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On classification of approximately inner actions of discrete amenable groups on strongly amenable subfactors

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1 Introduction

In the theory of operator algebras, study of automorphisms is one of the most important subjects. In [3] and [5], A. Connes has classified automorphisms of approximately finite dimensional (AFD) semifinite factors. After Connes' classification, V. F. R. Jones has introduced the characteristic invariant and classified actions of finite groups on the AFD type II_1 factor in [14]. Immediately after that work, A. Ocneanu succeeded in classifying actions of discrete amenable groups on AFD semifinite factors. Based on these results, actions of discrete amenable groups on AFD factors of type III have been classified in [30], [21] and finally [16].

In subfactor theory, P. H. Loi first studied the automorphisms of subfactors in [23]. He introduced the invariant of automorphisms $\Phi(\alpha)$, which we will call the Loi invariant, and obtained some structural results on type III_λ subfactors. In [28], S. Popa has introduced the proper outerness for automorphisms of subfactors, and proved that properly outer actions of discrete amenable groups on strongly amenable subfactors of type II_1 are classified by the Loi invariant. (In [1], Choda-Kosaki have introduced the same property independently and they call it strong outerness.)

So it is natural to ask if we can classify not necessary strongly outer actions of discrete amenable groups. Several people have noticed the similarity between theory of subfactors and (single) type III factors, and especially this similarity has been emphasized by Kawahigashi in [17]. Based on this similarity, Kawahigashi has introduced two outer conjugacy invariants for automorphisms, i.e., the higher obstruction and an algebraic ν invariant in [20]. The former is an analogue of the Connes obstruction (or modular obstruction) and the latter is an analogue of Sutherland-Takesaki's modular invariant. With these two invariants, Kawahigashi has classified approximately inner automorphisms of subfactors under some extra assumptions on subfactors. After this work, Loi removed these extra assumptions and generalized Kawahigashi's results by commuting square technique in [24].

The aim of this article is generalization of Kawahigashi's result to the case of actions of arbitrary discrete amenable groups. Out idea is based on [30] and [16], that is, we use duality technique. In the classification of group actions on type III factors, by taking the crossed product by modular automorphism group, the classification problem is reduced to the case of semifinite factors. In subfactor setting, non-strongly outer automorphisms correspond to the modular automorphism, and we take the crossed product by the relative χ group and reduce classification problem to easier case.

To classify actions, we consider two kinds of invariants. One is the characteristic invariant, and another is a ν invariant. Our characteristic invariant may or may not be different from the original one introduced by Jones, and is the generalization of Kawahigashi's higher obstruction. Our characteristic invariant has relation with the κ invariant, which has been introduced by Jones in [13]. In this article, we consider the "algebraic" version of the κ invariant, and if this κ invariant is trivial, then our characteristic invariant coincides with the usual characteristic invariant. The second invariant ν invariant is precisely Kawahigashi's algebraic ν invariant, and this is an analogue of Sutherland-Takesaki's modular invariant mentioned as above.

This article is a brief explanation of [25], and more details will be found in [25].

2 Preliminaries and notations

In this section, we recall several facts on automorphisms of subfactors and fix notations.

Let $N \subset M$ be a subfactor with finite index and $N \subset M \subset M_1 \subset M_2 \subset \cdots$ the Jones tower. (In this article, we consider only minimal index, and assume that every subfactor of type II is extremal in the sense of [29].) Let α be an automorphism of $N \subset M$. Then α can be extended onto M_k inductively by setting $\alpha(e_k) = e_k$, where e_k is the Jones projection for $M_{k-2} \subset M_{k-1}$.

First we recall the Loi invariant and the strong outerness for automorphisms.

Definition 2.1 ([23, Section 5]) *With above notations, put*

$$\Phi(\alpha) := \{\alpha|_{M' \cap M_k}\}_k.$$

We call this Φ the Loi invariant for α.

Definition 2.2 ([1, Definition 1], [28, Definition 1.5.1]) *An automorphism $\alpha \in \mathrm{Aut}\,(M, N)$ is said to be strongly outer or properly outer if we have no non-zero $a \in \cup_k M_k$ satisfying $\alpha(x)a = ax$ for every $x \in M$.*

According to the notation of [16], we use the notation $\mathrm{Cnt}_r\,(M, N)$ to denote the set of non-strongly outer automorphisms of $N \subset M$.

In [28], Popa proved the following important results.

