数理解析研究所講究録
1128 巻 2000 年 172-180

Weak Solutions of Cahn-Hilliard Equations having Forcing Terms and Optimal Control Problems

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1 Mathematical setting of Cahn-Hilliard equations

Let Ω be an open bounded set of $\mathbf{R}^n$, $n=1,2,3$, with a smooth boundary $\Gamma=\partial\Omega$. We denote $Q=(0,T)\times\Omega, T>0$. We consider an initial-boundary value problem involving a scalar function $u=u(t,x),(t,x)\in Q$, and u satisfies

$$\text{(C-H)}\qquad \begin{cases} \dfrac{\partial u}{\partial t}+\gamma\Delta^2 u-\Delta f(u)=g \quad in \ \ Q \\ u(x,0)=u_0(x) \quad x\in\Omega \\ \dfrac{\partial u}{\partial\eta}=\dfrac{\partial\Delta u}{\partial\eta}=0 \ \ on\ \Gamma, \end{cases} \tag{1.1}$$

where $\gamma>0, u_0\in H^1(\Omega)\cap L^{2p}(\Omega), g\in L^2(0,T;L^2(\Omega))$ and $f(u)$ is a polynomial given by

$$f(s)=\sum_{j=1}^{2p-1}a_j s^j, \quad p\in\mathbf{N},\quad p\geq 1. \tag{1.2}$$

The equation in (C-H) was called the generalized Cahn-Hilliard equation and was proposed as a continuum model for the description of the dynamics of pattern formation in phase transition. Strictly speaking, the Cahn-Hilliard equation corresponds to the case where $p=2, f(u)=-\alpha u+\beta u^3,\alpha,\beta>0$. We denote by $F(s)$ the primitive of $f(s)$ vanishing at $u=0$,

$$F(s)=\sum_{j=2}^{2p}b_j s^j, \quad jb_j=a_{j-1},\quad 2\leq j\leq 2p, \tag{1.3}$$

and we assume that the leading coefficient of f (and F) is positive $a_{2p-1}=2pb_{2p}>0$.
For the mathematical setting of the problem (C-H), we introduce $H=L^2(\Omega)$ with nomal scalar product $(\cdot,\cdot)$ and norm $|\cdot|$ and V defined by

$$V=\{\phi\mid\phi\in H^2(\Omega),\frac{\partial\phi}{\partial\eta}=0\ on\ \Gamma\}. \tag{1.4}$$

Then V is a closed subspace of $H^2(\Omega)$ and is equipped with the norm $\|\phi\|_V=|\Delta\phi|_{L^2(\Omega)}$.

The weak formulation of (C-H) is obtained by multiplying (1.1) by a test function $v\in V$, integrating over Ω, and using the Green formula and the boundary condition. Thus we give a definition of a weak solution of (C-H) as follows. A scalar function u is said to be a weak solution of (C-H) if and only if $u \in \{\phi \mid \phi \in L^2(0,T;V) \cap L^{2p}(0,T;L^{2p}(\Omega)), \phi' \in L^2(0,T;V')\}$ and u satisfies

$$\begin{cases} \frac{d}{dt}(u,v) + \gamma(\Delta u, \Delta v) - (f(u), \Delta v) = (g,v), & \forall v \in V \cap L^q(\Omega) \\ u(0) = u_0 \end{cases} \tag{1.5}$$

in the sense of $\mathcal{D}'(0,T)$, where $\frac{1}{2p} + \frac{1}{q} = 1$.

The bilinear form a associated with (C-H) is defined by $a(\psi,\phi) = \gamma(\Delta\psi, \Delta\phi)$, $\forall \psi, \phi \in V$. This form is continuous on V but not coercive, and this difficulty will be overcome by using further properties of the equation. The operator $A \in \mathcal{L}(V,V')$ is defined through $(A\psi,\phi) = a(\psi,\phi)$, $\forall \phi, \psi \in V$. It's domain is given by

$$D(A) = \{v \mid v \in H^4(\Omega), \frac{\partial v}{\partial \eta} = \frac{\partial \Delta v}{\partial \eta} = 0 \ \ on \ \ \Gamma\}. \tag{1.6}$$

The existence and uniqueness result for (C-H) is proved in Temam [1] under the condition $g = 0$. In this paper, we give an improved proof of (C-H) having nonzero forcing functions $g \neq 0$. Based on the result we establish the existence of optimal controls for the associated the quadratic cost problem. For the cost problem, we derive the necessary conditions of optimality for the two types of obsevations.

