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Error analysis of finite element solutions of convection problems and its application to the density-dependent Stokes problems

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1 Introduction

Let Ω be a bounded polygon in $\mathbf{R}^2$ or a polyhedron in $\mathbf{R}^3$. Let $T>0$ and $Q=\Omega\times(0,T)$ and $S=\partial\Omega\times(0,T)$. We would like to propose a convergent finite element scheme approximating the following convection problem (P): find unknown density $\rho(x,t)$ of some material : $\Omega\rightarrow\mathbf{R}$ satisfying

$$\frac{\partial\rho}{\partial t}+(u\cdot\mathrm{grad})\rho=0 \quad \text{in} \quad Q, \tag{1}$$

$$\rho(x,0)=\rho^0(x) \quad \text{on} \quad \Omega \tag{2}$$

with respect to known velocity $u(x,t)$ of some incompresible flows:

$$\mathrm{div}u=0 \quad \text{in} \quad Q, \tag{3}$$

$$u(x,t)=0 \quad \text{on} \quad S. \tag{4}$$

In the construction of a convergent finite element scheme for the problem (P), of courese, we assume some regulaity and boundedness on the velocities u besides the incompressibility (3). But, what regularity on velocities we have to assume? There may be many possibilties for the reply. Still, the most important one is depend on its aim, certainly, to which the advection problem (P) is applied.

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1 INTRODUCTION

In fact, the problem (P) is a subtarget of another propblem ($\hat{\mathrm{Q}}$), the incompressible Navier-Stokes problems with nonhomogeneous density, i. e., which is a system governed by the density-dependent Navie-Stokes equation :

$$\rho\left\{\frac{\partial u}{\partial t}(u\cdot\mathrm{grad})u\right\}-\mu\Delta u+\mathrm{grad}p=\rho f \quad \text{in} \quad Q,$$

together with two other equations, (1) and (3), where u, p and ρ are unknown velocities, pressures and densities of the non-homogeneous liquids, respectively.

This problem ($\hat{\mathrm{Q}}$) admits at least a weak solution ρ, u, p satisfying

$$u\in L^{\infty}\left(0,T;\{L^2(\Omega)\}^d\right)\cap L^2\left(0,T;\{H_0^1(\Omega)\}^d\right), \tag{5}$$

for the initial velocity $u_0 \in \{L^2(\Omega)\}^d$, the outer force $f \in L^1\left(0,T;\{L^2(\Omega)\}^d\right)$ and

$$0<M_1\leq\rho^0\leq M_2<\infty \tag{6}$$

where $M_i, i = 1,2$, are constants (cf., for example, A. S. N. Antontsev, A. V. Kazhikhov and V. N. Monakhov [1] and P. L. Lions [10]) . So, in our paper we shall choose conditions (5) and (6) on the velocity u and the density ρ^0, respectively.

Beside, there are another conditions in our paper to be needed, because we have to construct a finite element scheme for (P). Thus, conditions on discrete velocities are also needed. Unfortunately, there may be no convergent scheme, which approximates solutions of the problem ($\hat{\mathrm{Q}}$). So, from a convergent implicit scheme, which approximates solutions of the classical Navier-Stokes equations in the incompressible case with homogeneous density, we shall choose a suitable condition on discrete velocities. For references it is useful to see the book [11] by R. Temam.

In fact, let $\tau > 0$ be a time mesh and let U^n be an approximation of $u(t_n), 0 = t_0 < t_1 < t_2 < \cdots < t_n = n\tau, \cdots, N = [T/\tau]$. Let $\delta U^n = U^n - U^{n-1}$. Then we consider the condition

$$\|U^n\|^2+\textstyle\sum_{n=1}^N\left(\|\delta U^n\|^2+\tau\|U^n\|_h^2\right)\leq c_0<\infty, \tag{7}$$

where c_0 is independent of the time meshes τ and the space meshes $h > 0$ used in triangulations $\{\mathcal{T}_h\}_h$ of the domain Ω but depend on the initial value of u_0 and the force term f. Here the notation $\|\cdot\|$ is the L^2 norm over the domain Ω and

$$\|U^n\|_h=\sqrt{\sum_{K\in\mathcal{T}_h}\int_K|\mathrm{grad}U^n|^2dx}.$$

In this paper we propose a finite element scheme $(\mathrm{P})_h$: (12), and prove the convergence of their discrete solutions to a weak solution of the problem (P) in another paper [6] under these conditons from (5) to (7). The order of the error of discrete solutions of (12) is studied under sufficient regularity conditions on u and ρ in this paper.

