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Nonlinear Integral Equation and The Application for Trasmission System of AIDS
非線型積分方程式の応用

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In this paper we consider the differential equations on the models for transmission of AIDS. At first we shall treat the basic equation which was proposed by May, Anderson and Mclean on 1988.

$$\frac{dX}{dt} = B - (\lambda + \mu)X,$$
$$\frac{dY}{dt} = \lambda X - (v + \mu)Y, \tag{1}$$
$$\frac{dN}{dt} = B - \mu N - vY, (N = X + Y)$$

where N;total population,
X;suscepitibules,
Y;infecteds,
B; birth process,
λ ; force of infection,
λ is defined by the following equation,

$$\lambda = \frac{\beta c Y}{N}$$

v ; disease-related death rate
μ ; death rate related all other causes
β ; probability of acquiring infection from any one infected partner,
c ; average rate of acquireing partners.

The net birth rate B is given by,

$$B = \nu(N - (1-\epsilon)Y),$$

ν is theper capita birth rate in the absence of infection and ϵ is the fraction of all offspring born infected mothers who survive. By subsutituting this birth process these assumptions yield the following pair of differetial equations.

$$\frac{dN}{dt} = N((\nu - \mu) - (v + (1-\epsilon)\nu)\frac{Y}{N}),$$
$$\frac{dY}{dt} = Y((\beta c - \mu - v) - \beta c\frac{Y}{N})$$

This equation is seemed to be little complex but we can solve this by getting the function $\frac{Y}{N}$ explicitly and using the logistic curve. In this model (1) birth rate and mortarilty are all constants. This assumptions are convenient when we consider the case which is occured in the developing coutries. On the other hand, in the case for tne advanced countries and for the long time prediction we must consider the age-depended parameters. By appling the age-dependent population equation we can make the age-dependent model for transmission of AIDS. This model is expressed by the first order partial differential equations.

$$\begin{aligned} \frac{\partial X}{\partial a} + \frac{\partial X}{\partial t} &= -[\lambda(a,t) + \mu(a)]X, \\ \frac{\partial Y}{\partial a} + \frac{\partial Y}{\partial t} &= \lambda X - [v + \mu(a)]Y, \\ \frac{\partial N}{\partial a} + \frac{\partial N}{\partial t} &= -\mu(a)N - vY, \end{aligned} \tag{2}$$

where X, Y, N mean the disribution of susceptibles infecteds and populations, respectively, at time t. Hence,

$$\int_0^\infty X(a,t)da, \int_0^\infty Y(a,t)da, \int_0^\infty N(a,t)da, \tag{3}$$

express the total number of susceptibles, infecteds and population, respectively. The birth process is defined the next expression,

$$B(t) = \int_0^\infty m(a)[N(a,t) - (1-\epsilon)Y(a,t)]da.$$

In this case we do not consider the vartical transmission, then we define the initial data as follows.

$$X(0,t) = N(0,t) = B(t), Y(0,t) = 0, \tag{4}$$

The initial data for $X(a,0), Y(a,0), N(a,0)$ we shall put the real distribution whick we can get from the fieldwork. The parmeter λ is also the most important number of this age-dependent models. Solving methods are essentially depend on the type of λ. In general λ is given by,

$$\lambda(a,t) = \beta c \frac{\int p(a,a')Y(a,a')da'}{\int p(a,a')N(a,a')da'.} \tag{5}$$

β, c are same as in the first model(1). The function $p(a,a')$ defines the probability that a susceptible of age a will choose a partner of age a'. Precisely, the shape of the function $p(a,a')$ decides the treating method of this partial differential equation model There are two extreme cases.
Case(A);susceptibles will choose only the same age poeple,
Case(B);sexually active adults choose partners inedependent of age, that is, the function $p(a,a')$ has a constant value.
In the case(A), we must recogaize that λ is define $p = \delta$. That is,

$$\lambda(a,t) = \beta c \frac{Y(a,t)}{N(a,t)}.$$

In this case, the partial differential equation can be transfer to the linear integral equation with the convolutional kernel by appling the method of the linear population models.

$$\begin{aligned} B(t) &= \int_0^\infty m(a)B(t-a)\pi(a,0)da \\ &+ \int_t^\infty m(a)\phi(a-t)\pi(a,a-t)exp[\int_0^t -vZ(s)ds]da \\ &- (1-\epsilon)\int_t^\infty m(a)Z(t)\phi(a-t)\pi(a,a-t)exp[\int_0^t vZ(s)ds]da, \quad (6) \end{aligned}$$

where

$$X(a,0),\ N(a,0) = \phi(a), \tag{7}$$

$$\pi(b,a) = exp[-\int_a^b \mu(s)ds]. \tag{8}$$

The initial distribution of the population is defined the by ϕ, and the function π expresses the probability which one person of age a can alive until age b. Only the first term include tne unknown function B, the model can be treated as the linear Volterra integral equation of the second kind.