2 Lemmas and a priori estimates

Lemma 1 *Assume that* $u_0 \in H^1(\Omega) \cap L^{2p}(\Omega)$ *and* $g \in L^2(0,T;L^2(\Omega))$. *Then the weak solution* u *of* (C-H) *satisfies*

$$J(u(t)) \leq J(u_0) + \frac{1}{2}C\|g\|_{L^2(0,T;L^2(\Omega))}, \quad \forall t \in [0,T],$$

where $J(u)$ *is a Lyapunov function given by*

$$J(u) = \frac{\gamma}{2}|\nabla u|^2 + \int_\Omega F(u)dx. \tag{2.7}$$

Proof: We can prove this lemma by introducing a solution G of the following Neumann problem

$$\begin{cases} -\Delta G = g & in \ Q \\ \frac{\partial G}{\partial \eta} = 0 & on \ \Gamma, \end{cases} \tag{2.8}$$

and calculating the derivative of $J(u)$ with respect to t.

Lemma 2 *For every* $\eta > 0$, $\{|\Delta u|^2 + \eta|u|^2\}^{\frac{1}{2}}$ *is a norm on* V *which is equivalent to the* $H^2(\Omega)$ *norm. Similarly,* $\{|\Delta^2 u|^2 + \eta|u|^2\}^{\frac{1}{2}}$ *is a norm on* $D(A)$ *which is equivalent to the* $H^4(\Omega)$ *norm.*

Lemma 3 *Assume that* $p = 2$ *when* $n = 3$ *and that* p *is arbitrary when* $n = 1$ *or* 2. *Then we have*

$$|\Delta f(u)|^2 \leq k(1 + |\Delta^2 u|^{2\sigma}), \quad 0 < \sigma < 1,$$

for some σ *and* k *independent of* $u \in H^2(\Omega)$.

For proofs of Lemma 2 and Lemma 3, we refer to Temam [1; p.154, p.161].

A priori estimates on $L^\infty(0,T;L^2(\Omega)), L^2(0,T;H^2(\Omega))$ and $L^\infty(0,T;L^{2p}(\Omega))$. First we give some useful inequalities. Note that the leading term of $f(s)$ is $2pb_{2p}s^{2p-1}$, that of $f(s)s$ is $2pb_{2p}s^{2p}$, that of $F(s)$ is $b_{2p}s^{2p}$ and that of $f'(s)$ is $2p(2p-1)b_{2p}s^{2p-2}$. Since $b_{2p}>0$, it is verified that there exist a constants c_1, $c_2=c_2(\varepsilon)$, c_3, c_4 such that

$$f(s)s \geq pb_{2p}s^{2p} - c_1, \quad \forall s\in \mathbf{R} \tag{2.9}$$

$$|f(s)| \leq \varepsilon b_{2p}s^{2p} + c_2(\varepsilon), \quad \forall s\in \mathbf{R} \tag{2.10}$$

$$\frac{1}{2}b_{2p}s^{2p} - c_3 \leq F(s) \leq \frac{3}{2}b_{2p}s^{2p} + c_3, \quad \forall s\in \mathbf{R} \tag{2.11}$$

$$f'(s) \geq b_{2p}s^{2p-2} - c_4, \quad \forall s\in \mathbf{R}, \tag{2.12}$$

where $\varepsilon>0$ is arbitrary. By substituting $v=u(t)$ to the equation in (1.5), we obtain

$$\frac{1}{2}\frac{d}{dt}|u|^2 + \gamma|\Delta u|^2 + (\nabla f(u), \nabla u) = (g,u). \tag{2.13}$$

From (2.13) it follows by (2.12) that for $\varepsilon>0$,

$$\frac{1}{2}\frac{d}{dt}|u|^2 + \gamma|\Delta u|^2 + b_{2p}\int_\Omega u^{2p-2}|\nabla u|^2dx \leq c_4|\nabla u|^2 + \frac{1}{\varepsilon}|g|^2 + \varepsilon|u|^2. \tag{2.14}$$