It may be difficult to extend the scheme (12) to a finite element scheme approximating the density-dependent Navier-Stokes problem, although it is possible to extend (12) to a finite element scheme $(\bar{\mathrm{Q}})_{g,h}$, $g = \rho f$, which approximates the density-dependent Stokes problem $(\mathrm{Q})_f$ governed by the density-dependent Stokes equation

$$\rho\frac{\partial u}{\partial t} - \mu\Delta u + \mathrm{grad}p = \rho f \quad \text{in} \quad Q,$$

together with two other equations (1) and (3) under the two conditions (2) and (4). Further, it is possible to give the order of the error of their discrete solutions. Note that, in the density-dependent Stokes problem, u, p and ρ are unknown velocities, pressures and densities of non-homogeneous liquids, respectively. We shall consider a scheme $(\bar{\mathrm{Q}})_{g,h}$: (12) and (21), approximating the density-dependent Stokes problem, and show that the approximation U^n satisfies

$$\tau\sum_{n=1}^{N}\left\|\frac{\delta U^n}{\tau}\right\|^2 \leq c_0 < \infty, \tag{8}$$

where c_0 is independent of the time meshes τ and the space meshes h, but depend on the initial value of u_0 and the force term f. If we assume the condition (8) adding to the conditions (5), (6) and (7), then we get an increase in the order of the error of discrete densities of our scheme (12).

Notation

For a domain G we write $\left(f,g\right)_G = \int_G fgdx$. In particular, we write $(f,g) = \int_\Omega fgdx$. We denote by $\|f\|_{p,G}$ the L^p norm of f over G. In paticular, for $G = \Omega$, we write $\|f\|_p = \|f\|_{p,\Omega}$. For a $d-1$ dimensional simplex F, we write $\left\langle f,g\right\rangle_F = \int_F fgd\sigma$. Let $H = L^2(\Omega)$ and $\mathbf{H} = \{L^2(\Omega)\}^d, X = \{H_0^1(\Omega)\}^d$, $V = \{v \in X \mid \mathrm{div}v = 0\}$.

Further, for the usual Sobolev space, $W^{1,p}(G)$ with $1 \leq p \leq \infty$ we need semi-norms, $|v|_{l,p,G} = \left\{\sum_{|a|=l}\|D^a v\|_{p,G}^p\right\}^{1/p}$, and a norm, $\|v\|_{m,p,G} = \left\{\sum_{l=0}^{m}|v|_{l,p,G}^p\right\}^{1/p}$, where D^a are differential operators of order a, a denote multi-indexes. For the case $G = \Omega$ we drop the letter G from the surfixes of $|v|_{l,p,G}$ and $\|v\|_{m,p,G}$ and let $Z^{m,p} = \left\{v \in L^p(\Omega) \,\middle|\, v|_K \in W^{m,p}(K)\right\}$, m denotes a non-negative integer and p satisfies $1 < p < \infty$. Generally, we call a function $v \in Z^{m,p}$ for some m and p a piecewise smooth function. The dual space of a Banach space Z is denoted by Z^*.

Further, $W^{m,p}(0,T;L^q(\Omega)) = \left\{ v \in L^p(0,T;L^q(\Omega)) \mid \int_0^T \|D^s v\|_{L^q(\Omega)}^p ds < \infty \right\}$ where $D^s v$ denotes s -th derivative of v in the sense of distribution.

2 Discrete densities and velocities

Let $\mathcal{T}_h = \{K\}$ be a decomposition of Ω by triangulations for $d = 2$ or by tetrahedrons for $d = 3$. Here we denote by h the maximum of the diameters h_K of $K \in \mathcal{T}_h$. The sequence of decompositons of Ω, $\{\mathcal{T}_h\}$ is *regular* : there exists a constant c_1 such that

$$\limsup_{h \to 0} \sup_{K \in \mathcal{T}_h} \frac{h_K}{h_{0,K}} \le c_1 < \infty, \tag{9}$$

where $h_{0,K}$ is the largest diameter of spheres included in K.

