数理解析研究所講究録
1128 巻 2000 年 148-155

Keller-Segel 系の爆発解の挙動について

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0 Introduction

We consider the behavior of blow-up solutions for (KS)

$$(KS)\quad \begin{cases} \dfrac{\partial u}{\partial t} = \nabla\cdot(\nabla u - \chi u \nabla v) & in\ \Omega,\ t>0 \\ \tau\dfrac{\partial v}{\partial t} = \triangle v - \gamma v + \alpha u & in\ \Omega,\ t>0 \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & on\ \partial\Omega,\ t>0 \\ u(\cdot,0)=u_0, v(\cdot,0)=v_0 & on\ \Omega \end{cases}$$

Here Ω is a bounded domain in $\mathbf{R}^2$ with smooth boundary $\partial\Omega$, and τ,χ,γ, and α are positive constants, and u_0, v_0 are nonnegative, nontrivial, smooth functions on $\bar{\Omega}$.

In what follows we denote $\|\cdot\|_{L^p(\Omega)}=\|\cdot\|_p$, $M=\|u_0\|_1$, $⨍_\Omega f dx = \dfrac{1}{|\Omega|}\int_\Omega f dx$, $D:=\{x\in\mathbf{R}^2 \mid |x|<1\}$, and let T be the maximal existance time of solution (u,v).

Theorem1.1 in [1] states:
If $M<\dfrac{4\pi}{\alpha\chi}$, then the solution (u,v) exists globally in time and globally bounded.
If $\Omega=\{x\in\mathbf{R}^2 \mid |x|<L\}$ and (u_0,v_0) is radial in x, and $M<\dfrac{8\pi}{\alpha\chi}$, then the solution (u,v) exists globally in time and globally bounded.

Then what happens if $\dfrac{4\pi}{\alpha\chi}\le M<\dfrac{8\pi}{\alpha\chi}$ and (u_0,v_0) is non radially symmetric? For simplicity, we put $\alpha=\gamma=\chi=1$, and $\Omega=D$.

Theorem2 in [7] and Lemma9 in [7] states:
Let $\tau=0$, $\Omega=D$, and $M<8\pi$. If $T<\infty$, then there exists $x_0\in\partial D$

satisfying

$$\liminf_{t\to T}\int_{D\cap B(x_0,\epsilon)} u(x,t)dx \geq 4\pi \ for\ any\ \epsilon > 0.$$

In this paper, we consider to extend this result to $\tau > 0$. A main result is following.

Theorem *Let* $\tau > 0,\ \Omega = D,$ *and* $M < 8\pi$. *If* $T < \infty$, *then there exists a continuous map* $p(t) : [0,T) \to \partial D$ *satisfying*

$$\limsup_{t\to T}\int_{D\cap B(p(t),\epsilon)} u(x,t)dx \geq 2\pi \ for\ any\ \epsilon > 0.$$

1 Fundamental Lemmas for Theorem

Following Lemmas are known.

Lemma1 *The following holds:*

$$\| u(\cdot,t) \|_1 = \| u_0 \|_1,$$

and

$$\| v(\cdot,t) \|_1 = e^{-\frac{t}{\tau}} \| v_0 \|_1 + \| u_0 \|_1 (1 - e^{-\frac{t}{\tau}}).$$

Lemma2 *Put*

$$W(t) = \int_\Omega u\log u - uv + \frac{1}{2}(| \nabla v |^2 + v^2)dx.$$

Then we have

$$\frac{dW}{dt}(t) + \tau\int_\Omega v_t^2 dx + \int_\Omega u \mid \nabla(\log u - v) \mid^2 dx = 0,$$

and it follows that

$$\frac{dW}{dt}(t) \leq 0, \ and\ W(t) \leq W(0).$$

Lemma3 *Let* $M = \| u_0 \|_1$. *The following holds:*

$$a\int_\Omega uvdx \leq \int_\Omega u\log udx + M\log\frac{1}{M}\int_\Omega e^{av}dx \ for\ any\ a > 0.$$

Lemma4 (Corollary of Proposition[3]-2.3)
Let $\Omega = D$. *There exists* C_Ω *such that*

$$\int_\Omega e^w dx \leq C_\Omega \exp\Big(\frac{1}{8\pi} \| \nabla w \|_2^2 + \frac{1}{|\Omega|} \| w \|_1\Big) \ for\ any\ 0 \leq w \in W^{1,2}(\Omega).$$