By the interpolation inequality $|\nabla u|^2 \leq c_1'|u|\|u\|_{H^2(\Omega)}$ and taking $\eta=1$ in Lemma 2, we can see $\|u\|^2_{H^2(\Omega)} \leq C(|\Delta u|^2+|u|^2) \leq C(|\Delta u|+|u|)^2$, so that $\|u\|_{H^2(\Omega)} \leq c_2'(|\Delta u|+|u|)$ if we take $c_2'=\sqrt{C}$. Set $c_3'=c_1'c_2'$. Then we have

$$|\nabla u|^2 \leq c_3'|u|(|\Delta u|+|u|) \leq \varepsilon(|\Delta u|+|u|)^2 + \frac{1}{\varepsilon}(c_3')^2|u|^2 \leq (\frac{(c_3')^2}{\varepsilon}+2\varepsilon)|u|^2 + 2\varepsilon|\Delta u|^2.$$

If we take ε such that $2c_4\varepsilon = \frac{\gamma}{2}$, then we obtain from (2.14) that

$$\frac{1}{2}\frac{d}{dt}|u|^2 + \frac{\gamma}{2}|\Delta u|^2 + b_{2p}\int_\Omega u^{2p}|\nabla u|^2dx \leq c_4(\frac{(c_3')^2}{\varepsilon}+2\varepsilon)|u|^2 + \frac{1}{\varepsilon}|g|^2 + \varepsilon|u|^2.$$

Since $b_{2p}\int_\Omega u^{2p}|\nabla u|^2dx > 0$, by setting $c_5 = 2c_4(\frac{(c_3')^2}{\varepsilon}+2\varepsilon)+2\varepsilon$, and $c_6=\frac{2}{\varepsilon}$, we have

$$\frac{d}{dt}|u|^2 + \gamma|\Delta u|^2 \leq c_5|u|^2 + c_6|g|^2. \tag{2.15}$$

By applying Bellman-Gronwall inequality, we derive from (2.15)

$$|u(t)|^2 \leq |u_0|^2exp(c_5T) + c_6exp(c_5T)\|g\|_{L^2(0,T;L^2(\Omega))}, \quad \forall\, t\in[0,T].$$

Therefore $u\in L^\infty(0,T;L^2(\Omega))$. Similarly, by integrating both sides of (2.15), we have

$$\gamma\int_0^T |\Delta u|^2dt \leq |u_0|^2 + c_5\|u\|^2_{L^2(0,T;H)} + c_6\|g\|^2_{L^2(0,T;L^2(\Omega))}.$$

This implies $u\in L^2(0,T;H^2(\Omega))$. On the other hand, by Lemma 1 and (2.11), we obtain

$$\frac{1}{2}b_{2p}\int_\Omega u^{2p}dx \leq c_3|\Omega| + J(u_0) + \frac{1}{2}C\|g\|_{L^2(0,T;L^2(\Omega))}, \quad \forall t\in[0,T]. \tag{2.16}$$

Note that $J(u_0)$ is finite by $u_0 \in H^1(\Omega)\cap L^{2p}(\Omega)$. Then,from (2.16) we have $u \in L^\infty(0,T;L^{2p}(\Omega))$.

A priori estimates on $L^\infty(0,T;H^2(\Omega))$ and $L^2(0,T;H^4(\Omega))$.

Assume that $u_0 \in H^2(\Omega)$. Substituting $v = \Delta^2 u \in \mathcal{D}(A) \subset H^4(\Omega)$ to (1.5), this yields

$$(\frac{du}{dt}, \Delta^2 u) + \gamma(\Delta^2 u, \Delta^2 u) = (\Delta f(u), \Delta^2 u) + (g, \Delta^2 u).$$

Then by using Schwartz inequality we can derive

$$\frac{1}{2}\frac{d}{dt}|\Delta u|^2 + \frac{\gamma}{3}|\Delta^2 u|^2 \le \frac{1}{2\gamma}|\Delta f(u)|^2 + \frac{3}{2\gamma}|g|^2. \tag{2.17}$$

