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Ranks of algebras of continuous C*-algebra valued functions

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1 Introduction and Main Results

The (topological) stable rank of Rieffel[11] and the real rank of Brown and Pedersen[2] are noncommutative generalizations of the dimension of a compact Hausdorff space. In fact, when X is a compact Hausdorff space, the stable rank of $C(X)$ is $\left[\frac{\dim X}{2}\right]+1$, and the real rank of $C(X)$ is $\dim X$, where $\dim X$ is a covering dimension of X. While it has been known for some time that the covering dimension satisfies $\dim(X\times Y)\leq\dim(X)+\dim(Y)$ for compact Hausdorff spaces X and Y (see Proposition 9.3.2 of [9]), little is known about the analogous situation for C*-algebras, namely the stable and real ranks of tensor products of C*-algebras. In the case of real rank we can not hope such a product type theorem for general C*-algebras as Kodaka and Osaka pointed out: In [4] and [8] there are examples of two separable nuclear C*-algebras A and B such that

$$RR(A)=RR(B)=0 \quad \text{and} \quad RR(A\otimes B)=1.$$

In this talk we report results about the stable and particularly the real ranks of tensor products of C*-algebras under the assumption that one of the factors is commutative.

This is a joint work[7] with M. Nagisa, H. Osaka, and N. C. Phillips.

Our main results are as follows:

(1) If X is any locally compact σ-compact Hausdorff space and A is any C*-algebra, then $RR(C_0(X)\otimes A)\leq\dim(X)+RR(A)$.

(2) If X is any locally compact Hausdorff space and A is any purely infinite simple C*-algebra, then $RR(C_0(X)\otimes A)\leq 1$.

(3) $RR(C([0,1])\otimes A)\geq 1$ for any nonzero C*-algebra A, and $sr(C([0,1]^2)\otimes A)\geq 2$ for any unital C*-algebra A.

(4) If A is a unital C*-algebra such that $RR(A) = 0$, such that $sr(A) = 1$, and such that $K_1(A) = 0$, then $sr(C([0,1]) \otimes A) = 1$.

(5) There is a simple separable unital nuclear C*-algebra A such that $RR(A) = 1$ and $sr(C([0,1]) \otimes A) = 1$.

The result (1) is an analog and generalization of the inequality $\dim(X \times Y) \leq \dim(X) + \dim(Y)$. We do not expect equality because this can fail even in the case of compact metric spaces (see [10]), and also for $A = M_n$ ([1]) or for purely infinite simple A (result (2) above).

As corollaries to (1), we give several related results. The one most closely resembling the inequality for dimensions of products is the following: $RR(C_0(X) \otimes A) \leq RR(C_0(X)) + RR(A)$ for any unital A and any X.

The result (2) on purely infinite simple C*-algebras is mainly proved by N. C. Phillips. So we skip over explaining about it.

The results (3), (4), and (5) are the main part of a closer investigation of tensor products with $C[0,1]$. We show that $sr(C[0,1] \otimes A) = 1$ implies that both $sr(A) = 1$ and $K_1(A) = 0$. One might therefore hope that $sr(C([0,1]) \otimes A) = 1$ would also imply $RR(A) = 0$. Unfortunately, as our result (5) shows, this is not true.

2 Real rank of $C_0(X) \otimes A$

The essential point is that it suffices to show that

$$RR(C(X) \otimes A) \leq \dim X + RR(A)$$

for any unital C*-algebra A and a compact Hausdorff space X. The various formulations involving spaces that are only locally compact and C*-algebras without identities are then derived from this result by compactifying and passing to ideals.

The basic case is $X = [0,1]$, which is done by a direct argument. The case $X = [0,1]^n$ follows by induction, and the case of a finite complex follows by attaching cells. We pass to a general compact space X by realizing it as an approximate inverse limit of finite CW-complexes with dimension at most $\dim(X)$, following Mardešić and Rubin [5].

