C^*-algebra K-theory $K_*(C^*(X))$.**

Let $C_0(X)$ denotes the C^*-algebra of continuous functions $f : X \to \mathbb{C}$ vanishing at ∞.

Definition. (1) An X-module is a separable Hilbert space H_X with a $*$-representation $\rho : C_0(X) \to \mathcal{B}(H_X)$.
(2) H_X is said to be adequate if (i) any $f \in C_0(X)$, $f \neq 0$, dose not act on H_X as a compact operator and (ii) $\overline{C_0(X)H_X} = H_X$.

A typical example of an X-module is the L^2-space of sections $L^2(S, \mu)$, where S is a hermitian vector bundle over X and μ is a Radon measure on X.

Definition. Suppose $T : H_X \to H_Y$ is a bounded operator.
(1) $\mathrm{Supp}(T) = \{(x, y) \in X \times Y \mid f \in C_0(X), f(x) \neq 0,\ g \in C_0(Y), g(y) \neq 0 \implies gTf \neq 0\}$.
(2) Propagation of $T = \sup\{d(x, y) : (x, y) \in \mathrm{Supp}(T)\}$
(3) $T \in \mathcal{B}(H_X)$ is locally compact if fT and Tf are compact for any $f \in C_0(X)$.

Definition. $C^*(X) \subset \mathcal{B}(H_X)$ denotes the C^*-algebra generated by all locally compact finite propagation operators on H_X.

Lemma. Suppose $f : X \to Y$ is a (Borel) coarse map and H_X, H_Y are adequate X and Y-modules.
(i) For any $\varepsilon > 0$ there exists an isometry $V_f : H_X \to H_Y$ such that $\mathrm{Supp}(V_f) \subset \{(x, y) \in X \times Y : d(f(x), y) \leq \varepsilon\}$.
(ii) V_f induces morphisms $\mathrm{ad}(V_f) : C^*(X) \to C^*(Y)$ and $\mathrm{ad}(V_f)_* : K_*(C^*(X)) \to K_*(C^*(Y))$. $\mathrm{ad}(V_f)_*$ does not depend on the choice of ε and V_f.

Proposition. The functor $K_*(C^*(X)) : \mathcal{C} \to$ Abelian groups is invariant under coarse equivalence.

4.2.2. **Analytic description of $K_*^{lf}(X)$ by G. G. Kasparov.**

Definition.
(i) An even Fredholm module for X is a pair (H_X, F) such that (a) $F \in \mathcal{B}(H_X)$, (b) $F^*F - I$ and $FF^* - I$ are locally compact and (c) $[f, F] = fF - Ff$ is compact for any $f \in C_0(X)$.
(ii) An odd Fredholm module for X is a pair (H_X, F) such that (a) $F \in \mathcal{B}(H_X)$ is self-adjoint, (b) $F^2 - I$ is localy compact and (c) $[f, F]$ is compact for any $f \in C_0(X)$.
(iii) $K_0^{lf}(X) = \{\text{even adequate Fredholm modules}\}/\sim$
$K_1^{lf}(X) = \{\text{odd adequate Fredholm modules}\}/\sim$.

4.2.3. **Index map** $\mathrm{Ind} : K_i^{lf}(X) \to K_i(C^*(X))$.

We consider the case for $i = 0$. The case for $i = 1$ can be defined similarly.

Given $[(H_X, F)] \in K_0^{lf}(X)$. Choose a locally finite, uniformly bounded open cover $\mathcal{U} = \{U_i\}_i$ of X and a continuous partition of unity $\{\varphi_i\}$ subordinated to $\mathcal{U}$. Then $F' = \sum_i \varphi_i^{1/2} F \varphi_i^{1/2} \in \mathcal{B}(H_X)$ converges in strong topology and $[(H_X, F')] = [(H_X, F)]$ in $K_0(X)$. Let $D^*(X)$ denote the multiplier algebra of $C^*(X)$ in $\mathcal{B}(H)$ and let $\partial : K_1(D^*(X)/C^*(X)) \to K_0(C^*(X))$ denote the boundary map in the 6-term cyclic exact sequence. Then $F' \in D^*(X)$ and $\overline{F'} \in D^*(X)/C^*(X)$ is unitary, so we have $[F'] \in K_1(D^*(X)/C^*(X))$.

Definition. $\operatorname{Ind}[(H_X, F)] = \partial([F']) \in K_0(C^*(X))$

4.2.4. **Coarse Index map** $\operatorname{Ind} : \widehat{K}_*^{lf}(X) \to K_*(C^*(X))$.