Theorem 2.3 ([28, Theorem 1.6]) *Non-strongly outer automorphisms are centrally trivial. Moreover if $N \subset M$ is a strongly amenable subfactor of type II_1, then these two notions are equivalent.*

Theorem 2.4 ([28, Theorem 3.1]) *Let $N \subset M$ be a strongly amenable subfactor of type II_1 and G a discrete amenable group. If α and β are strongly outer actions of G on $N \subset M$, then α and β are cocycle conjugate if and only if $\Phi(\alpha) = \Phi(\beta)$ hold.*

In [23, Theorem 5.4], Loi gave the following characterization of approximately inner automorphisms.

Theorem 2.5 *If $N \subset M$ is a strongly amenable subfactor of type II_1, then* $\operatorname{Ker}\Phi = \overline{\operatorname{Int}}(M, N)$ *holds.*

Loi proved above theorem when $N \subset M$ is of finite depth. But his proof works when $N \subset M$ is strongly amenable.

In [17], Kawahigashi defined the relative Connes invariant $\chi(M, N)$ for a subfactor of type II$_1$ $N \subset M$ as an analogue of Connes' χ group [4]. Based on this, Goto defined the algebraic χ group for subfactors in [11] as follows.

Definition 2.6 *Set*

$$\chi_a(M, N) := \frac{\operatorname{Ker}\Phi \cap \operatorname{Cnt}_r(M, N)}{\operatorname{Ad}\mathcal{N}(M, N)},$$

where $\mathcal{N}(M, N)$ is the normalizer group. We call this group an algebraic χ group for $N \subset M$.

In [11, Theorem 2.1], Goto proved that $\chi_a(M, N)$ is an abelian group.

When $N \subset M$ is strongly amenable and has the trivial normalizer, $\chi(M, N) = \chi_a(M, N)$ holds because of Theorem 2.3 and Theorem 2.5.

3 Connes-Radon-Nikodym type cocycle theorem

Throughout the rest of this article, we always assume that $N \subset M$ has the trivial normalizer.

Theorem 3.1 *For every $\sigma \in \operatorname{Cnt}_r(M, N)$ and $\alpha \in \operatorname{Ker}\Phi$, there exists a unitary $u_{\alpha,\sigma} \in U(N)$ such that*
(1) $\operatorname{Ad} u_{\alpha,\sigma}\sigma = \alpha\sigma\alpha^{-1}$,
(2) $u_{\alpha,\sigma_1\sigma_2} = u_{\alpha,\sigma_1}\sigma_1(u_{\alpha,\sigma_2})$,
(3) $u_{\alpha\beta,\sigma} = \alpha(u_{\beta,\sigma})u_{\alpha,\sigma}$,
(4) $u_{\alpha,\operatorname{Ad} v} = \alpha(v)v^*$, $u_{\operatorname{Ad} v,\sigma} = u\sigma(u^*)$, $v \in U(N)$,
(5) *Take nonzero $a \in M_k$ satisfying $\sigma(x)a = ax$. Then a unitary $u_{\alpha,\sigma}$ satisfies $\alpha(a) = u_{\alpha,\sigma}a$.*

We only explain how to construct $u_{\alpha,\sigma}$. Since σ is non-strongly outer, there exist $k > 0$ and $0 \neq a \in M_k$ such that $\sigma(x)a = ax$ holds for every $x \in M$. This implies ${}_M M_{kM} \succ {}_{M\sigma^{-1}} M_M$. Fix an intertwiner $T \in \operatorname{Hom}({}_M M_{kM}, {}_{M\sigma^{-1}} M_M)$ with $TT^* = 1$. Let U_α be the canonical implementing unitary of α on $L^2(M_k)$. Set $u_{\alpha,\sigma} := \sigma(TU_\alpha T^*\hat{1})$. By using $\alpha \in \operatorname{Ker}\Phi$, we can prove that this $u_{\alpha,\sigma}$ does not depend on the choice of $k > 0$ or an intertwiner T. Hence $u_{\alpha,\sigma}$ is well-defined and satisfies the properties in the above proposition.

Remark. When α itself is non-strongly outer and has the trivial Loi invariant, $u_{\alpha,\alpha}$ is equal to the higher obstruction $\gamma_h(\alpha)$. So in general, $u_{\alpha,\alpha}$ fails to be 1. Examples of such automorphisms are non-strongly outer automorphisms of subfactors with Dynkin diagram A_{4n-1}. See [18], [33] and [9].