Theorem 2.1 *Let A be a unital C*-algebra. Then,*

$$RR(C[0,1] \otimes A) \leq 1 + RR(A).$$

Sketch of Proof. Case 1: Take any elements f_0, f_1 in $C[0,1] \otimes A$, where we assume $RR(A)$. Let $\varepsilon > 0$ be an arbitrary positive number. Since $[0,1]$ is compact, there is a $\delta > 0$ such that

$$|s - t| < \delta \quad \text{implies} \quad |f_j(s) - f_j(t)| < \varepsilon/3 \ (j = 0, 1).$$

Devide $[0,1]$ into 2N-intervals with $\frac{1}{N} < \delta$. Set $t_k = \frac{k}{2N}$ $(k = -1, 0, 1, \cdots, 2N+1)$. Consider two open coverings of $(0,1)$: $\{U_i\}_{i=1}^N$ such that $U_i = (t_{2i-3}, t_{2i})$, and $\{V_i\}_{i=1}^N$ such that $V_i = (t_{2i-2}, t_{2i+1})$. We know that

$$\begin{aligned} U_i \cap U_{i+1} &= (t_{2i_1}, t_{2i}) \subset V_i, \\ V_i \cap V_{i+1} &= (t_{2i}, t_{2i+1}) \subset U_{i+1}. \end{aligned}$$

Set $a_{2k+j} = f_j(t_{2k+j})$ $(j = 0, 1,\ k = 0, 1, \cdots, N-1)$, and $a_{2N} = f_0(1)$, $a_{2N+1} = f_1(1)$. Since $RR(A) = 0$, there exist invertible elements $b_0, b_1, \cdots, b_{2N+1}$ such that $||a_j - b_j|| < \frac{\varepsilon}{3}$ for all j. Choose continuous functions $\{h_i\}_{i=1}^N$ such that each support of h_i is contained in U_i and $\sum_{i=1}^N h_i = 1$ on $[0,1]$. Similarly, choose continuous function $\{k_i\}_{i=1}^N$ such that each support of k_i is contained V_i and $\sum_{i=1}^N k_i = 1$ on $[0,1]$. Then, define

$$\begin{aligned} g_0(t) &= \textstyle\sum_{i=1}^N h_i(t) b_{2i-2}, \\ g_1(t) &= \textstyle\sum_{i=1}^N k_i(t) b_{2i-1}. \end{aligned}$$

Then, for $t \in [t_{2i_1}, t_{2i}]$,

$$\begin{aligned} ||f_0(t) - g_0(t)|| &= ||f_0(t) - h_{i+1}(t) b_{2i} - h_{i+2} b_{2i+2}|| \\ &= ||h_{i+1}(t)(f_0(t) - b_{2i}) + h_{i+2}(f_0(t) - b_{2i+2})|| \\ &\leq \varepsilon/3 + \varepsilon/3 < \varepsilon. \end{aligned}$$

Similarly, $||f_1(t) - g_1(t)|| < \varepsilon$. Moreover, since $g_1(t) = b_{2i-1}$, $g_0(t)^2 + g_1(t)^2 \geq b_{2i-1}^2$, hence $g_0(t)^2 + g_1(t)^2$ is invertible. Similarly, when $t \in [t_{2i}, t_{2i+1}]$, we have $h_{i+1}(t) = 1$, hence $g_0(t)^2 + g_1(t)^2 \geq b_{2i}^2$. Therefore, these imply that

$$RR(C[0,1] \otimes A) \leq 1.$$

Case 2: $RR(A) = n (\geq 1)$. We do the same argument as in Case 1 using the following lemma:

Lemma 2.2 *Let A be a unital C^*-algebra with $RR(A) = n$. For any $\varepsilon > 0$, $N \geq n$, and $a_0, a_1, \cdots, a_N \in A_{sa}$, there exist $b_0, b_1, \cdots, b_n \in A_{sa}$ such that $||a_i - b_i|| < \varepsilon$ for $0 \leq i \leq N$ and $\sum_{j=k}^{k+n} b_j^2$ is invertible for $0 \leq k \leq N - n$.*

□.

