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OSCILLATION CRITERIA FOR SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS OF EULER TYPE

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Consider the second order nonlinear differential equation

$$t^2x'' + g(x) = 0, \qquad t > 0, \tag{1}$$

where $' = d/dt$, and $g(x)$ is continuous on $\mathbf{R}$ and satisfies

$$xg(x) > 0 \quad \text{if } x \neq 0. \tag{2}$$

We assume that the uniqueness is guaranteed for the solutions of (1) to the initial value problem. We can prove that all solutions of (1) are continuable in the future (for the proof, see [1]).

A solution $x(t)$ of (1) (or (5) below) is said to be *oscillatory* if there exists a sequence $\{t_n\}$ tending to ∞ such that $x(t_n) = 0$. Otherwise, $x(t)$ is said to be *nonoscillatory*. In case $g(x) = \lambda x$, equation (1) is called the Euler differential equation and it is well known that all nontrivial solutions are oscillatory if $\lambda > 1/4$ and are nonoscillatory if $\lambda \leq 1/4$.

Sugie and Hara [1] investigated the oscillation problem for the nonlinear differential equation (1) and gave the following result without requiring such monotonicity of $g(x)$ as sublinear or superlinear.

THEOREM A. *Let $\lambda > 0$. Then all nontrivial solutions of* (1) *are oscillatory if*

$$\frac{g(x)}{x} \geq \frac{1}{4} + \frac{\lambda}{\log|x|} \tag{3}$$

for $|x|$ sufficiently large.

THEOREM B. *Assume* (2) *and suppose that there exists a λ with $0 < \lambda < 1/16$ such that*

$$\frac{g(x)}{x} \leq \frac{1}{4} + \frac{\lambda}{(\log|x|)^2} \tag{4}$$

for $x > 0$ or $x < 0$, $|x|$ sufficiently large. Then all nontrivial solutions of (1) *are nonoscillatory.*

Clearly, Theorems A and B are complete extensions of the result for the linear case and can be applied to sublinear and superlinear cases. Hence, it is safe to say that the classification into sublinear and superlinear cases is not important to the oscillation problem for equation (1). Since equation (1) is nonlinear, we cannot use Sturm's separation theorem. For this reason, it is possible that oscillatory solutions and nonoscillatory solutions exist together in equation (1). Theorems A and B show, however, that there is no possibility of coexistence. As to our problem, the most difficult case is

$$\frac{g(x)}{x} \searrow \frac{1}{4} \quad \text{as } |x| \to \infty.$$

Previous results except Theorems A and B are inapplicable to this critical case.

Wong [2] studied the equation

$$x'' + a(t)g(x) = 0, \qquad t > 0, \tag{5}$$

which includes the Emden-Fowler differential equation. Using Sturm's comparison theorem, he improved Theorems A and B as follows:

THEOREM C. *Assume that $a(t)$ is continuously differentiable and satisfies*

$$t^2 a(t) \geq 1 \tag{6}$$

for t sufficiently large, and that there exists a λ with $\lambda > 1/4$ such that

$$\frac{g(x)}{x} \geq \frac{1}{4} + \frac{\lambda}{(\log|x|)^2} \tag{7}$$

for $|x|$ sufficiently large. Then all nontrivial solutions of (5) *are oscillatory.*

THEOREM D. *Assume that $a(t)$ is continuously differentiable, and satisfies*

$$0 \leq t^2 a(t) \leq 1 \tag{8}$$

for t sufficiently large and

$$A(t) \stackrel{\text{def}}{=} \frac{a'(t)}{2a^{3/2}(t)} + 1 = o(1) \quad \text{as } t \to \infty. \tag{9}$$

If, in addition, $A(t) \leq 0$ and there exists a λ with $0 < \lambda \leq 1/16$ such that

$$\frac{g(x)}{x} \leq \frac{1}{4} + \frac{\lambda}{(\log|x|)^2} \tag{10}$$

for $x > 0$ or $x < 0$, $|x|$ sufficiently large, then all nontrivial solutions of (5) *are nonoscillatory.*

Since equation (5) coincides with equation (1) when $a(t) = 1/t^2$, it seems reasonable to assume (6) and (8) in Theorems C and D, respectively. But condition (9) on $A(t)$ is considerably strict. Although it is known that all nontrivial solutions of (5) are nonoscillatory if $a(t) = 1/t^3$ and $g(x)$ is linear or sublinear, condition (9) is not satisfied.

Condition (7) completely contains (3), and condition (10) is slightly weaker than (4). Unfortunately, the case

$$\frac{g(x)}{x} = \frac{1}{4} + \frac{\lambda}{(\log|x|)^2} \tag{11}$$

with $1/16 < \lambda \leq 1/4$ remains unsetted. Wong [2] expected that if $1/16 < \lambda \leq 1/4$, then equation (1) with (11) has both oscillatory solutions and nonoscillatory solutions.

In this note, we give a perfect answers to the unsolved problem above and show that Wong's conjecture is not true. To see this, we assume that $a(t)$ and $g(x)$ satisfy suitable smoothness conditions for the uniqueness of solutions of the initial value problem. We also assume that every solution of (5) exists in the future. As is well known, the uniqueness of solutions of (5) is guaranteed if $a(t)$ is continuous and $g(x)$ is locally Lipschitz continuous. In case $a'(t) \leq 0$, we can show global existence of solutions of (5) by using Proposition 2.2 in [1].

We first state the following oscillation theorem for equation (5).

