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Unconventional DDM&parallel method for fluid computation

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A new finite element Domain decomposition method, which is based on a point-by-point scheme, domain decomposition method (DDM) and the matrix-storage free formulation, is developed and implemented to the model equation for Navier-Stokes equations, convection-diffusion equation. Numerical experiments demonstrated that the proposed method is efficiently to solve the model equation.

Key words: Domain Decomposition Method, Locally Implicit Finite Element Scheme, Point by Point, Stiffness Matrix free , Convection-Diffusion Equation

1. Introduction

The finite element method (FEM) has been widely used in engineering science computing for solving different equations arisen from many field such as structure, heat and fluid etc..The excellent properties of FEM compared with the finite difference method are its applicability to an unstructured mesh, simplicity for managing boundary conditions and etc.. But , the general FEM should create the stiffness matrix, and there are some strict limits to create the stiffness matrix. The defects of FEM are larger storage requirements and higher CPU cost than the finite difference method and would especially become a barrier when handling large scale problems.

Domain decomposition method (DDM) is interesting for several reasons. One is the possibility to use different physical models on the subdomains in order to get a more accurate modelization. It also reduces complex geometry to simpler subregions. Last but not least, as this method is easily parallelizable, we can take advantage of parallel computers.

The locally implicit finite element scheme is a unique kind of finite element method. The distinct feature of this scheme, comparing with the general FEM, is not need to build the stiffness matrix and the numerical computational procedure carried with point-by-point scheme, which provides us with more space to improve numerical computation procedure and more flexibility in applications. on the other hand, Theoretical analysis and numerical simulations show that the proposed locally implicit finite element scheme has good stability.

2. Governing Equation and Discretization

we discuss the following model equation, convection-diffusion equation, for Navier-Stokes :

$$\frac{\partial u}{\partial t}+\frac{\partial(\alpha u)}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2}\quad(0\le x\le L)\qquad(2.1)$$

with the time initial condition is,

$$u(x,0)=H(x)\qquad(2.2)$$

and boundary conditions

$$\begin{aligned}&u(0,t)=a,u(L,t)=b\\&u'(0,t)=0,u'(L,t)=0\end{aligned}\qquad(2.3)$$

a, b are two constants.

Assume [0, L] is divided uniformly by M_1+M_2 points $0=x_0\le x_1\le\ldots\le x_{M_1+M_2-1}=L$, $\Delta x=x_i-x_{i-1}$ let

$$\psi_1=1-\xi,\quad \psi_2=\xi\quad(0\le\xi\le1)\qquad(2.5)$$

For every interval $[x_{j-1}, x_j]$, the coordinate transformation is

$$x=x_{j-1}\psi_1+x_j\psi_2\qquad(2.6)$$

Let φ_j be the shape function responding to point x_j,

$$\begin{aligned}u_j&=u(x_{j-1},t)\psi_1+u(x_j,t)\psi_2\qquad\forall x\in[x_{j-1},x_j]\\u_j&=u(x_j,t)\psi_1+u(x_{j+1},t)\psi_2\qquad\forall x\in[x_j,x_{j+1}]\end{aligned}\qquad(2.7)$$

then the finite element approximation problem of (2.1) is

$$\frac{\partial}{\partial t}\int_0^L u_j\varphi_j dx-\int_0^L(\alpha u_j-\nu\frac{\partial u_j}{\partial x})\frac{\partial\varphi_j}{\partial x}dx=0\qquad(2.8)$$

After integration, we have

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{1}{6}u_{j+1}+\frac{2}{3}u_j+\frac{1}{6}u_{j-1}\right)+\left(\frac{\alpha}{2\Delta x}\right)\left(u_{j+1}-u_{j-1}\right)\\&-\left(\frac{\nu}{\Delta x^2}\right)\left(u_{j+1}-2u_j+u_{j-1}\right)=0\end{aligned}\qquad(2.9)$$