Corollary 2.3 *Let A be a unital C^*-algebra. Then,*

$$RR(C[0,1]^n \otimes A) \leq n + RR(A).$$

Next, we consider the case of that X is a finite CW-complex.
Recall that the definition of the pullback.

Definition 2.4 *Let A, B, and C be C^*-algebras, and let $\phi : A \to C$ and $\psi : B \to C$ be $*$-homomorphisms. Define*

$$A \oplus_{(C,\phi,\psi)} B = \{(a,b) \in A \oplus B : \phi(a) = \psi(b)\}.$$

When ϕ and ψ are understood, we simply write $A \oplus_C B$.

One of examples for the pullback is the following:

Lemma 2.5 *Let X_0 be a compact Hausdorff space, and let $X = X_0 \cup_h D^n$ be the compact Hausdorff space obtained by attaching an n-cell D^n to X_0 via the attaching map $h : S^{n-1} \to X_0$. (Here S^{n-1} is the boundary of D^n.) Let A_0 be any C^*-algebra, set $A = C(X_0) \otimes A_0$, $B = C(D^n) \otimes A_0$, and $C = C(S^{n-1}) \otimes A_0$, and define $\phi : A \to C$ and $\psi : B \to C$ by $\phi(f) = f \circ h$ for $f : X_0 \to \mathbf{C}$ continuous and $\psi(f) = f|_{S^{n-1}}$ for $f : D^n \to \mathbf{C}$ continuous. Then*

$$A \oplus_{(C,\phi,\psi)} B \cong C(X_0 \cup_h D^n) \otimes A_0.$$

We need a result on the real rank of pullbacks. The first version of the next lemma contains an error, that is, too much surjectivity is assumed. We are grateful to Takashi Sakamoto for calling our attention to this.

Proposition 2.6 *Let A, B, and C be unital C^*-algebras, let $\phi : A \to C$ be a unital $*$-homomorphism, and let $\psi : B \to C$ be a surjective unital $*$-homomorphism. Then*

$$RR(A \oplus_C B) \leq \max(RR(A), RR(B)).$$

Using this proposition we can get the following result:

Proposition 2.7 *Let A be a unital C^*-algebra, and let X be a finite CW-complex of dimension n. Then*

$$RR(C(X) \otimes A) = RR(C([0,1]^n) \otimes A).$$

We now pass from finite CW-complexes to compact Hausdorff spaces. For this, we use the notion of an approximate inverse system of compact metric spaces, due to Mardešić and Rubin ([5], Definition 1). An approximate inverse system of compact metric spaces consists of a directed set Λ with no maximal element, for each $\lambda \in \Lambda$ a compact metric space X_λ with metric d_λ and a real number $\epsilon_\lambda > 0$, and for each λ, $\lambda' \in \Lambda$ with $\lambda \leq \lambda'$ a not necessarily continuous function $p_{\lambda\lambda'} : X_{\lambda'} \to X_\lambda$. Moreover, the following conditions must be satisfied:

(1) $d_{\lambda_1}(p_{\lambda_1\lambda_2} \circ p_{\lambda_2\lambda_3}(x), p_{\lambda_1\lambda_3}(x)) \leq \epsilon_{\lambda_1}$ for $\lambda_1 \leq \lambda_2 \leq \lambda_3$ and $x \in X_{\lambda_3}$.

(2) $p_{\lambda\lambda} = \mathrm{id}$ for all λ.

(3) For all $\lambda \in \Lambda$ and all $\eta > 0$ there is $\lambda' \geq \lambda$ such that for all $\lambda_2 \geq \lambda_1 > \lambda'$ and all $x \in X_{\lambda_2}$, we have $d_\lambda(p_{\lambda\lambda_1} \circ p_{\lambda_1\lambda_2}(x), p_{\lambda\lambda_2}(x)) \leq \eta$.

(4) For all $\lambda \in \Lambda$ and all $\eta > 0$, there is $\lambda' \geq \lambda$ such that for all $\lambda'' \geq \lambda'$ and all x, $x' \in X_{\lambda''}$, if $d_{\lambda''}(x, x') \leq \epsilon_{\lambda''}$ then $d_\lambda(p_{\lambda\lambda''}(x), p_{\lambda\lambda''}(x')) \leq \eta$.