Proposition[4]-8.1 *Let F be a set of $w(\cdot,t)(0 \le t < T)$ such that $t \mapsto w(\cdot,t) \in H^1(D)$ is continuous and $\sup_{0\le t<T} \| w(\cdot,t) \|_{L^1(D)} < \infty$, then either one of the following holds:*
(1) There exists $\{t_k\} \nearrow T$ such that $w_k = w(\cdot,t_k) \in F$ satisfying the following.
For any ϵ, there exists C_ϵ such that

$$\log\left(\fint_D e^{w_k} dx\right) \le \frac{1+\epsilon}{16\pi}\int_D |\nabla w_k|^2\, dx + C_\epsilon.$$

(2) There exists a continuous map $t \mapsto q(t) \in \partial D$ such that

$$\liminf_{t\to T} \frac{\int_{D\cap B(q(t),\epsilon)} \exp(w(x,t))P_*(x)dx}{\int_D \exp(w(x,t))P_*(x)dx} \ge \frac{1}{2} \text{ for any } \epsilon > 0,$$

where $P_*(x) = \dfrac{8}{(1+|x|^2)^2}$.

Brézis-Merle Type Inequality for Parabolic Equations of Second Order
We consider the following problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + \displaystyle\sum_{j=1}^{2} b_j(x,t)\frac{\partial u}{\partial x_j} + c(x,t)u = f & \text{in } \Omega\times(0,T) \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega\times(0,T) \\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases}$$

Let $b_j, c \in H^{\alpha,\frac{\alpha}{2}}(\bar\Omega\times[0,T])$ and $q \in \partial\Omega$, where α is a real number with $0 < \alpha < 1$ and h belongs to $H^{\alpha,\frac{\alpha}{2}}(\bar\Omega\times[0,T])$ if

$$|h(x,t) - h(y,s)| \le Const.(|x-y|^\alpha + |t-s|^{\frac{\alpha}{2}})$$

for any $(x,t),(y,s) \in \bar\Omega\times[0,T]$. Given $0 < \tau < T$ and $0 < \epsilon < 2\pi\nu$, there exists positive constants η_0 with $\eta_0 \in (0,\frac{1}{4})$ and $C > 0$ depending on $\tau, \epsilon, \eta \in (0,\eta_0)$, $\| u_0 \|_{L^1(\Omega)}$, and $\| f \|_{L^1(\Omega\times(0,T))}$ such that $\eta \in (0,\eta_0)$ and $\sup_{0<t<T} \| f^+(t) \|_{L^1(\Omega\cap B(q,3\eta))} \le 2\pi\nu - \epsilon$ imply

$$\int_{B(q,\eta)} e^{u(x,t)}dx \le C \text{ for } \tau \le t \le T,$$

where u denote the solution of the above problem.

Proposition1 *The following holds:*
(1) $T < \infty$ implies $\lim_{t\to T}\int_\Omega uv dx = \infty$.

(2)$T<\infty$ *implies* $\lim_{t\to T}\int_{\Omega} e^{av}dx=\infty$ *for any* $a>\frac{M+\sqrt{M^2-4\pi M}}{2M}$.

(3)$T<\infty$ *implies* $\lim_{t\to T}\int_{\Omega}|\nabla v|^2 dx=\infty$.

2 Proof of Proposition1

Before proving Proposition1, we remark that $T<\infty$ implies $M\geq 4\pi$ by the controposition of Theorem1.1 in [1], so in the root sign $M^2-4\pi M$ is not negative.

Proof of Proposition1

Theorem1 in [5] shows that $T<\infty$ implies

$$\lim_{t\to T}\| uv \|_1=\lim_{t\to T}\| e^{av}\|_1=\lim_{t\to T}\| \nabla v\|_2^2=\lim_{t\to T}\| u\log u\|_1=\infty \text{ for any } a>1.$$

So we prove only (2). From Lemma3 and Lemma4 with $w=av$, we have

$$a\int_{\Omega} uvdx\leq\int_{\Omega} u\log udx+\frac{Ma^2}{8\pi}\int_{\Omega}|\nabla v|^2 dx+C \text{ for any } a>0. \qquad (2.1)$$