Let k be a non-negative integer and P_k be the totality of polynomials of degree k. Our approximation method for discrete densities relies on the discontinuous, finite element method induced by L. Lesaint and P.-A. Raviart [7], analyzed by C. Johnson and J. Pitkäranta [5]. So we choose an approximating space G_h of discrete densities as follows.

$$G_h = \left\{ \alpha : \Omega \to \mathbf{R} \;\middle|\; \alpha|_K \in P_k \; K \in \mathcal{T}_{h_b} \right\}.$$

We should construct an approximation space X_h which approximates velocities well. For such a space we shall choose a special kind of spaces introduced by M. Crouzeix and P.-A. Raviart [4].

Let l be a non-negative integer and let us consider a family of subspaces $\{P_K \mid K \in \mathcal{T}_h\}$ such that $P_l \subset P_K \subset C^1(K)$. We introduce a space defined by

$$\Phi_h = \left\{ \phi : \Omega \to \mathbf{R} \;\middle|\; \phi|_K \in P_{l-1} \right\}.$$

We call Φ_h a discontinuous finite element space of degree $l-1$. Let W_h be the totality of functions v such that $v|_K \in P_K$ for all $K \in \mathcal{T}_h$ and

$$\int_F (v_1 - v_2)\phi d\sigma = 0 \quad \forall \phi \in \Phi_h \tag{10}$$

for $v_i = v|K_i, K_i \in \mathcal{T}_h, i = 1,2$, such that $F = \partial K_1 \cap \partial K_2$. Further, let $W_{h,0}$ be the totality of functions v of W_h such that

$$\int_F v\phi d\sigma = 0 \quad \forall \phi \in \Phi_h \tag{11}$$

for $K \in \mathcal{T}_h$ and F such that $F = \partial\Omega \cap \partial K$. We define an approximation space $X_h = \{W_{h,0}\}^d$, which approximate velocities. For $v \in X_h$ we define $\mathrm{div}_h : X_h \to \Phi_h$ defined by

$$(\mathrm{div}_h v, \phi) = \sum_{K \in \mathcal{T}_h} \left(\mathrm{div} v, \phi\right)_K \quad \forall \phi \in \Phi_h.$$

By this operator div_h we define $V_h = \{v \in X_h \mid \mathrm{div}_h v = 0\}$, which approximates the solenoidal space V. Here we call an element v of X_h a Crouzeix-Raviart velocity of degree l.

In our paper, the Crouzeix-Raviart velocities of degree $l = 2k+1$ are applied to discrete velocities in the scheme described below. Further let us notice that the linear span of $\{\alpha_1\alpha_2 \mid \alpha_i \in G_h, i = 1,2\}$ coincides with a discontinuous finite element space Φ_h of degree $l-1 = 2k$.

The upwind element and downwind element

For a discrete velocity $U \in X_h$ and a face element F, $F \subset \partial K, K \in \mathcal{T}_h$, we can choose a constant vector $U_{F\nu}$ and a unit normal ν_F to F determined uniquely by

$$\int_F U \cdot \nu_F d\sigma = \int_F U_{F\nu} \cdot \nu_F d\sigma \geq 0$$

for the sake of conditions (10) and (11). Here, if $\int_F U \cdot \nu d\sigma = 0$, then $U_{F\nu} = \vec{0}$ and ν_F can be set any of unit normals to F. By ν_F both of the upwind element K_U and the downwind associated with F are determined. Here, we see $F = K_U \cap K_D$. By these elements, K_U and K_D, we can define the gap of α on F by

$$[\alpha]_D^U = \alpha|_{K_U} - \alpha|_{K_D}.$$

3 A scheme and its stability

First we propose a finite element scheme which approximates advection equations well. Let $s_h, t_h : V_h \times G_h \times G_h \to \mathbf{R}$ be trilinear functionals defined by

$$\begin{cases} s_h(v,\alpha_1,\alpha_2) = \sum_{K\in\mathcal{T}_h} \sum_{j=1}^{d} \left(v_j \frac{\partial \alpha_1}{\partial x_j}, \alpha_2\right)_K, \\ \\ t_h(v,\alpha_1,\alpha_2) = \sum_{F\subset\Omega} \left\langle v_{MF}, [\alpha_1]_U^D\, \alpha_{2D} \right\rangle_F. \end{cases}$$

Now, our scheme is described as the following way.