Discretizing the time by the Euler implicit scheme for every $[t_n, t_{n+1}]$, $\Delta t=t_n-t_{n-1}$,

$$\frac{1}{6}\left(u_{j+1}^{n+1}-u_{j+1}^{n}\right)+\frac{2}{3}\left(u_j^{n+1}-u_j^{n}\right)$$
$$+\frac{1}{6}\left(u_{j-1}^{n+1}-u_{j-1}^{n}\right)+\left(\frac{\alpha}{2\Delta x}\right)\left(u_{j+1}^{n+1}-u_{j-1}^{n+1}\right) \quad (2.10)$$
$$-\left(\frac{\nu}{\Delta x^2}\right)\left(u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}\right)=0$$

where $u_j^l = u(x_j, t_l)$

$u_j^0 = H(x_j) \qquad j=0,1,2,\cdots,M=(M_1+M_2-1)$

$u_0^n = a \qquad u_{M_1+M_2-1}^n = b \qquad n=0,1,\cdots,[\frac{T}{\Delta t}]$

$C=\alpha\Delta t/\Delta x$, $\quad Q=\nu\Delta t/\Delta x^2$

Equation (2.10) is similar to the finite difference scheme.

let $\Delta u_j = u_j^{n+1}-u_j^n$ then (2.10)can be written as

$$\frac{1}{6}\Delta u_{i+1}+\frac{2}{3}\Delta u_i+\frac{1}{6}\Delta u_{i-1}$$
$$+\frac{C}{2}(\Delta u_{i+1}-\Delta u_{i-1})-Q(\Delta u_{i+1}-2\Delta u_i+\Delta u_{i-1})$$
$$=\mathrm{Re}\,s_j^n \quad (2.11)$$

with $\mathrm{Re}\,s_j^n = -\frac{C}{2}\left(u_{j+1}^n-u_{j-1}^n\right)+Q\left(u_{j+1}^n-2u_j^n+u_{j-1}^n\right)$

We construct the following iteration scheme. For the $m+1$ iteration, let

$$\Delta u_j^{(m+1)}=\Delta u_j^{(m)}+du_j\,,\quad \Delta u_j^{(0)}=0\ , du_0=0$$

(m=0, 1, 2, ...) (2.12)

and du_j computed by the following formulas (2.14) and (2.16).

From (2.11), the left to right scheme is

$$\frac{2}{3}du_j+\frac{1}{6}du_{j+1}+\frac{C}{2}du_{j+1} \quad (2.13)$$
$$-Q\left(-2du_j+du_{j+1}\right)=RHS(u)$$

here

$$RHS(u)=\mathrm{Re}\,s_j^n-\left[\frac{1}{6}\Delta u_{j+1}^{(m)}+\frac{2}{3}\Delta u_j^{(m)}+\frac{1}{6}\Delta u_{j-1}^{(m+1)}\right.$$
$$\left.+\frac{C}{2}\left(\Delta u_{j+1}^{(m)}-\Delta u_{j-1}^{(m+1)}\right)-Q\left(\Delta u_{j+1}^{(m)}-2\Delta u_j^{(m)}+\Delta u_{j-1}^{(m+1)}\right)\right]$$

Resume $du_j \approx du_{j+1}$ and let $|C|$ instead of the C

$$\left(\frac{5}{6}+\frac{|C|}{2}+Q\right)du_j=RHS(u) \quad (2.14)$$

Similarly, the right to left scheme is

$$\frac{2}{3}du_j+\frac{1}{6}du_{j-1}-\frac{C}{2}du_{j-1} \quad (2..15)$$
$$-Q\left(-2du_j+du_{j-1}\right)=LHS(u)$$

here

$$LHS(u)=\mathrm{Re}\,s_j^n-\left[\frac{1}{6}\Delta u_{j+1}^{(m+1)}+\frac{2}{3}\Delta u_j^{(m)}+\frac{1}{6}\Delta u_{j-1}^{(m)}\right.$$
$$\left.+\frac{C}{2}\left(\Delta u_{j+1}^{(m+1)}-\Delta u_{j-1}^{(m)}\right)-Q\left(\Delta u_{j+1}^{(m+1)}-2\Delta u_j^{(m)}+\Delta u_{j-1}^{(m)}\right)\right]$$

Resume $du_{j-1}\approx du_j$ and let $|C|$ instead of the C

$$\left(\frac{5}{6}-\frac{|C|}{2}+Q\right)du_j=LHS(u) \quad (2.16)$$

So we get the locally implicit finite element scheme (2.11),(2.14) and (2.16) for the convection-diffusion equation.