By Lemma 3 and using Young inequality, we can deduce $|\Delta f(u)|^2 \le \frac{\gamma^2}{2}|\Delta^2 u|^2 + k_1'$ for some $k_1' > 0$. Set $k_2 = \dfrac{k_1'}{2\gamma}$. Then it follows from (2.17) that

$$\frac{d}{dt}|\Delta u|^2 + \frac{\gamma}{12}|\Delta^2 u|^2 \le k_2 + \frac{3}{2\gamma}|g|^2. \tag{2.18}$$

Now we integrate the both sides of (2.18) to obtain

$$|\Delta u(t)|^2 + \frac{\gamma}{12}\int_0^T |\Delta^2 u|^2 dt \le |\Delta u_0|^2 + k_2 T + \frac{3}{2\gamma}\|g\|^2_{L^2(0,T;L^2(\Omega))}.$$

Since $u_0 \in H^2(\Omega)$, this implies $u \in L^\infty(0,T;H^2(\Omega)) \cap L^2(0,T;H^4(\Omega))$.

Lemma 4 *The following two statements hold true:*

$$\text{(A):}\quad \text{If } u_0 \in H_0^1(\Omega)\cap L^{2p}(\Omega), \text{ then } f(u) \in L^2(0,T;L^2(\Omega)) \text{ for} \begin{cases} n=1, & p \le 5 \\ n=2, & p \le 3 \\ n=3, & p \le 2 \end{cases}$$

(B): If $u_0 \in V \subset H^2(\Omega)$, then $f(u) \in L^2(0,T;L^2(\Omega))$ for all $p \ge 1$ and $n = 1,2,3$.

Proof of (A). Since $u_0 \in H_0^1(\Omega) \cap L^{2p}(\Omega)$, we have by the above a priori estimates

$$u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;L^{2p}(\Omega)).$$

Now we recall the Gagliardo-Nirenberg inequality:

$$\left(\int_\Omega |\phi|^{\frac{2q}{q-1}} dx\right)^{\frac{4(q-1)}{n}} \le C\left(\int_\Omega |\phi|^2 dx\right)^s |\Delta\phi| , \qquad s = \frac{4q}{n} - 1 . \tag{2.19}$$

Since $u \in L^\infty(0,T;L^2(\Omega))$, then $(\int_\Omega |u(t)|^2 dx)^s < +\infty$, $\forall t \in [0,T]$, so that by (2.19), we have

$$\int_\Omega |u(t)|^{\frac{2q}{q-1}} dx \le C'\|u(t)\|_V^{\frac{n}{4(q-1)}}, \qquad \forall t \in [0,T] . \tag{2.20}$$

Since $u \in L^2(0,T;H^2(\Omega))$, then $\int_0^T\int_\Omega |\Delta u|^2 dxdt < +\infty$. Hence if $n \le 8(q-1)$, then

$$\int_0^T\int_\Omega |u|^{\frac{2q}{q-1}} dxdt \le C'\int_0^T \|u\|_V^2 dt < +\infty . \tag{2.21}$$

And if we take $q = 1 + \frac{1}{l}$, the inequlity (2.21) implies

$$\int_0^T \int_\Omega |\phi|^{2l+2} dxdt \le C' \|\phi\|^2_{L^2(0,T;V)} < +\infty . \tag{2.22}$$

Since $|f(u)|^2 \approx u^{4p-2}$, we have :

$$\begin{aligned}
n = 1 \Rightarrow \quad & l \le 8 \Rightarrow \|u\|_{L^{18}(0,T;L^{18}(\Omega))} < +\infty \quad \Rightarrow 4p - 2 \le 18 \Rightarrow p \le 5; \\
n = 2 \Rightarrow \quad & l \le 4 \Rightarrow \|u\|_{L^{10}(0,T;L^{10}(\Omega))} < +\infty \quad \Rightarrow 4p - 2 \le 10 \Rightarrow p \le 3; \\
n = 3 \Rightarrow \quad & l \le 2 \Rightarrow \|u\|_{L^{6}(0,T;L^{6}(\Omega))} < +\infty \quad \Rightarrow 4p - 2 \le 6 \Rightarrow p \le 2.
\end{aligned}$$