$(\mathbf{P})_h$: For $U^{n-1} \in V_h$ and $r^{n-1} \in G_h$ find $r^n \in G_{h_b}$ such that

$$\frac{(\delta r^n, \beta)}{\tau} + s_h(U^{n-1}, r^n, \alpha)$$

$$+t_h(U^{n-1}, r^n, \alpha) =< g^n, \alpha > \quad \forall \alpha \in G_h, \tag{12}$$

where g^n belongs to the dual space of G_h for each n. For the case: $g^n \equiv 0$ we have a discrete scheme approximating to the problem (P) as $h \to 0$ and $\tau \to 0$.

The stability to the solutions $r^n, n = 1,2,3,\cdots,N$, is described below. We omit the proofs of lemmas.

Lemma 1 *(The discrete maximum principle) For the case : $g^n \equiv 0$, and for each $m = 1, 2, 3, \cdots, N$, together with $k = 0$, we have*

$$\min \left\{ r^{m-1}|_K \;\middle|\; K \in \mathcal{T}_h \right\} \le \min \left\{ r^m|_K \;\middle|\; K \in \mathcal{T}_h \right\}$$

$$\le \max_K \left\{ r^m|_K \;\middle|\; K \in \mathcal{T}_h \right\} \le \max_K \left\{ r^m\big|_K \mid K \in \mathcal{T}_h \right\}.$$

Lemma 2 *(L^2-stability) For each $m = 1, 2, 3, \cdots, N$, together with $0 \le k$ we have*

$$\|r^m\|^2 + \sum_{n=1}^{m} \|\delta r^n\|^2 + \tau \sum_{n=1}^{m} \left\langle U_F^{m-1}, \left|[r^m]_U^D\right|^2 \right\rangle_F = \|r^0\|^2$$

$$+ 2\tau \sum_{n=1}^{m} < g^n, r^n > .$$

4 Truncation terms

Let π_h be the orthogonal projection from $L^2(\Omega)$ onto the subspace G_h. We write $\overline{\rho}^n = \pi_h(\rho^n)$. Le us consider the errors $e^n = r^n - \overline{\rho}^n$ of r^n.

To represent trucation terms it is convienient to introduce trilinear forms s_h^* and t_h^* adjoint formally to s_h and t_h, respectively. In fact, they are defined by

$$\begin{cases} s_h^*(v, \alpha_1, \alpha_2) &= -\displaystyle\sum_{K \in \mathcal{T}_h} \sum_{j=1}^{d} \left(v_j \alpha_1, \frac{\partial \alpha_2}{\partial x_j} \right)_K, \\ t_h^*(v, \alpha_1, \alpha_2) &= \displaystyle\sum_{F \subset \Omega} \left\langle v_{MF}, \alpha_{1U} \left[\alpha_2\right]_D^U \right\rangle_F. \end{cases}$$

Clearly, these trilinear forms may be defined for piecewise smooth functions β_1 and β_2 instead of α_1 and α_2, respectively. Other trilinear forms L_h and l_h are also useful to describe our truncation terms for discrete densities r^n.

$$\begin{cases} L_h(v_1, v_2, v_3) &= \displaystyle\sum_{K \in \mathcal{T}_h} \left(v_1, v_2 v_3 \right), \\ l_h(v_1, v_2, v_3) &= \displaystyle\sum_{F \subset \Omega} \left\langle v_1, v_2 v_3 \right\rangle_F, \end{cases}$$

where v_1, v_2 and v_3 are piecewise smooth scalar functions over domain Ω. By using these trilinear forms we can descrive truncation terms as follows.

Lemma 3 *Let ρ be the exact solution of the problelm (P) and $\rho^n(x) = \rho(x, t_n)$. Then*

$$\frac{(\delta e^n, \alpha)}{\tau} + s_h^*(U^{n-1}, e^n, \alpha) + t_h^*(U^{n-1}, e^n, \alpha) =< g^n, \alpha >, \qquad (13)$$

$$
\begin{aligned}
< g^n, \alpha >= & \left(\dot{\rho} - \frac{\delta \overline{\rho}^n}{\tau}, \alpha\right) + s_h(u^n - U^{n-1}, \rho^n, \alpha) \\
& + s_h^*(U^{n-1}, \rho^n - \overline{\rho}^n, \alpha) + t_h^*(U^{n-1}, \rho^n - \overline{\rho}^n, \alpha) \\
& - t_h(U^{n-1}, \rho^n, \alpha) - L_h(\mathrm{div}_h U^{n-1}, \rho^n - \overline{\rho}^n, \alpha) \\
& + l_h\left(\left[U^{n-1}\right]_{F\nu}^{U\nu}, \rho^n - \overline{\rho}^n, \alpha\right) + l_h\left(\left[U^{n-1}\right]_{D\nu}^{F\nu}, \rho^n - \overline{\rho}^n, \alpha\right).
\end{aligned}
$$

5 Error analysis

We can prove the order of errors for the above discrete densities r^n provided that the exact solution ρ is smooth enough.