We compare the above theorem and Connes-Radon-Nikodym cocycle theorem. Let M be a type III factor, $\mathcal{F}(M)$ the flow of weights of M and ϕ a dominant weight of M. (See [6] for these notations.) For each $c \in Z^1(\mathcal{F}(M))$, we can associate an extended modular automorphism $\sigma_c^\phi \in \mathrm{Aut}\,(M)$. Take $\alpha \in M$. Then we have $\alpha\sigma_c^\phi\alpha^{-1} = \sigma_{\mathrm{mod}\,(\alpha)(c)}^{\phi\circ\alpha^{-1}} = \mathrm{Ad}\,[D\phi\circ\alpha^{-1} : D\phi]_c\sigma_{\mathrm{mod}\,(\alpha)(c)}^{\phi}$. Especially if $\mathrm{mod}\,(\alpha)$ is trivial, then α and σ_c^ϕ commute in $\mathrm{Aut}\,(M)/\mathrm{Int}(M)$. Hence we have the correspondence as in Table 1.

Table 1:

Subfactor case	$\alpha \in \mathrm{Ker}\,\Phi$	$\sigma \in \mathrm{Cnt}_r\,(M,N)$	$u_{\alpha,\sigma}$
Type III factor case	$\alpha \in \mathrm{Ker}\,\mathrm{mod}$	$\sigma_c^\phi,\ c \in Z^1(\mathcal{F}(M))$	$[D\phi\circ\alpha^{-1} : D\phi]_c$

Here we take α, $\beta \in \mathrm{Ker}\,\Phi \cap \mathrm{Cnt}_r\,(M,N)$ and assume that α and β commute. In this case $u_{\alpha,\beta}$ is a scalar. We set $\kappa_a(\alpha,\beta) := u_{\beta,\alpha}^*$ and call this scalar the algebraic κ invariant, or the κ invariant simply. The origin of this notation is Jones' κ invariant in [13] and when $N \subset M$ is strongly amenable, it can be proved that this coincides with the relative Jones κ invariant defined in [19, Section 2] in the same way as in the proof of [20, Theorem 4.1].

Here we make following assumptions.

(A1) $\chi_a(M,N)$ is a finite group.

(A2) There exists a lifting from $\chi_a(M,N)$ to $\mathrm{Aut}\,(M,N)$. We fix one of the lifting σ. This σ can be regarded as an action of $\chi_a(M,N)$.

(A3) $\mathrm{Ker}\,\Phi = \mathrm{Aut}\,(M,N)$.

Set $K := \chi_a(M,N)$. By using Theorem 3.1, we can extend $\alpha \in \mathrm{Ker}\,\Phi$ onto the crossed product subfactor $N \rtimes_\sigma K \subset M \rtimes_\sigma K$ in the same way as [12].

Proposition 3.2 *Let w_g be an implementing unitary in $N \rtimes_\sigma K \subset M \rtimes_\sigma K$. For each $\alpha \in \mathrm{Ker}\,\Phi$, there exists a unique automorphism $\tilde{\alpha} \in \mathrm{Aut}\,(M \rtimes_\sigma K, N \rtimes_\sigma K)$ satisfying $\tilde{\alpha}(x) = \alpha(x)$ for $x \in M$ and $\tilde{\alpha}(w_g) = u_{\alpha,\sigma_g}w_g$.*

Moreover we have the following.

(1) *For $\alpha, \beta \in \mathrm{Ker}\,\Phi$, $\tilde{\alpha}\tilde{\beta} = \widetilde{\alpha\beta}$ and $\tilde{\alpha}^{-1} = \widetilde{\alpha^{-1}}$ hold.*

(2) *For $\alpha = \mathrm{Ad}\,u$, $u \in U(N)$, $\tilde{\alpha} = \mathrm{Ad}\,u$ holds.*

(3) *The extension $\tilde{\alpha}$ commutes with the dual action of $\hat{K}$.*

(4) *If $\kappa(\sigma_g, \sigma_h) = 1$ hold for every $h \in G$, then $\widetilde{\sigma_g}$ is an inner automorphism and $\widetilde{\sigma_g} = \mathrm{Ad}\,w_g$ holds.*

4 Cocycle conjugacy invariants

Let G be a discrete group and α be an action of G with trivial Loi invariant. We consider the cocycle conjugacy invariants for α.

Set $H := \{n \in G \,|\, \alpha_n \in \mathrm{Cnt}_r\,(M,N)\}$. Obviously H is a normal subgroup of G and a cocycle conjugacy invariant. Define a homomorphism ν from H to $\chi_a(M,N)$ by $\nu(n) = [\alpha_n] \in \chi_a(M,N)$. Since we assume that α has the trivial Loi invariant, this ν is well defined. This ν satisfies $\nu(gng^{-1}) = \nu(n)$ for every $g \in G$ and $n \in H$. In [20], Kawahigashi introduced an algebraic ν invariant ν_{alg} and an analytic ν invariant ν_{ana} as an analogue of Sutherland-Takesaki's modular invariant in [30, Definition 5.8]. Obviously our ν is a generalization of an algebraic ν invariant ν_{alg}.