The (inverse) limit ([5], Definition 2) $X = \lim(X_\lambda, \epsilon_\lambda, p_{\lambda\lambda'}, \Lambda)$ is the subspace of $\Pi_{\lambda\in\Lambda} X_\lambda$ defined by

$$X = \left\{ x = (x_\lambda) \in \Pi_{\lambda\in\Lambda} X_\lambda \ : \ x_\lambda = \lim_{\lambda' \geq \lambda} p_{\lambda,\lambda'}(x_{\lambda'}) \ \text{ for all } \lambda \in \Lambda \right\},$$

with the relative product topology. (See also Theorem 2 of [5].)

Lemma 2.8 *Let $(X_\lambda, \epsilon_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an approximate inverse system of compact metric spaces, with limit X. Let $p_\lambda : X \to X_\lambda$ be the restriction to X of the projection $\Pi_{\lambda\in\Lambda} X_\lambda \to X_\lambda$. Let A be a C^*-algebra, and let $\alpha_\lambda : C(X_\lambda) \otimes A \to C(X) \otimes A$ be given by $\alpha_\lambda(f) = f \circ p_\lambda$. Then for any $f_1, f_2, \ldots, f_n \in C(X) \otimes A$ and any $\varepsilon > 0$, there exist $\lambda \in \Lambda$ and $g_1, g_2, \ldots, g_n \in C(X_\lambda) \otimes A$ such that $\|\alpha_\lambda(g_m) - f_m\| < \varepsilon$ for $1 \leq m \leq n$.*

In the following result, $\tilde{A}$ denotes A if A is unital and the unitization A^+ of A if A is not unital. By definition, we have $RR(A) = RR(\tilde{A})$.

Theorem 2.9 *Let X be a normal locally compact Hausdorff space (in particular, a σ-compact locally compact Hausdorff space), and let $n = \dim(X)$. Then for any C^*-algebra A we have*

$$RR(C_0(X) \otimes A) \leq RR(C([0,1]^n) \otimes \tilde{A}) \leq \dim(X) + RR(A).$$

Sketch of Proof. The inequality $RR(C[0,1]^n \otimes \tilde{A}) \leq \dim X + RR(A)$ follows from Theorem 2.1. Since $RR(A) = RR(\tilde{A})$, this gives the second half of the inequality. For the first half of the inequality we may assume that X is compact and A is unital. Indeed, since X is normal, $\dim X = \dim \beta X$, where βX is Stone-Cěch compactification. So, we have

$$RR(C_0(X) \otimes A) \leq RR(C(\beta X) \otimes \tilde{A}) \leq RR(C[0,1]^n \otimes \tilde{A}).$$

Note that $RR(C_0(X) \otimes A)$ is a closed two-sided ideal of $RR(C(\beta X) \otimes \tilde{A})$, and that real rank of a C*-algebra is greater than or equal to real rank of any its closed two-sided ideal.

By Theorem 5 of [5], there exists an approximate inverse system of compact metric spaces $(X_\lambda, \epsilon_\lambda, p_{\lambda\lambda'}, \Lambda)$, with limit X, such that each X_λ is a polyhedron (and thus in particular a finite CW-complex) of dimension at most n. It follows from Proposition 2.7 that $RR(C(X_\lambda) \otimes A) \leq RR(C([0,1]^n) \otimes A)$.