$$B(t) = \int_0^t m(a)\pi(a,0)B(t-a)da + F(t). \tag{9}$$

Then the standerd method of the linear integral equation with the convolutional kernel bring the conclusion about the existence of the solution, uniquness and the asymptotic behavior as time goes to the infinity. We can calculate not only the qualitative property but also the quantitative measure. In the integral equation we can see the function Z, this function is a kind of the logistic curve and is appeared when the model(2) is solved along the characteristic curve of this equation.

For the case (B), if we calculate the solution of the first order partial differential(2) along the characteristid curve, we cannot have the integral equation. The main reason of this fact is that the power of infection λ includes the functional of the distribution $Y(a,t), N(a,t)$. The following equation is reduced along the characteristic line.

$$U_C'(t) = \lambda(a,t)U_C(t) - [\lambda(a,t) + v + \mu(a)]W_C(t), \quad (10)$$

$$W_C'(t) = -vU_C(t) - \mu(a)W_C(t), \quad (11)$$

where,

$$W_C(t) = N(t+C,t), U_C(t) = Y(t+C,t).$$

With the assumption that λ, μ are Lipshitz continuous, we can prove that the equation has only one solution on the real line. But we cannot get more informations from this method. Recently some auther could established the proof of the existence of the periodic solutions for the nonlinear populational problems with the semigoup theory. There is possibility that we can apply this method for our models. It will be clear in the future obserbation.

The analysing method for the model(2) with the assumption (B) can be apply for the nonlinear model of the population problem, because in the case (B) the parameter include the functional of the distribution. The paper of Gurtin and MacCamy did the epoc making, before this method was used in the theory of the epidemic models. This paper is the first one in which the population problems were treated under the rather general assumption about the total number of population. The following equation is the prptotype of the nonlinear population problem.

$$\begin{aligned} &\frac{\partial n}{\partial a} + \frac{\partial n}{\partial t} + \mu(a, N(t))n(a,t) = 0, \quad a > 0, 0 < t < T \\ &n(0,t) = \int_0^\infty m(a, N(t))n(a,t)da, \quad 0 < t \leq T, \\ &n(a,0) = \varphi(a), \quad a \geq 0. \end{aligned} \quad (12)$$

where n is the distribution of the population and N is the total number of the population, that is,

$$N(t) = \int_0^\infty n(a,t)da. \quad (13)$$

As in the previous case the birth process B satisfies the equation,

$$B(t) = n(0,t).$$

For we considering the population model, $\varphi \in L^1(R_+), \mu(a,N), m(a,N)$ are all nonnegative function. Especially μ, m have the integral term of n, so μ, m are the functional of n. In the paper of Gurtin, MacMamy they putted the hypotheses on μ, m that those functional have the continuous patial derivative with respect to N. We can remove this assumption instead of the Lipshitz continuous. Then we can get the same theorem with Gurtin and MacCamy under the following two assumptions, that is, under these assumption there exists only one positive solution $n(a,t)$ for the equaton(12).
$(H1)$ φ is piecewise continuous,
$(H2)$ $\mu, m \in C(R^+ \times R^+)$ and with respect to N these functional are unifomly Lipshitz continuous.
The proof for this theorem is similar as the proof of the case for the equation(2). The integral equation along the characteristic line is following.

$$N(t) = \int_0^t K(t-a;t;N)B(a)da + \int_0^\infty L(a,t;N)\varphi(a)da,$$
$$B(t) = \int_0^t m(t-a, N(t))K(t-a,t;N)B(a)da + \int_0^\infty m(t+a, N(t))L(a,t;N)\varphi(a)da,$$
$$K(\alpha,t;N) = exp(-\int_{t-a}^t \mu(\alpha+\tau-t, N(\tau))d\tau),$$
$$L(\alpha,t;N) = exp(-\int_0^t \mu(\tau+\alpha, N(\tau))d\tau).$$

By using iterational method, that is, using Banach contraction method, we can prove the exisetence of the unique solution on the nonnegative real half line. From making process for this equation it is so difficult to observe the qualitative proparty of the solution. Recently under the special hypotheses it proved that the excicetence of the periodic solution of the equation (12). There are many problems upon this nonlinear populational problems.

参考文献

[1] M.E.Gurtin and R.C.MacCamy(1974),Non-linear age-dependent population dynamics,Archive for Rational Mechanics and Analysis 54:281-300