3 Error Estimation

Let $\overline{u}_j^n$ is the exact solution of (2.1)—(2.3),

$\rho_j^l=\overline{u}_j^l-u_j^l$, $\qquad e_j^n=O(\Delta t+\Delta x^2)$,

$\rho^l=(\rho_1^l,\rho_2^l,\ldots,\rho_{N-1}^l)^T$, $\; e^l=(e_1^l,e_2^l,\ldots,e_{N-1}^l)^T$,

then ρ_j^l satisfies the following equation:

$$\begin{cases}A\rho^{n+1}=B\rho^n+\Delta t e^n\\ \rho^0=0\end{cases},\ (n=0,1,..,\left[\frac{T}{\Delta t}\right]-1) \quad (3.1)$$

where A ,B are $(N_1-1)\times(N_1-1)$ matrices

$$A=\begin{pmatrix}\frac{2}{3}+2Q & \frac{1}{6}+\frac{c}{2}-Q & 0 & 0 & . & . & . & 0\\ \frac{1}{6}-\frac{c}{2}-Q & \frac{2}{3}+2Q & \frac{1}{6}+\frac{c}{2}-Q & 0 & . & . & . & 0\\ 0 & \frac{1}{6}-\frac{c}{2}-Q & \frac{2}{3}+2Q & \frac{1}{6}+\frac{c}{2}-Q & . & . & . & 0\\ . & . & . & . & . & . & . & .\\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6}-\frac{c}{2}-Q & \frac{2}{3}+2Q\end{pmatrix}$$

$$B=\begin{pmatrix}\frac{2}{3} & \frac{1}{6} & 0 & 0 & 0 & . & . & 0\\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & 0 & . & . & 0\\ 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & . & . & 0\\ . & . & . & . & . & . & . & .\\ 0 & 0 & 0 & 0 & 0 & . & \frac{1}{6} & \frac{2}{3}\end{pmatrix}$$

and so

$$\rho^{n+1}=\Delta t[A^{-1}e^n+A^{-1}BA^{-1}e^{n-1}$$
$$+(A^{-1}B)^2A^{-1}e^{n-2}+\ldots+(A^{-1}B)^nA^{-1}e^0]$$

It is easy to see that

$$\|B\|_\infty=1,\ \|A\|_\infty\ge 1,\ \|A^{-1}\|_\infty\le 1.$$
$$\|A^{-1}B\|_\infty\le\|A^{-1}\|_\infty\|B\|_\infty\le 1,$$
$$\|A^{-1}e^n\|_\infty\le\|A^{-1}\|_\infty\|e^n\|_\infty=O(\Delta t+\Delta x^2).$$

because $0\le N\le\frac{T}{\Delta t}$,

$$\|\rho^n\|_\infty\le\Delta tNO(\Delta t+\Delta x^2)\le O(\Delta t+\Delta x^2)$$

So when step ratio $s=\frac{\Delta t}{\Delta x^2}=cons.$ $\|\rho^n\|_\infty \le O(\Delta x^2)$.

so $|\rho_j^i| = O(\Delta x^2)$. (3.2)

(3.6) indicate that approximation scheme (2.14) is convergent for arbitrary value S. So the error of the locally implicit finite element scheme is first-order for time, and is second-order for space. As to the stability analysis, we refer to reference [4].