Thus we can prove the assertion $f(u) \in L^2(0,T;L^2(\Omega))$ under the condition on n and p.
Proof of (B). Since $u_0 \in V \subset H^2(\Omega)$, we have $u \in L^\infty(0,T;V)$ by the second a priori estimates. Since $f(u) \approx u^{2p-1}$, it is sufficient to prove $\int_0^T \int_\Omega (u^{2p-1})^2 dxdt < +\infty$ for the assertion $f(u) \in L^2(0,T;L^2(\Omega))$. By Sobolve embedding theorem, we have :

$$H^2(\Omega) \hookrightarrow C^{1,\frac{1}{2}}(\Omega), \quad n = 1; \tag{2.23}$$
$$H^2(\Omega) \hookrightarrow C^{1,0}(\Omega), \quad n = 2; \tag{2.24}$$
$$H^2(\Omega) \hookrightarrow C^{0,\frac{1}{2}}(\Omega), \quad n = 3. \tag{2.25}$$

This implies $\|\phi\|_{C^0} \le C\|\phi\|_V$, $\forall \phi \in H^2(\Omega)$. Hence using this inequality we reach

$$\int_0^T \int_\Omega |f(u)|^2 dxdt \le \int_0^T \int_\Omega (2b_{2p}^2 |u|^{4p-2} + 2c_4^2) dxdt \le 2b_{2p}^2 \|u\|^{4p-2}_{L^\infty(0,T;H^2(\Omega))} + 2Tc_4^2 < +\infty.$$

This means that $f(u) \in L^2(0,T;L^2(\Omega))$.

3 Uniqueness and existence of weak solutions

Theorem 1 *Assume that* $g \in L^2(0,T;L^2(\Omega))$.

1. *For the initial value* $u_0 \in H^1(\Omega) \cap L^{2p}(\Omega)$, *suppose that* $\begin{cases} p \le 5, \ n = 1 \\ p \le 3, \ n = 2 \\ p \le 2, \ n = 3 \end{cases}$. *Then the problem* (C-H) *possesses a unique weak solution* u *which belongs to*

$$\mathcal{C}([0,T];H) \cap L^2(0,T;V) \cap L^\infty(0,T;L^{2p}(\Omega)), \quad \forall T > 0 . \tag{3.1}$$

2. *For* $u_0 \in V$, *suppose that* $\begin{cases} p = 2, n = 3 \\ p \ arbitrary, n = 1,2 \end{cases}$. *Then the problem* (C-H) *possesses a unique weak solution* u *satisfying*

$$u \in \mathcal{C}([0,T], V) \cap L^2(0,T;D(A)), \quad \forall T > 0 \tag{3.2}$$

Proof: We use the orthonormal basis of H consisting of the eigenvectors of A, which is denoted by $\{w_j\}_{j=1}^{\infty}$. Then we have $Aw_j = \lambda_j w_j$, $\forall j$ and we implement the Faedo-Galerkin method with these functions. For each integer m, we look for an approximate solution u_m of the form $u_m(t) = \sum_{i=1}^{m} g_{im}(t)w_i$ satisfying the following m-th order equation of a vector $g_{jm}(t)$:

$$\begin{cases} (\frac{du_m}{dt}, w_j) + (\Delta^2 u_m(t), w_j) - (\Delta f(u_m(t)), w_j) = (g, w_j), \ t \in [0,T] \quad j = 1, 2, ..., m \\ u_m(0) = u_{0m}, \end{cases} \tag{3.3}$$

where $u_{0m} = \sum_{j=1}^{m}(u_0, w_j)w_j$ is the orthogonal projection in H of u_0 onto the space spanned by $w_1, w_2, ...w_m$. The existence of u_m on some interval $[0, T_m]$ follows from the standard theorem on the existence of solutions of ordinary differential equations. That $T_m = T$ is a consequence of the following a priori estimates. Applying the above priori estimates to $u_m(t)$, we have that u_m is bounded independently of m in the spaces

$$L^\infty(0,T;H) \cap L^2(0,T;V) \cap L^\infty(0,T;L^{2p}(\Omega)). \tag{3.4}$$

By the weak compactness of these spaces, we find a subsequence $\{u_{mk}\}$ of $\{u_m\}$ and a u in $L^\infty(0,T;H) \cap L^2(0,T;V) \cap L^\infty(0,T;L^{2p}(\Omega))$ such that