Let $N = RR(C([0,1]^n) \otimes A)$, let $a_0, a_1, \ldots, a_N \in (C(X) \otimes A)_{sa}$, and let $\epsilon > 0$. By Lemma 2.8, there is $\lambda \in \Lambda$, a unital *-homomorphism $\alpha_\lambda : C(X_\lambda) \otimes A \to C(X) \otimes A$, and $b_0, b_1, \ldots, b_N \in C(X_\lambda) \otimes A$, such that $\|\alpha_\lambda(b_j) - a_j\| < \frac{\epsilon}{2}$ for $0 \leq j \leq N$. Replacing b_j by $\frac{1}{2}(b_j + b_j^*)$, we may assume each b_j is selfadjoint without increasing $\|\alpha_\lambda(b_j) - a_j\|$. By Proposition 2.7, there are $c_0, c_1, \ldots, c_N \in (C(X_\lambda) \otimes A)_{sa}$ such that $\|c_j - b_j\| < \frac{\epsilon}{2}$ for $0 \leq j \leq N$ and such that $\sum_{j=0}^N c_j^2$ is invertible. Then the elements $\alpha_\lambda(c_0), \alpha_\lambda(c_1), \ldots, \alpha_\lambda(c_N)$ are in $(C(X) \otimes A)_{sa}$, and satisfy $\|\alpha_\lambda(c_j) - a_j\| < \epsilon$ and $\sum_{j=0}^N \alpha_\lambda(c_j)^2$ is invertible. This proves that $RR(C(X) \otimes A) \leq N$. □

3 Lower bounds on rank

In this section we explain about the result (3) briefly.

Proposition 3.1 *Let A be a nonzero C*-algebra. Then* $RR(C([0,1]) \otimes A) \geq 1$.

Sketch of Proof. Suppose that $RR(C[0,1] \otimes A) = 0$. We try to get a contradiction from this assumption. We may assume that A is unital. Since A is a quotient C*-algebra, A is non-zero C*-algebra with real rank zero. Take non-zero projection p, and consider a corner algebra $C[0,1] \otimes pAp$ of $C[0,1] \otimes A$. Then $C[0,1] \otimes pAp$ has real rank zero from the fact that any non-zero hereditary C*-subalgebra of a C*-algebra with real rank zero has also real rank zero [2]. Replacing A by pAp, we may assume that A is unital.

Define $f \in C([0,1], A)_{sa}$ by $f(t) = (2t-1) \cdot 1_A$ for $0 \leq t \leq 1$. By assumption, there is an invertible selfadjoint element $g \in C([0,1], A)$ such that $\|f - g\| < \frac{1}{2}$. From the spectral argument we can conclude that there exists a point $t_0 \in [0,1]$ such that $g(t_0)$ has 0 as a spectral point. This is a contradiction to the invertibility of g. □

The following result is easily to be proved.

Proposition 3.2 *Let A be any C*-algebra. Suppose that* $sr(C([0,1]) \otimes A) = 1$. *Then* $sr(A) = 1$ *and* $K_1(A) = 0$.

Proposition 3.3 *Let A be a unital C*-algebra. Then* $sr(C([0,1]^2) \otimes A) \geq 2$.

Proof. Suppose that $sr(C([0,1]^2) \otimes A) = 1$. Then $sr(C(S^1) \otimes C([0,1]) \otimes A) = 1$ by Proposition 2.7. So $sr(C(S^1) \otimes A) = 1$ and $K_1(C(S^1) \otimes A) = 0$ by Proposition 3.2. Therefore $0 = K_1(C(S^1) \otimes A) \cong K_1(A) \oplus K_0(A)$, whence $K_0(A) = 0$. Since A is stably finite (because $sr(A) \leq sr(C(S^1) \otimes A) \leq 1$) and unital, this is a contradiction. □

4 Stable rank of $C([0,1]) \otimes A$)

In this section we explain about results (4) and (5). We need the following two technical lemmas.

Lemma 4.1 *For every $\epsilon > 0$ there is $\delta > 0$ such that whenever A is a unital C^*-algebra, $u, v \in A$ are unitaries, and $p \in A$ is a projection, with $\|up - vp\| < \delta$, then there is a path $t \mapsto z_t$ of unitaries in A with $z_0 = 1$, $z_1 up = vp$, and $\|z_t - 1\| < \epsilon$ for all $t \in [0,1]$.*