Let $\{K_i, g_{i+1}^i\}$ be an anti-Čech-approximation of a coarse space X. The coarse locally finite K-homology group $\widehat{K}_*^{lf}(X)$ and the coarse index map $\operatorname{Ind} : \widehat{K}_*^{lf}(X) \to K_*(C^*(X))$ are defined by the following diagram:

$$\begin{array}{ccc}
\{K_*^{lf}(K_i)\}_i & \xrightarrow{\text{Ind}} & \{K_*(C^*(K_i))\}_i \\
\varinjlim \downarrow & & \downarrow \varinjlim \\
\widehat{K}_*^{lf}(X) & \xrightarrow{\text{Ind}} & K_*(C^*(X))
\end{array}$$

5. Conjectures in Coarse Geometry

Geometric/Topological conjectures in §1 have their coarse counterparts. A coarse analog of the Novikov higer signature conjecture is the coarse Baum-Connes conjecture:

(1) The coarse Baum-Connes conjecture: Suppose X is a proper metric space of UC & BG. Then the coarse index map $\operatorname{Ind} : \widehat{K}_*^{lf}(X) \to K_*(C^*(X))$ is an isomorphism (a rational isomorphism/monomorphism).

(2) Gromov's conjecture: A Riemannian manifold of UC & BG is (rationally) hypereuclidean (hyperspherical).

Definition. (1) An open Riemannian n-manifold is called (rationally) hypereuclidean if there is an $\mathcal{A}$-morphism $f : X \to \mathbb{R}^n$ of degree one (nonzero).

(2) It is called (rationally) hyperspherical if for every $R > 0$ there is a short 'proper map $f : X \to S^n(R)$ onto the standard sphere of radius R of degree one (nonzero). ('proper means that f maps $X \setminus C$ to a point for some compact subset C of X.)

The Higson corona ν is a functor from the coarse category to the topological category of compact Hausdorff spaces. This functor translates the above coarse conjectures to the corresponding topological problems: Consider the boundary homomorphism $\delta : H^i(\nu(X); \mathbb{Q}) \to H_c^{i+1}(X, \mathbb{Q})$ in the exact sequence of the pair $(\overline{X}, \nu(X))$.

(3) The Weinberger's conjecture: $\delta : H^*(\nu(X); \mathbb{Q}) \to H_c^{*+1}(X, \mathbb{Q})$ is an epimorphism.

Suppose Γ is a finitely generated group with a word length metric and such that $B\Gamma$ is a finite complex. Let X be a universal cover of $B\Gamma$ with a Γ-invariant metric. In this setting we have Γ-equivariant versions of the above problems:

(4) FW: $\delta : H^{n-1}(\nu(X); \mathbb{Q}) \to H^n(X; \mathbb{Q})$ is an equivariant split surjection.

(5) CP: There exists an equivariant rationally acyclic compactification $\widehat{X}$ of X on which the action of Γ is small at ∞

(6) HR: There exists an equivariant rationally acyclic Higson dominated compactification $\widehat{X}$ of X.

The general implications between geometric problems and their coarse versions are presented in the next statement:

The Descent Principle: The original geometric conjectures can be derived from their coarse versions.

Proposition. Suppose Γ is a finitely generated group with a word length metric and such that $B\Gamma$ is a finite complex. Then each statement of the coarse Baum–Connes conjecture, FW, CP and HR for Γ implies the Novikov conjecture for Γ.

6. Main results for coarse conjectures

G. Yu [5] obtained the following results:

Main Theorem. Suppose X is a proper metric space of UC & BG. If $as\dim X < \infty$ (or there exists a coarsely uniform embedding of X into a Hilbert space), then the coarse Baum-Connes conjecture holds for X.

A. N. Dranishnikov obtained the following generalizations:

Theorem. (1) If X has the slow dimensin growth, then the coarse Baum-Connes conjecture holds for X.

(2) If $X \in \mathrm{ANE}(\mathcal{A})$, and $as\dim_* X < \infty$, then the coarse Baum-Connes conjecture holds for X.

(3) If X is a Riemannian n-manifold of UC and $as\dim X = n$, then X is hyperspherical.

The statement (2) is verified in the following steps (the notation BC(X) means that the coarse Baum-Connes conjecture holds for X):

(a) BC (finite dim asymptotic polyhedra)

(b) BC(Y) & Y $\mathcal{A}$-homotopy dominates $X \Longrightarrow$ BC(X)

(c) $X \in ANE(\mathcal{A})$ & $as\,\dim_* X < \infty \Longrightarrow$ There exists a finite dim asymptotic polyhedron K which $\mathcal{A}$-homotopy dominates X.

This natural argument shows that the ANE/Dimension theory in the asymptotic topology is well-formulated.

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