For velocities $v \in \{H^2(\Omega)\}^d$ we assume that

(H.1) there exists an operator

$$
\Pi_h \in \mathcal{L}\left(\left\{H^2(\Omega)\right\}^d ; \{W_h\}^d\right) \cap \mathcal{L}\left(\left\{H^2(\Omega)\right\}^d \cap \left(H_0^1(\Omega)\right\}^d ; \{W_{h,0}\}^d\right)
$$

such that

$$
\begin{aligned}
\mathrm{div}_h \Pi_h v &= \mathrm{div}_h v, \\
\|\Pi_h v - v\|_h &\leq c_2 h^m |v|_{m+1}, \qquad (14)
\end{aligned}
$$

for all $v \in \{H^{m+1}(\Omega)\}^d, 1 \leq m \leq 2k+1$, where c_2 is independent of h and τ, but depend on c_1.

The hypothesis (H.1) is due to M. Crouzeix and P.-A. Raviart [4] and the existence of finite elements velocities for $k = 1, 3$ satisfying this hypothesis see the examples in [4]. Let $\tilde{u}^n = \Pi_h(u^n)$. For the error E^n of the discrete velocity U^n defined by

$$
E^n = U^n - \tilde{u}^n, n = 1, 2, 3, \cdots, N.
$$

We assume that the discrete velocities U^n satisfy the estimates

(H.2).

$$
\|E^m\|_h^2 \leq \|E^0\|_h^2 + C_1 h^{2(2k+1)} + C_2 \tau^2 \quad m = 1, 2, 3, \cdots, N, \qquad (15)
$$

where $C_i, i = 1, 2$, are independent of τ and h.

When we apply an implicit standard finite element scheme to the Stokes equation we get the above estimates (15). Considering each truncation terms in Lemma 3 the following lemma holds true.

Lemma 4 *Let g^n be the right hand side of (13). Under the assumptions on ρ in Theorem 1 there exists a constant C, which is independent of h and τ, such that*

$$
\begin{aligned}
2\tau \sum_{n=1}^{m} \left\langle g^n, e^n \right\rangle \leq & \ \tau \sum_{F \subset \Omega} \left\langle U_{F\nu}^{n-1}, \left|[e^n]_D^U\right|^2 \right\rangle_F \\
& + C\left(\tau^{2a} + h^{2(2k+1)} + \tau \sum_{n=1}^{m} \|E^n\|_h^2\right), \qquad (16)
\end{aligned}
$$

where a denotes a constant described in the next theorem.

After substituing $\alpha = 2\tau e^n$ into the first identity in Lemma 3 and applying the proof of Lemma 2 , adding from $n = 1$ to m we get

$$\|e^m\|^2 + \sum_{n=1}^{m} \|\delta e^n\|^2 + \tau \sum_{F\subset\Omega} \left\langle U_{F\nu}^{n-1}, \left|[e^n]_D^U\right|^2 \right\rangle_F$$
$$\leq \|e^0\|^2 + 2\tau \sum_{n=1}^{m} < g^n, e^n > . \tag{17}$$

Applying Lemma 4 and the discrete Gronwall inequality we can prove the next theorem below.

Theorem 1 *Assume (5), (7), (15) and that a solution ρ of the problem (P) satisfies*

$$\rho \in C\left([0,T]; C^{k+1}\left(\overline{\Omega}\right)\right) \cap C^1\left([0,T]; W^{k+1,4}(\Omega)\right) \cap C^2\left([0,T]; H\right). \tag{18}$$

In the case: $k = 0$, we further assume (6). Then there exists a constant C, independent of τ and h such that

$$\|e^m\|^2 \leq C\Big(\|e^0\|^2 + \|E^0\|_h^2 + h^{2(2k+1)} + \tau^{2a}\Big) \quad m = 1, 2, 3, \cdots, N,$$
$$\text{where } C \text{ is independent of } \quad \tau \quad \text{and} \quad h, \tag{19}$$

where $a = 1/2$ for the case: (8) does not hold, and $a = 1$ for the other case: (8) holds.