Lemma 4.2 *Let A be a unital C^*-algebra with $K_1(A) = 0$, $sr(A) = 1$, and $RR(A) = 0$. Then for every $\epsilon > 0$ there is $\delta > 0$ such that whenever $a, b \in inv(A)$ satisfy $\|a\|, \|b\| \leq 1$ and $\|a - b\| < \delta$, then there is a continuous path $t \mapsto c_t$ in $inv(A)$ such that*

$$c_0 = a, \quad c_1 = b, \quad \text{and} \quad \|c_t - a\| < \epsilon.$$

Theorem 4.3 *Let A be a unital C^*-algebra with $K_1(A) = 0$, $sr(A) = 1$, and $RR(A) = 0$. Then $sr(C([0,1]) \otimes A) = 1$.*

Proof. Let $a \in C([0,1]) \otimes A$, and let $\epsilon > 0$. We have to approximate a within ϵ by an invertible element of $C([0,1]) \otimes A$. Scaling both a and ϵ, we may assume that $\|a\| \leq 1$.

Choose $\delta > 0$ as in the previous lemma for $\frac{\epsilon}{3}$ in place of ϵ. Choose $0 = t_0 < t_1 < \cdots < t_n = 1$ such that

$$\|a(t_j) - a(t_{j-1})\| < \tfrac{\delta}{3} \quad \text{and} \quad \|a(t) - a(t_{j-1})\| < \tfrac{\epsilon}{3}$$

for $1 \leq j \leq n$ and $t \in [t_{j-1}, t_j]$. Using the fact that $sr(A) = 1$, choose $c_0, c_1, \ldots, c_n \in inv(A)$ such that

$$\|c_j - a(t_j)\| < \min(\tfrac{\epsilon}{3}, \tfrac{\delta}{3}).$$

Then $\|c_j - c_{j-1}\| < \delta$. For each j, use the previous lemma to choose a continuous path $t \mapsto b(t) \in inv(A)$, defined for $t \in [t_{j-1}, t_j]$, such that

$$b(t_{j-1}) = c_{j-1}, \quad b(t_j) = c_j, \quad \text{and} \quad \|b(t) - c_{j-1}\| < \tfrac{\epsilon}{3}.$$

The two definitions at t_j (one from the j-th interval, one from the $(j+1)$-st interval) agree, so $t \mapsto b(t)$ is a continuous invertible path defined for $t \in [0,1]$. Moreover, for $t \in [t_{j-1}, t_j]$ we have

$$\|b(t) - a(t)\| \leq \|b(t) - c_{j-1}\| + \|c_{j-1} - a(t_{j-1})\| + \|a(t_{j-1}) - a(t)\| < \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} = \epsilon.$$

□

We now give a example of a simple separable unital C*-algebra which satisfy the hypotheses of this theorem but are not AF. In particular, $sr(C([0,1]) \otimes A) = 1$ does not imply that A is AF, even if A is nuclear.

Example 4.4 *Example 4.11 of [6] gives a simple separable unital nuclear C*-algebra A satisfying $K_1(A) = 0$ and $RR(A) = 0$. It also has $sr(A) = 1$. It thus satisfies the hypotheses of Theorem 4.3. It is not AF because $K_0(A)$ contains torsion.* □

The following result induces the fact that $sr(C[0,1] \otimes A) = 1$ does not imply that $RR(A) = 0$.

Theorem 4.5 *Let $A = \varinjlim A_n$ be a direct limit of interval algebras of the following form. Let $(y_0, y_1, \ldots)$ be a dense sequence in $[0,1]$, let $1 = k(0) < k(1) < \cdots$ be integers such that $k(n)|k(n+1)$ for all n, let $A_n = C([0,1], M_{k(n)})$, and let $\phi_{n,n+1} : A_n \to A_{n+1}$ be the unital maps given by*

$$\phi_{n,n+1}(a) = \mathrm{diag}(a, a, \ldots, a, a(y_n)),$$

where $a(y_n)$ stands for the constant function on $[0,1]$ with that value. Then we have $sr(C([0,1]) \otimes A) = 1$.

Example 4.6 *By Theorem 9 of [3], there is a simple C*-algebra A of the form considered in Theorem 4.5 such that $RR(A) = 1$. The theorem gives $sr(C([0,1]) \otimes A) = 1$.* □

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