4. Domain Decomposition Method

Suppose the initial domain Ω is divided into two sub-domains, denoted by $\Omega_1=[x_0, x_{M_1}]$, and $\Omega_2=[x_{M_1}, x_{M_1+M_2-1}]$. $\Omega_3=[x_{M_1-1}, x_{M_1}]$ is the overlapping area. $u_i = u|_{\Omega_i}$ $(i=1,2)$ For the Ω_1, the node number is M_1, and for the Ω_2, the node number is M_2. Then we construct the iterations (u_1^k, u_2^k) for arbitrary time interval

[t_{n-1}, t_n] with the locally implicit finite element scheme as follow:

1). When n=1, Case 1.k=1,

$x \in \Omega_1$

$$\begin{cases}\begin{cases}\left(\frac{5}{6}+\frac{|C|}{2}+Q\right)du_1^k(j,1)=RHS(u_1)\\ \left(\frac{5}{6}-\frac{|C|}{2}+Q\right)du_1^k(j,1)=LHS(u_1)\end{cases}\\ u_1(x)=H(x)\\ u_1^k(M_1-1,n)=u_1(M_1-1,0)+C(u_1(M_1-2,0)-u_1(M_1-1,0))\\ u_1^k(M_1,n)=u_1(M_1,0)+C(u_1(M_1-1,0)-u_1(M_1,0))\\ u_1(0,n)=a\end{cases} \tag{4.1}$$

for $x \in \Omega_2$

$$\begin{cases}\begin{cases}\left(\frac{5}{6}+\frac{|C|}{2}+Q\right)du_2^k(j,1)=RHS(u_2)\\ \left(\frac{5}{6}-\frac{|C|}{2}+Q\right)du_2^k(j,1)=LHS(u_2)\end{cases}\\ u_2(x)=H(x)\\ u_2^k(M_1-1,n)=u_1^k(M_1-1,n)\\ u_2^k(M_1,n)=u_1^k(M_1,n)\\ u_2^k(M_1+M_2-1,n)=b\end{cases} \tag{4.2}$$

Case 2. k>1,

for $x \in \Omega_1$

$$\begin{cases}\left(\frac{5}{6}+\frac{|C|}{2}+Q\right)du_1^k(j,1)=RHS(u_1^{k-1})\\ \left(\frac{5}{6}-\frac{|C|}{2}+Q\right)du_1^k(j,1)=LHS(u_1^{k-1})\\ u_1^k(M_1-1,1)=u_2^{k-1}(M_1-1,1)\\ u_1^k(M_1,1)=u_2^{k-1}(M_1,1)\\ u_1^{k-1}(0,1)=a\end{cases} \tag{4.3}$$

for $x \in \Omega_2$

$$\begin{cases}\left(\frac{5}{6}+\frac{|C|}{2}+Q\right)du_2^k(j,1)=RHS(u_2^{k-1})\\ \left(\frac{5}{6}-\frac{|C|}{2}+Q\right)du_2^k(j,1)=LHS(u_2^{k-1})\\ u_2^k(M_1-1,1)=u_1^k(M_1-1,1)\\ u_2^k(M_1,1)=u_1^k(M_1,1)\\ u_2^k(M_1+M_2-1,1)=b\end{cases} \tag{4.4}$$

2). When n=2,3,...

Case 1. k=1,

$x \in \Omega_1$

$$\begin{cases}\begin{cases}\left(\frac{5}{6}+\frac{|C|}{2}+Q\right)du_1^k(j,n)=RHS(u_1)\\ \left(\frac{5}{6}-\frac{|C|}{2}+Q\right)du_1^k(j,n)=LHS(u_1)\end{cases}\\ u_1^k(M_1-1,n)=\tilde{u}_1(M_1-1,n-1)+C(\tilde{u}_1(M_1-2,n-1)-\tilde{u}_2(M_1-1,n-1))\\ u_1^k(M_1,n)=\tilde{u}_1(M_1,n-1)+C(\tilde{u}_1(M_1-1,n-1)-\tilde{u}_2(M_1,n-1))\end{cases} \tag{4.5}$$

Where $\tilde{u}_1$, $\tilde{u}_2$ are the approximation values of above step for several iteration steps.

for $x \in \Omega_2$

$$\begin{cases}\begin{cases}\left(\frac{5}{6}+\frac{|C|}{2}+Q\right)du_2^k(j,n)=RHS(u_2)\\ \left(\frac{5}{6}-\frac{|C|}{2}+Q\right)du_2^k(j,n)=LHS(u_2)\end{cases}\\ u_2^k(M_1-1,n)=u_1^k(M_1-1,n)\\ u_2^k(M_1,n)=u_1^k(M_1,n)\\ u_2^k(M_1+M_2-1,n)=b\end{cases} \tag{4.6}$$