$$u_{mk} \to u \ \ in\ L^2(0,T;V) \cap L^{2p}(0,T;L^{2p}(\Omega))\ weakly \tag{3.5}$$

$$u_{mk} \to u \ \ in\ L^\infty(0,T;H)\ weakly *. \tag{3.6}$$

For simplity of notations we rewrite u_{mk} by u_m. By Lemma 4, we know $f(u_m) \in L^2(0,T;L^2(\Omega))$ for some n and p in Lemma 4. So we can deduce $\Delta f(u_m) \in L^2(0,T;V')$. By $u_m' = g - \Delta^2 u_m + \Delta f(u_m)$, we use that $\{u_m'\}$ is bounded in $L^2(0,T;V')$. Therefore $u_m \in W(0,T;V,V')$. Since the embedding $V \hookrightarrow H$ is compact, by the Aubin compactness theorem, we can have

$$u_m \to u \ \ in\ L^2(0,T;L^2(\Omega))\ \ strongly. \tag{3.7}$$

Hence we can deduce $u_m \to u$ *a.e. in* Q by taking a subsequence of $\{u_m\}$ if necessary. Since $f(u_m)$ is a polynomial, it is easy to see that

$$f(u_m) \to f(u)\ \ a.e.\ in\ Q. \tag{3.8}$$

From Lemma 4, we have

$$\int_0^T \int_\Omega |f(u_m) - f(u)| dxdt \le 2\|f(u_m)\|^2_{L^2(0,T;L^2(\Omega))} + 2\|f(u)\|^2_{L^2(0,T;L^2(\Omega))} < +\infty \tag{3.9}$$

By the Lebesgue dominated convergence theorem, we have from (3.8) and (3.9)

$$\lim_{m\to\infty} \int_0^T \int_\Omega |f(u_m) - f(u)|^2 dxdt \le \int_0^T \int_\Omega \lim_{m\to\infty} |f(u_m) - f(u)|^2 dxdt = 0 \tag{3.10}$$

That is,

$$f(u_m) \to f(u)\ in\ L^2(0,T;H)\ \ strongly. \tag{3.11}$$

This implies

$$\int_0^T \int_\Omega (f(u_m) - f(u))\phi dxdt \leq (\int_0^T \int_\Omega |f(u_m) - f(u)|^2 dxdt)^{\frac{1}{2}} \|\phi\|_{L^2(0,T;H)} \to 0$$

for all $\phi \in L^2(0,T;H)$. Set $\phi = \psi(t) \otimes \Delta v, \psi(t) \in L^2(0,T), \forall v \in V \subset H^2(\Omega)$, then we have

$$\int_0^T \int_\Omega |f(u_m) - f(u)|\Delta v \psi(t) dxdt \to 0, \quad \forall v \in V \cap L^q(\Omega). \tag{3.12}$$

We can rewrite (3.3) as

$$\langle \frac{du_m}{dt}, v\rangle + (\Delta u_m(t), \Delta v) - (f(u_m(t)), \Delta v) = (g, v), \quad \forall v \in V \cap L^q(\Omega) \tag{3.13}$$

We pass to the limit in (3.13) and use (3.12) to find that

$$\langle \frac{du}{dt}, v\rangle + (\Delta u(t), \Delta v) - (f(u(t)), \Delta v) = (g, v), \quad \forall v \in V \cap L^q(\Omega). \tag{3.14}$$

That $u(0) = u_0$ follows from $u_m(0) \to u_0$ in $H^2(\Omega)$. This proves the existence of a weak solution u to (C-H). We omit the proof of uniqueness. The regularity results for the solution u are proved by a priori estimates given above.

4 Optimal control problems

In this section we study the optimal control problems associated with (C-H) (cf. Yong and Zheng [2]). Let $\mathcal{U}$ be a Hilbert space of control variables. $B \in \mathcal{L}(\mathcal{U}, L^2(0,T;L^2(\Omega)))$ is called the controller. We consider the following control system.

$$\begin{cases} \frac{\partial y}{\partial t} + \gamma\Delta^2 y - \Delta f(y) = Bv \quad in \ Q \\ y(x,0) = y_0(x) \quad x \in \Omega \\ \frac{\partial y}{\partial n} = \frac{\partial \Delta y}{\partial n} = 0 \quad on \ \Gamma \end{cases} \tag{4.1}$$