From Lemma2,

$$\int_{\Omega} u\log u-uv+\frac{1}{2}(|\nabla v|^2+v^2)dx\leq W(0). \qquad (2.2)$$

By (2.1)+$\frac{Ma^2}{4\pi}$(2.2),

$$\left(a-\frac{Ma^2}{4\pi}\right)\int_{\Omega} uvdx\leq\left(1-\frac{Ma^2}{4\pi}\right)\int_{\Omega} u\log udx+C \text{ for any } a>0.$$

Put $a=\frac{M+\sqrt{M^2-4\pi M}}{M}$ in the above inequality, then

$$\int_{\Omega} u\log udx\leq\frac{M+\sqrt{M^2-4\pi M}}{2M}\int_{\Omega} uvdx+C.$$

Using this and Lemma3, we have

$$\left(a-\frac{M+\sqrt{M^2-4\pi M}}{2M}\right)\int_{\Omega} uvdx\leq M\log\frac{1}{M}\int_{\Omega} e^{av}dx+C \text{ for any } a>0.$$

Since $\lim_{t\to T}\int_{\Omega} uvdx=\infty$,

$$\lim_{t\to T}\int_{\Omega} e^{av}dx=\infty \text{ for any } a>\frac{M+\sqrt{M^2-4\pi M}}{2M}.$$

Remark

1. Proposition3.1 in [6] shows that $\| v(\cdot,t) \|_{W^{1,q}(\Omega)} \leq C$ for any $q \in (1,2)$. By using this and Hölder's inequality and Sobolev's imbedding theorem, we have

$$\int_\Omega uv dx \leq \| u \|_p \| v \|_{p'} \leq C \| u \|_p \quad \text{for any } p > 1.$$

So, it follows from Proposition1(1) that $T < \infty$ implies

$$\lim_{t \to T} \| u(\cdot,t) \|_p = \infty \text{ for any } p > 1.$$

3 Proof of Theorem

Proof of Theorem

Suppose the first alternative (1) of Proposition[4]-8.1 holds, then there exists $\{t_k\} \nearrow T$ such that $v_k = v(\cdot,t_k)$ satisfy the following:

$$\log\Big(\frac{1}{\pi}\int_D e^{v(x,t_k)}dx\Big) \leq \frac{1+\epsilon}{16\pi}\int_D | \nabla v(x,t_k) |^2 dx + C_\epsilon \text{ for any } \epsilon > 0. \qquad (3.1)$$

From Lemma2 and Lemma3 with $a = 1$, we have

$$\frac{1}{2}\int_D | \nabla v |^2 dx \leq W(0) + M \log \frac{1}{M}\int_D e^v dx \qquad (3.2)$$

By $M(3.1) + (3.2)$,

$$\Big(\frac{1}{2} - \frac{1+\epsilon}{16\pi}M\Big)\int_D | \nabla v(x,t_k) |^2 dx \leq W(0) - M\log M + M\log\pi + C_\epsilon M.$$

Since $M < 8\pi$, We can take ϵ such that

$$\frac{1}{2} - \frac{1+\epsilon}{16\pi}M > 0.$$

Then

$$\int_D | \nabla v(x,t_k) |^2 dx < \infty.$$

This contradicts to Proposition1.

Therefore the second alternative (2) of Proposition[4]-8.1 holds. Then there exists a continuous map $t \in [0,T) \mapsto q(t) \in \partial D$ such that

$$\liminf_{t \to T} \frac{\int_{D \cap B(q(t),\epsilon)} e^v P_*(x)dx}{\int_D e^v P_*(x)dx} \geq \frac{1}{2} \text{ for any } \epsilon > 0. \qquad (3.3)$$

Since $P_*(x) = \dfrac{8}{(1+ | x |^2)^2}$, $x \in D$ implies $2 \leq P_*(x) \leq 8$.