Case 2. k=2,3,...

for Ω_1

$$\begin{cases}\begin{cases}\left(\frac{5}{6}+\frac{|C|}{2}+Q\right)du_1^k(j,n)=RHS(u_1^{k-1})\\ \left(\frac{5}{6}-\frac{|C|}{2}+Q\right)du_1^k(j,n)=LHS(u_1^{k-1})\end{cases}\\ u_1^k(M_1-1,n)=u_2^{k-1}(M_1-1,n)\\ u_1^k(M_1,n)=u_2^{k-1}(M_1,n)\\ u_1^k(0,n)=a\end{cases} \tag{4.7}$$

for $x \in \Omega_2$

$$
\begin{cases}
\begin{cases}
\left(\dfrac{5}{6}+\dfrac{|C|}{2}+Q\right)du_2^k(j,n)=RHS(u_2^{k-1}) \\
\left(\dfrac{5}{6}-\dfrac{|C|}{2}+Q\right)du_2^k(j,n)=LHS(u_2^{k-1})
\end{cases} & (4.8) \\
u_2^k(M_1-1,n)=u_1^k(M_1-1,n) \\
u_2^k(M_1,n)=u_1^k(M_1,n) \\
u_2^k(M_1+M_2-1,n)=b
\end{cases}
$$

Here we mention that the boundary conditions imposed on the intersection are different from the typical DD method. In order to impose the boundary conditions more exactly during the procedure, we take into account of the convection effect in (4.5)

5. Numerical Results

Two numerical experiments were done for the discontinues problem, and $\Delta t = 0.4$, $\Delta x = 1.0$, $\alpha = 1.0$. In the first numerical experiment, u is varied about 20.0 to 10.0. In the second numerical experiment, the u is varied about 10.0 to 20.0.

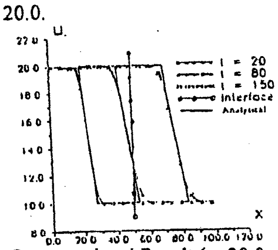

Fig. 1 Computational Result (u=20.0 ~ 10.0)

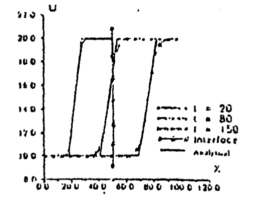

Fig. 2 Computational Result (u=10.0 ~ 20.0)

In Fig. 1 and Fig. 2, the vertical line is a symbol of two domains. there are three states, t=20, t=80, t=150.When t=20, the discontinuity is in the part 1, not to cross the vertical line to part 2. The curve is smooth both in part 1 and part 2. When t=80, the discontinuity is crossing the vertical line to the part2. There is some undulation in the curve. Because of the error of this scheme, the boundary condition of part 1 on the vertical line would be imposed to exactly true value. The imposed value would be undulating around the exact value. Then the curve is not very smooth. After the discontinuity crossed to the part2 (t=150), the curve is smooth. But accumulating the historical effect, and under the dissipation and the convection, there is a wave on the inlet of the part 1. So the numerical results is satisfied. The numerical results are coincide with the error analysis. i.e., for the space, the error is second-order, for the time, the error is first-order.

6. Concluding Remarks

In the present study, we discussed the locally implicit finite element scheme, which is base on a point-by-point scheme, domain decomposition method (DDM) and the matrix-storage free formulation, then developed and implemented it to the model equation for Navier-Stokes equation, convection-diffusion equation. By virtue of the point-by-point scheme, the implementation is quite simple.

The validity of the Scheme was verified through two numerical experiments which convection is dominated. In order to impose the boundary conditions more exactly during the procedure, we take into account of the convection effect in (3.3) for the DD method. The results of the numerical experiments were satisfied. And the error is coincided with the theoretical analysis.

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