Here in (4.1), it is assume that $v \in \mathcal{U}$ and $y_0 \in H^1(\Omega) \cap L^{2p}(\Omega)$. We define the space $Y = L^2(0,T;V) \cap L^{2p}(0,T;L^{2p}(\Omega))$ and its dual space $Y' = L^2(0,T;V') \cap L^q(0,T;L^q(\Omega))$, where $\frac{1}{2p} + \frac{1}{q} = 1$. By virtue of Theorem 1, we can define the solution map $v \to y(v)$ of $\mathcal{U}$ into $W(0,T;Y,Y')$. The solution $y(v)$ is called the state of the control system (4.1). The observation of the state is assumed to be given by $z(v) = Cy(v)$, where $C \in \mathcal{L}(W(0,T;Y,Y'),M)$ is an operator called the observer, and M is a Hilbert space of observations. The cost function associated with the control system (4.1) is given by

$$J(v) = \|Cy(v) - z_d\|_M^2 + (Nv, v)_{\mathcal{U}} \text{ for all } v \in \mathcal{U}, \tag{4.2}$$

where $z_d \in M$ is a desired value of $z(v)$ and $N \in \mathcal{L}(\mathcal{U})$ is symmetric and non-negative, i.e.,

$$(Nv, v)_{\mathcal{U}} = (v, Nv)_{\mathcal{U}} \geq \kappa\|v\|_{\mathcal{U}}^2, \tag{4.3}$$

for some $\kappa > 0$. Let $\mathcal{U}_{ad}$ be a closed convex subset of $\mathcal{U}$, which is called the admissible set. The quadratic cost optimal control problem subject to (4.1) and (4.2) is:

(i) Find an element $u \in \mathcal{U}_{ad}$ such that

$$\inf_{v\in\mathcal{U}_{ad}} J(v) = J(u); \tag{4.4}$$

(ii) Give a characterization of such u.

We shall call u the optimal control for the optimal control problem. We can prove the following existence theorem on the optimal control for the system (4.1).

Theorem 2 *Suppose that $\mathcal{U}_{ad}$ is bounded or $\kappa > 0$. Then there exists at least one optimal control $u \in \mathcal{U}_{ad}$ for* (4.1) *with* (4.2).

Next we consider the problem (ii). It is well known that the optimality condition for u is given by the variational inequality

$$J'(u)(v-u) \geq 0 \ \text{ for all } \ v \in \mathcal{U}_{ad}, \tag{4.5}$$

where $J'(u)$ denotes the Gateaux derivative of $J(v)$ in (4.2) at $v = u$.

The following theorem is essential in deriving necessary optimality conditions.

Theorem 3 *Assume that all the conditions of Theorem* 1 *hold. Then the map $v \to y(v)$ of $\mathcal{U}$ into $W(0,T;Y,Y')$ is weakly Gateaux differentiable at $v = u$ and such the Gateaux derivative of $y(v)$ at $v = u$ in the direction $v - u \in \mathcal{U}$, say $z = Dy(u)(v-u)$, is a unique weak solution of the following equation*

$$\begin{cases} \dfrac{\partial z}{\partial t} + \gamma\Delta^2 z = \Delta(f'(y(u))z) + B(v-u) \quad in \ \ Q \\ z(0) = 0 \quad x \in \Omega \\ \dfrac{\partial z}{d\eta} = \dfrac{\partial \Delta z}{\partial \eta} = 0 \quad on \ \ \Gamma \end{cases} \tag{4.6}$$

By calculating the Gateaux derivative of (4.2) via Theorem 3, we see that the cost $J(v)$ is weakly Gateaux differentiable at u in the direction $v - u$. Then the optimality condition (4.5) can be rewritten as

$$\langle C^*\Lambda_M(Cy(u) - z_d), Dy(u)(v-u)\rangle_{W(0,T;Y,Y')',W(0,T;Y,Y')} + (Nu, v-u)_{\mathcal{U}} \geq 0, \quad \forall v \in \mathcal{U}_{ad}, \tag{4.7}$$

where Λ_M is the cannonical isomorphism from M to M'.

Now we study the necessary conditions of optimality. To avoid the complexity of observation states, we consider the two types of distributive and terminal value observations.