From Proposition1

$$\lim_{t \to T}\int_D e^v dx = \infty,$$

it follows from (3.3) that

$$\lim_{t\to T}\int_{D\cap B(q(t),\epsilon)} e^v dx = \infty \text{ for any } \epsilon > 0. \qquad (3.4)$$

(a)In case that there exists $q \in \partial D$ such that $q(t) \to q\ (t \to T)$.
We suppose for this $q(t)$ there exists η_1 such that

$$\limsup_{t\to T}\int_{D\cap B(q(t),\eta_1)} udx < 2\pi.$$

Then there exists $\epsilon > 0$ such that

$$\limsup_{t\to T}\int_{D\cap B(q(t),\eta_1)} udx \leq 2\pi - \epsilon,$$

and there exists T_0 such that $T_0 < t < T$ implies

$$\int_{D\cap B(q(t),\eta_1)} udx \leq 2\pi - \frac{\epsilon}{2}.$$

Because of the continuity of $q(t)$, for this η_1 there exists T_1 such that $t > T_1$ implies $\mid q(t) - q \mid< \frac{\eta_1}{2}$. Since $B(q, \frac{\eta_1}{2}) \subset B(q(t), \eta_1)$,
$t > \max\{T_0, T_1\} =: T_2$ implies

$$\int_{D\cap B(q,\frac{\eta_1}{2})} udx \leq 2\pi - \frac{\epsilon}{2}.$$

That is

$$\int_{D\cap B(q,\frac{\eta}{2})} udx \leq 2\pi - \frac{\epsilon}{2} \text{ for any } \eta \in (0, \eta_1).$$

By using Brézis-Merle's inequality, given $t_0 \in (T_2, T)$ there exists $\eta_0 \in (0, \min\{\eta_1, \frac{1}{4}\})$ and $C = C(t_0, \epsilon, \eta) > 0(\eta \in (0, \eta_0))$ such that $\eta \in (0, \eta_0)$ implies

$$\int_{D\cap B(q,\frac{\eta}{6})} e^v dx \leq C \text{ for any } t \in [t_0, T].$$

This contradicts to (3.4). Therefore

$$\limsup_{t\to T}\int_{D\cap B(q(t),\eta)} udx \geq 2\pi \text{ for any } \eta > 0.$$

Put $p(t) = q(t)$.
(b)In case that there doesn't exist $q \in \partial D$ such that $q(t) \to q\ (t \to T)$.
Put

$$A := \{\gamma \in \partial D \mid \mathit{for\,any}\ T_0 < T\ \mathit{there\,exists}\ t \in (T_0, T)\ \mathit{such\,that}\ q(t) = \gamma\}.$$

For any $\gamma \in A$, by the definition of A and (3.4), we have

$$\limsup_{t\to T}\int_{D\cap B(\gamma,\epsilon)} e^v dx = \infty \text{ for any } \epsilon > 0. \qquad (3.5)$$

We suppose for this γ there exists η_1 such that

$$\limsup_{t\to T} \int_{D\cap B(\gamma,\eta_1)} udx < 2\pi.$$

Then there exists $\epsilon > 0$ such that

$$\limsup_{t\to T} \int_{D\cap B(\gamma,\eta_1)} udx \leq 2\pi - \epsilon,$$

and there exists T_0 such that $T_0 < t < T$ implies

$$\int_{D\cap B(\gamma,\eta_1)} udx \leq 2\pi - \frac{\epsilon}{2}.$$

That is

$$\int_{D\cap B(\gamma,\eta)} udx \leq 2\pi - \frac{\epsilon}{2} \text{ for any } \eta \in (0,\eta_1).$$

By using Brézis-Merle's inequality, given $t_0 \in (T_0, T)$ there exists $\eta_0 \in (0, \min\{\eta_1, \frac{1}{4}\})$ and $C = C(t_0, \epsilon, \eta) > 0 (\eta \in (0, \eta_0))$ such that $\eta \in (0, \eta_0)$ implies

$$\int_{D\cap B(\gamma,\frac{\eta}{3})} e^v dx \leq C \text{ for any } t \in [t_0, T].$$

This contradicts to (3.5). Therefore

$$\limsup_{t\to T} \int_{D\cap B(\gamma,\eta)} udx \geq 2\pi \text{ for any } \eta > 0.$$

Put $p(t) = \gamma$.

Remark

1. We use Proposition1(2) with $a = 1$ to prove Theorem. But using $a > \dfrac{M + \sqrt{M^2 - 4\pi M}}{2M}$, we can improve the constant 2π to a larger one in Theorem, which is now studying.
2. If $M = 4\pi$, then $W(t)$ is bounded from below by putting $a = 1$, $M = 4\pi$ in (2.1). So when this, it follows from [6] that lim sup can be changed to lim inf in Theorem.
3. Theorem is correct even if Ω is a simply connected bounded domain in $\mathbf{R}^2$ with smooth boundary.

References

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