6 The Stokes equations

Here we consider a scheme for the modified density-dependent Stokes problem $(\bar{\mathrm{Q}})_g$ governed by

$$\begin{cases} \rho\dfrac{\partial u}{\partial t} + \mu\Delta u + \mathrm{grad}p = g \quad \text{in} \quad Q, \\[2ex] \dfrac{\partial\rho}{\partial t} + (u\cdot\mathrm{grad})\rho = 0 \quad \text{in} \quad Q, \\[2ex] \mathrm{div}u = 0 \quad \text{in} \quad Q, \end{cases} \tag{20}$$

with the boundary condition: $u = 0$ on S and the initial conditions: $\rho(x, 0) = \rho^0(x)$ and $u(x, 0) = u^0(x)$ on Ω, where $g \in L^2(0, T; X^*)$. The modified density-dependent Stokes problem $(\bar{\mathrm{Q}})_g$ with the case $g \equiv \rho f$ reduces to the density-dependent Stokes problem $(\mathrm{Q})_f$ previously described in the introduction.

6 THE STOKES EQUATIONS

Let us consider bilinear forms $a_h : X_h \times X_h \to \mathbf{R}$ and $b_h : X_h \times G_{h,0} \to \mathbf{R}$ defined by

$$a_h(v,w) = \sum_{K\in\mathcal{T}_h} \Big(\text{grad}v, \text{grad}w\Big)_K,$$

$$b_h(v,\alpha) = -\sum_{K\in\mathcal{T}_h} \Big(\text{div}_h v, \alpha\Big)_K,$$

where $G_{h,0} = \{\alpha \in G_h \mid (1,\alpha) = 0\}$.

For the above problem $(\bar{\mathrm{Q}})_g$ we consider a scheme $(\bar{\mathrm{Q}})_{g,h}$:
(1) for $r^{n-1} \in G_h$ and $U^{n-1} \in V_h$ find $r^n \in G_h$ satisfying (12) with the case $k = 0$,
(2) then, for $r^{n-1}, r^n \in G_h$ and $U^{n-1} \in V_h$, find $U^n \in X_h$ and $P^n \in G_{h,0}$ satisfying

$$\begin{cases} \dfrac{(r^n\delta U^n, v)}{\tau} + \mu a_h(U^n, v) + b_h(v, P^n) =< \hat{g}^n, v > \quad \forall v \in X_h, \\ \\ b_h(U^n,\alpha) = 0 \quad \forall \alpha \in G_{h,0}. \end{cases} \tag{21}$$

In the above, $\hat{g}^n$ approximates $g(x,t_n)$, in some sense, and satisfying

$$|< \hat{g}^n, v >| \le C_1^n \|v\|_h + C_0^n \|v\| \quad \forall v \in X_h, \tag{22}$$

with some constants C_n for $n = 1,2,3,\cdots,N$, where each C_n may depend on $\hat{g}^n$.
We have the stabiliy for the above scheme as follows.

Lemma 5 *Assume (22). Then, for each n, we have a unique solution r^n, U^n and P^n of the problem $(Q)_h$. Further, using G_h with $k = 0$ and $m = 1,2,3,\cdots,N$, we have*

$$2M_1\tau\sum_{n=1}^{m}\left\|\frac{\delta U^n}{\tau}\right\|^2 + 2\mu\|U^m\|_h^2 + \mu\sum_{n=1}^{m}\|\delta U^n\|_h^2$$

$$\le 2\mu\|U^0\|_h^2 + C_1'\sum_{n=1}^{m}|C_1^n|^2 + C_0'\tau\sum_{n=1}^{m}|C_0^n|^2.$$

Here C_1', C_0' are independent of h and τ.