Hence for $n \in H$ we have $\alpha_n = \mathrm{Ad}\, v_n \sigma_{\nu(n)}$ for some unitary $v_n \in U(N)$.

From equations $\alpha_m \alpha_n = \alpha_{mn}$ and $\alpha_g \alpha_{g^{-1}ng} \alpha_g^{-1} = \alpha_n$, $m, n \in H$ and $g \in G$, we get scalars $\lambda(g,n)$ and $\mu(m,n)$ by the following equations in a similar way as in the single factor case.

$$v_m \sigma_{\nu(m)}(v_n) = \mu(m,n) v_{mn},$$
$$\alpha_g(v_{g^{-1}ng}) u_{\alpha_g, \sigma_{\nu(n)}} = \lambda(g,n) v_n.$$

Proposition 4.1 *For $h, k, l \in H$ and $g, g_1, g_2 \in G$, we have the following equalities.*
(1) $\mu(h,k)\mu(hk,l) = \mu(k,l)\mu(h,kl)$,
(2) $\lambda(g_1 g_2, n) = \lambda(g_1, n)\lambda(g_2, g_1^{-1} n g_2)$,
(3) $\lambda(h,k) = \mu(h, h^{-1}kh)\mu(k,h)\kappa_a(\nu(k), \nu(h))$,
(4) $\lambda(g, hk)\overline{\lambda(g,h)\lambda(g,k)} = \mu(h,k)\overline{\mu(g^{-1}hg, g^{-1}kg)}$,
(5) $\lambda(g,1) = \lambda(1,h) = \mu(h,1) = \mu(1,h) = 1$.

By $Z(G, H|\kappa_a)$, we denote the set of (λ, μ) satisfying the above conditions. The above definition of λ and μ are depend on the choice of unitaries v_n. To get rid of this dependence, we define the equivalence relation in $Z(G, H|\kappa_a)$ as follows. Two elements (λ_1, μ_1) and (λ_2, μ_2) are equivalent if and only if there exists a map c from H to $\mathbf{T}$ with $c_e = 1$ such that $\lambda_1(g,n) = \overline{c_{g^{-1}ng}} \lambda_2(g,n) c_n$ and $\mu_1(m,n) = \overline{c_m c_n} \mu_2(m,n) c_{mn}$ hold. We set $\Lambda(G, H|\kappa_a) = Z(G, H|\kappa_a)/\sim$, where $\sim$ denotes the above equivalence relation. If the κ-invariant is trivial, then this is a usual characteristic invariant $\Lambda(G, N)$.

For a given action α of G, we get $[\lambda, \mu] \in \Lambda(G, H|\kappa_a)$ and $\nu \in \mathrm{Hom}_G(H, \chi_a(M,N))$. When we have to specify an action α, we also use notations H_α, $\Lambda(\alpha) = [\lambda_\alpha, \mu_\alpha]$ and ν_α.

We have the following proposition.

Proposition 4.2 *The triplet $(H_\alpha, \Lambda(\alpha), \nu_\alpha)$ is a cocycle conjugacy invariant.*

5 Classification results

Here we can state the main theorem in this article. We always assume (A1), (A2) and (A3) in Section 3.

Theorem 5.1 *Let $N \subset M$ be as above and G a discrete amenable group. Let α and β be approximately inner actions of G, whose range of ν are cyclic groups. Then α and β are cocycle conjugate if and only if $(H_\alpha, \Lambda(\alpha), \nu_\alpha) = (H_\beta . \Lambda(\beta), \theta(\nu_\beta))$ hold for some $\theta \in \mathrm{Aut}_{N \subset M}(\chi(M,N))$.*

Examples of subfactors satisfying the assumptions of the above theorem are $SU(n)_k$ subfactors, where n is a odd number, or n is even number and $2n$ divides k.sec See [7], [33] and [10].

To prove Theorem 5.1, we use the following theorem.