THEOREM 1. *Suppose that* $a(t)$ *satisfies*

$$t^2 a(t) \geq 1 \tag{12}$$

for t *sufficiently large, and that there exists a* λ *with* $\lambda > 1/16$ *such that*

$$\frac{g(x)}{x} \geq \frac{1}{4} + \frac{\lambda}{(\log|x|)^2} \tag{13}$$

for $|x|$ *sufficiently large. Then all nontrivial solutions of* (5) *are oscillatory.*

Remark 1. Clearly, Theorem 1 is a generalization of Theorem C.

Next, we give a nonoscillation theorem for equation (5). For this purpose, in addition to (2), we make the following assumption:

$$G(x) \stackrel{\text{def}}{=} \int_0^x g(\xi)d\xi \leq \frac{1}{2}x^2 \quad \text{for } x \in \mathbf{R}. \tag{14}$$

Then we have

THEOREM 2. *Let* (2) *and* (14) *hold. Suppose that* $a(t)$ *satisfies*

$$0 \leq t^2 a(t) \leq 1 \tag{15}$$

for t *sufficiently large, and that*

$$\frac{g(x)}{x} \leq \frac{1}{4} + \frac{1}{16(\log|x|)^2} \tag{16}$$

for $x > 0$ *or* $x < 0$, $|x|$ *sufficiently large. Then all nontrivial solutions of* (5) *are nonoscillatory.*

Remark 2. Assumption (2) implies that $G(x) > 0$ for $x \neq 0$ and condition (16) implies that $G(x) \leq x^2/2$ for $|x|$ sufficiently large.

Remark 3. Since condition (9) in Theorem D is considerably strict, we cannot apply Theorem D to even the case that $a(t) = 1/t^3$ and $g(x)$ is linear. Contrary to this, it is easy to apply Theorem 2 to many cases because the requirement on $a(t)$ is weak.

From Theorems 1 and 2 we see that if $\lambda > 1/16$, then all nontrivial solutions of (1) with (11) are oscillatory; otherwise, all of them are nonoscillatory.

Finally, we give some examples to illustrate Theorems 1 and 2.

EXAMPLE 1. Let N be a fixed positive integer. Consider equation (5) with

$$g(x) = \begin{cases} \dfrac{\lambda}{2}x(5 - 3\cos 2x) & \text{for } |x| \le N\pi, \\ \lambda x & \text{for } |x| > N\pi. \end{cases} \tag{17}$$

Then we have:

(i) if $\lambda > 1/4$ and $t^2 a(t) \ge 1$, then all nontrivial solutions are oscillatory;
(ii) if $0 < \lambda \le 1/4$ and $0 \le t^2 a(t) \le 1$, then all nontrivial solutions are nonoscillatory.

Note that $g(x)$ is continuously differentiable for $x \in \mathbf{R}$. Since conditions (12) and (13) hold in the case (i), by Theorem 1 all nontrivial solutions of (5) with (17) are oscillatory. It is clear that $g(x)$ satisfies (2). Since $g(x)/x \le 4\lambda$ for $x \in \mathbf{R}$, if $\lambda \le 1/4$, then condition (14) is satisfied. If, in addition, $0 \le t^2 a(t) \le 1$, then conditions (15) and (16) are also satisfied. Hence, by Theorem 2 all nontrivial solutions of (5) with (17) are nonoscillatory in the case (ii).

EXAMPLE 2. Consider equation (5) with

$$g(x) = \begin{cases} \left(\dfrac{1}{4} + \dfrac{\lambda}{(\log\omega)^2}\right)x + \dfrac{2\lambda}{\pi(\log\omega)^3}x\sin\left(\dfrac{\pi}{\omega}|x|\right) & \text{for } |x| \le \omega, \\ \dfrac{1}{4}x + \dfrac{\lambda x}{(\log|x|)^2} & \text{for } |x| > \omega, \end{cases} \tag{18}$$

where ω is a constant satisfying

$$12\pi(\log\omega)^3 = \pi\log\omega + 2. \tag{19}$$

Then we have:

(i) if $\lambda > 1/16$ and $t^2 a(t) \ge 1$, then all nontrivial solutions are oscillatory;
(ii) if $0 < \lambda \le 1/16$ and $0 \le t^2 a(t) \le 1$, then all nontrivial solutions are nonoscillatory.

From (19), the constant ω is uniquely determined (ω exists between 1.5 and 1.6). It is easy to verify that $g(x)$ is a continuously differentiable function and $xg(x) > 0$ if $x \neq 0$. In case (i), conditions (12) and (13) are satisfied, and therefore, all nontrivial solutions of (5) with (18) are oscillatory by Theorem 1. If $0 < \lambda \le 1/16$, then (19) implies

$$\begin{aligned} \frac{g(x)}{x} &\le \frac{1}{4} + \frac{\lambda}{(\log\omega)^2} + \frac{2\lambda}{\pi(\log\omega)^3} \\ &\le \frac{1}{4} + \frac{1}{16(\log\omega)^2} + \frac{1}{8\pi(\log\omega)^3} \\ &= \frac{1}{4} + \frac{3}{4}\left(\frac{\pi\log\omega + 2}{12\pi(\log\omega)^3}\right) = 1 \end{aligned}$$

for $x \in \mathbf{R}$, and therefore, condition (14) holds. It is clear that conditions (15) and (16) is satisfied in the case (ii). Hence, by Theorem 2 all nontrivial solutions of (5) with (18) are nonoscillatory.

REFERENCES

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2. J. S. W. Wong, Oscillation theorems for second-order nonlinear differential equations of Euler type, *Methods Appl. Anal.* **3** (1996), 476–485. MR **98e:34063**