This is obtained by substituting $v = 2\delta U^n$ into (21) through the discrete maximum principle on r^n. Applying the discrete Poincaré inequality we get

Lemma 6 *There exists a constant C_3, which is independent of h and τ, such that, for $m = 1,2,3,\cdots,N$,*

$$\|U^m\|^2 \le C_3\left(\mu\|U^0\|_h^2 + C_1'\sum_{n=1}^{m}|C_1^n|^2 + C_0'\tau\sum_{n=1}^{m}|C_0^n|^2\right).$$

7 Errors on discrete velocities

We consider the errors of discrete velocities and densities of the density-dependent Stokes problem $(\mathrm{Q})_{f,h}$. Let U^n be the discrete solution of the scheme $(\bar{\mathrm{Q}})_{g,h}$, where $g = \rho f$. Already we have define the errors E^n of discrete velocities U^n. The error of the discrete pressure P^n is defined by $E_P^n = P^n - p^n$. Then, for $v \in X_h$,

$$\frac{(r^n \delta E^n, v)}{\tau} + \mu a_h(E^n, v) + b_h(v, E_P^n) = \sum_{j=1}^{5} \left\langle \eta_j^n, v \right\rangle,$$

$$< \eta_1^n, v >= \Big((r^n - \rho^n) f^n, v \Big)$$

$$< \eta_2^n, v >= \left(\rho^n \dot{u}^n - \frac{\delta \tilde{u}^n}{\tau}, v \right),$$

$$< \eta_3^n, v >= -\mu \Big((\Delta u^n, v) + (\mathrm{grad} u^n, \mathrm{grad} v) \Big),$$

$$< \eta_4^n, v >= \mu \Big(\mathrm{grad}(\tilde{u}^n - u^n), \mathrm{grad} v \Big)$$

$$< \eta_5^n, v >= \Big(\mathrm{grad} p^n, v \Big) + \Big(\mathrm{div} v, p^n \Big).$$

Now, we see

$$| < \eta_1^n, v > | \leq \|f\|_{\infty,Q} \Big(\|e^n\| + h C_1(\eta_1) |\rho^n|_{1,2,\infty,Q} \Big) \|v\|,$$

$$| < \eta_2^n, v > | \leq C_0(\eta_2) \|\dot{u}\|_{\infty,Q} \Big(\|\delta r^n\| + \|e^n\| \Big) \|v\|$$
$$+ h C_0'(\eta_2) \|v\| \Big(\|\dot{u}\|_{\infty,Q} |\rho|_{1,2,\infty,Q} + \|\dot{u}\|_{1,2,\infty,Q} \|\rho\|_{\infty,Q} \Big)$$
$$+ |\rho|_{\infty,Q} \sqrt{\frac{\tau^3 \int_{t_{n-1}}^{t_n} \|\ddot{u}\|^2 ds}{3}} \|v\|,$$

$$| < \eta_3^n, v > | \leq \mu h C_1(\eta_3) |u^n|_{2,\infty,Q} \|v\|_h,$$
$$| < \eta_4^n, v > | \leq \mu h C_1(\eta_4) |u^n|_{2,\infty,Q} \|v\|_h,$$
$$| < \eta_5^n, v > | \leq h C_1(\eta_5) |p^n|_{1,\infty,Q} \|v\|_h,$$

where the constants $C(\eta_j), j = 1, 2, \cdots,$ are independent of $\tau, h, n = 1, 2, 3, \cdots, N$. Applying Lemma 5 to the above estimates we get the result below.

Lemma 7 *Assume (18) on* ρ *and*

$$u \in C\left([0,T],\left\{C^2(\overline{\Omega}^d)\right\}^d\right) \cap C^1\left([0,T],\left\{C(\overline{\Omega})\right\}^d\right) \cap W^{2,2}(0,T;\mathbf{H}). \qquad (23)$$

Then there exist constants C_1, C_2 *and* C_3*, which are independent of* h *and* τ*, such that*

$$\|E^m\|_h^2 \le \|E^0\|_h^2 + C_1\tau\sum_{n=1}^{m}\|e^n\|^2$$
$$+C_2\tau + C_3\frac{h^2}{\tau}.$$

Combining (16), (17), (23) and the discrete Gronwall inequality, we get the theorem below.

Theorem 2 *Assume (18) and (23). Then there exists a constant* C*, which is independent of* $m = 1, 2, 3, \cdots, N, h, \tau$*, such that*

$$\|e^m\|^2 + \|E^m\|_h^2 \le C\left(\|e^0\|^2 + \|E^0\|_h^2 + \tau + \frac{h^2}{\tau}\right).$$

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