Theorem 5.2 *Let $Q \subset P$ be a strongly amenable subfactor of type II_1, G a discrete amenable group. (Here we never assume* $\mathrm{Ker}\,\Phi = \mathrm{Aut}\,(P,Q)$.*) Let α and β be actions of G on $Q \subset P$ such that $H := \alpha^{-1}(\mathrm{Int}(P,Q)) = \alpha^{-1}(\mathrm{Cnt}\,(P,Q)) = \beta^{-1}(\mathrm{Int}(P,Q)) = \beta^{-1}(\mathrm{Cnt}\,(P,Q))$. Then α and β are cocycle conjugate if and only if $(\Phi(\alpha), \Lambda(\alpha)) = (\Phi(\beta), \Lambda(\beta))$ holds.*

Outline of proof of Theorem 5.1. Set $Q \subset P := N \rtimes_\sigma K \subset M \rtimes_\sigma K$. By [32, Theorem 6.1], $Q \subset P$ is strongly amenable. By Proposition 3.2, we have an action of $G \times \hat{K}$ by $(g,p) \mapsto \tilde{\alpha}_g \hat{\sigma}_p$, which we denote $\tilde{\alpha}_{g,p}$ for simplicity. We also get an action $\tilde{\beta}$ of $G \times \widehat{K}$ on $Q \subset P$. Here by assumptions $\Lambda(\alpha) = \Lambda(\beta)$ and $\nu_\alpha = \nu_\beta$, we can show $\Phi(\tilde{\alpha}) = \Phi(\tilde{\beta})$, $H_\alpha = \tilde{\alpha}^{-1}(\mathrm{Int}(P,Q)) = \tilde{\beta}^{-1}(\mathrm{Int}(P,Q)) = \tilde{\alpha}^{-1}(\mathrm{Cnt}\,(P,Q)) = \tilde{\beta}^{-1}\mathrm{Cnt}\,(P,Q)$ and $\Lambda(\tilde{\alpha}) = \Lambda(\tilde{\beta})$. (Here we use (4) in Proposition 3.2 and the triviality of algebraic κ invariant to determine inner part of $\tilde{\alpha}$ and $\tilde{\beta}$.)

By Theorem 5.2, $\tilde{\alpha}$ and $\tilde{\beta}$ are cocycle conjugate. Then we take the partial crossed product $Q \rtimes_{\hat{\sigma}} \hat{K} \subset P \rtimes_{\hat{\sigma}} \hat{K}$, extend the actions $\tilde{\alpha}$ and $\tilde{\beta}$ of G canonically and we denote them $\tilde{\tilde{\alpha}}$ and $\tilde{\tilde{\beta}}$ respectively. Then by [30, Proposition 1.1], $\tilde{\tilde{\alpha}}$ and $\tilde{\tilde{\beta}}$ are cocycle conjugate. With Takesaki's duality theorem, we can conclude α and β are cocycle conjugate. □

To prove Theorem 5.2, we need some preparations. The following lemma can be verified in the same way as in the proof of [26, Lemma 2.3].

Lemma 5.3 *Let $N \subset M$, G and α be as in Theorem 5.2. Assume $\Lambda(\alpha)$ is trivial. Then we can choose α-cocycle w_g such that $\mathrm{Ad}\, w_h \alpha_h = id$ for $h \in H$.*

Lemma 5.4 *Let $N \subset M$, G, α and β be as in Theorem 5.2 and assume that the characteristic invariants of α and β are trivial. Then α and β are cocycle conjugate.*

Proof. By the previous lemma, we can choose an α-cocycle w_g^1 and a β-cocycle w_g^2 such that $\mathrm{Ad}\, w_h^1 \alpha_h = \mathrm{Ad}\, w_h^2 \beta_h = id$ for $h \in H$. So we can regard these action as the centrally free actions of G/H. Since they have the same Loi invariant, these two actions are cocycle conjugate by [28, Theorem 3.1]. Since this means that they are cocycle conjugate as actions of G, original actions α and β are cocycle conjugate. □

The following lemma is the subfactor analogue of [26].

Lemma 5.5 *Let $N \subset M$, G and α be as in Theorem 5.2. Then α is cocycle conjugate to the action $\alpha \otimes \sigma^{(0)}$, where $\sigma^{(0)}$ is the model action of G/H on the AFD type II_1 factor R_0 constructed by Ocneanu in* [26], *and we regard $\sigma^{(0)}$ as the action of G in the natural way.*

Proof of Theorem 5.2. Let σ be an action of G on R_0 whose characteristic invariant is an inverse of $\Lambda(\alpha)$ and $\bar{\sigma}$ an action of G on R_0 with the characteristic invariant $\Lambda(\alpha)$. Then the characteristic invariants of $\alpha \otimes \sigma$ and $\beta \otimes \sigma$ are trivial, so they are cocycle conjugate by Lemma 5.4. Then we have

$$\begin{aligned} \alpha &\sim \alpha \otimes \sigma^{(0)} \sim \alpha \otimes \sigma \otimes \bar{\sigma} \\ &\sim \beta \otimes \sigma \otimes \bar{\sigma} \sim \beta \otimes \sigma^{(0)} \\ &\sim \beta. \end{aligned}$$

Hence α and β are cocycle conjugate. □

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