1. Case of $C \in \mathcal{L}(L^2(0,T;V), M)$. In this case, the cost function is given by

$$J(v) = \|Cy(v) - z_d\|_M^2 + (Nv, v)_{\mathcal{U}}, \quad \forall v \in \mathcal{U}. \tag{4.8}$$

Then it is easily verified that the optimality condition (4.7) is written as

$$\int_0^T \langle C^*\Lambda_M(Cy(u;t) - z_d(t)), z\rangle_{V',V} dt + (Nu, v-u)_{\mathcal{U}} \geq 0, \quad \forall v \in \mathcal{U}_{ad}, \tag{4.9}$$

where u is the optimal control for (4.8), z is the solution of (4.6), Λ_M is isomorphism map from M to M'.

Theorem 4 *Assume that all conditions of Theorem* 1 *hold. Assume further that* C *satisfy* $C \in \mathcal{L}(L^2(0,T;V),M)$. *Then the optimal control* $u \in \mathcal{U}_{ad}$ *for* (4.1) *with* (4.8) *is characterized by the following system of equations and inequality:*

$$\begin{cases} \dfrac{\partial y}{\partial t} + \gamma\Delta^2 y - \Delta f(y) = Bu \quad in \ Q \\ y(x,0) = y_0(x) \quad x \in \Omega \\ \dfrac{\partial y}{\partial n} = \dfrac{\partial \Delta y}{\partial n} = 0 \quad on \ \Gamma \end{cases} \tag{4.10}$$

$$\begin{cases} -\dfrac{\partial p}{\partial t} + \gamma\Delta^2 p(u) = f'(y(u))\Delta p(u) + C^*\Lambda_M(Cy(u) - z_d) \ in \ Q \\ p(u;T,x) = 0 \quad x \in \Omega \\ \dfrac{\partial p}{\partial \eta} = \dfrac{\partial \Delta p}{\partial \eta} = 0 \quad on \ \Gamma \end{cases} \tag{4.11}$$

$$\int_0^T \int_\Omega B^*p(t,x)(v-u)dxdt + (Nu, v-u)_{\mathcal{U}} \geq 0, \quad \forall v \in \mathcal{U}_{ad}. \tag{4.12}$$

2. Case of $C \in \mathcal{L}(H,M)$. In this case, we observe $z(v) = Cy(v;T)$. The associsted cost function is expressed as

$$J(v) = \|Cy(v;T) - z_d\|_M^2 + (Nv,v)_{\mathcal{U}}, \quad v \in \mathcal{U}_{ad}. \tag{4.13}$$

Then the optimal control u for (4.13) is characterized by

$$(C^*\Lambda_M(Cy(u;T) - z_d), Cz(T))_{V',V} + (Nu, v-u)_{\mathcal{U}} \geq 0, \quad \forall v \in \mathcal{U}_{ad}, \tag{4.14}$$

where z is the solution of equation (4.6).

Theorem 5 *Assume that all conditions in Theorem* 1 *hold. Then the optimal control* u *for* (4.1) *with* (4.13) *is characterized by the following system of equations and inequality:*

$$\begin{cases} \dfrac{\partial y}{\partial t} + \gamma\Delta^2 y - \Delta f(y) = Bu \quad in \ Q \\ y(x,0) = y_0(x) \quad x \in \Omega \\ \dfrac{\partial y}{\partial n} = \dfrac{\partial \Delta y}{\partial n} = 0 \quad on \ \Gamma \end{cases} \tag{4.15}$$

$$\begin{cases} -\dfrac{\partial p}{\partial t} + \gamma\Delta^2 p(u) = f'(y(u))\Delta p(u) \quad in \ Q \\ p(u;T,x) = C^*\Lambda_M(Cy(T,u) - z_d) \quad x \in \Omega \\ \dfrac{\partial p}{\partial \eta} = \dfrac{\partial \Delta p}{\partial \eta} = 0 \quad on \ \Gamma \end{cases} \tag{4.16}$$

$$\int_0^T \int_\Omega B^*p(t,x)(v-u)dxdt + (Nu, v-u)_{\mathcal{U}} \geq 0, \quad \forall v \in \mathcal{U}_{ad}. \tag{4.17}$$

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