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Continuous Accumulation Games - An Overview

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0.1 Introduction

In a continuous accumulation game on a continuous region (CAGCR), a HIDER distributes material over a continuous region at each moment of discrete time (turn), and a SEEKER examines the region. If the SEEKER locates any of the material hidden, the SEEKER confiscates it. The goal of the HIDER is to accumulate a certain amount of material before a given time and the goal of the SEEKER is to prevent this. In this paper we shall describe some results on continuous accumulation games that are treated in greater detail in the paper *Continuous accumulation games on continuous regions.*

The issues raised when the region is continuous are substantially different from the discrete case as we illustrate in the following two examples.

Example 1 *The HIDER hides h units of material on the circumference C of a circle having length one in such a way that the upper boundary forms a continuous nonnegative function f on C with*

$$\int_C f(t)\ dt = h.$$

The SEEKER may search one time and win if it finds any part of the HIDER's material. The SEEKER may search a connected arc of the circumference having length $s < 1$. If the payoff is to the HIDER then this game has value s. For if the SEEKER chooses the starting point for the search arc according to a random distribution on the circle, then with probability at least s this search arc will contain a point u on the circle with $f(u) > 0$. If the search arc contains such a point u then the SEEKER will find a positive amount of the HIDER's material, due to the nature of continuity. Thus the SEEKER will win with probability at least s. On the other hand, if the HIDER chooses a point x on the circumference at random and then concentrates its material over an arc of length t beginning at x, for instance by using a function with graph an isosceles triangle of height $2h/t$ then this arc will intersect the arc of the SEEKER with probability at most $(t+s)$. As t converges to 0 this quantity converges to s. Thus the HIDER can hold the expected payoff to the SEEKER as close to s as the HIDER desires.

Example 2 *We shall slightly modify the model of Example 1 and obtain a totally different result. The HIDER can use the same strategies as in Example 1, but the SEEKER is allowed to use any open set. By a well known exercise in real analysis there is a dense open set S of*

C such that the measure of S is equal to s no matter how small s is. Suppose the SEEKER chooses the set S. If the maximum of the function f on C is M then S has a nonempty open intersection with the set

$$D = \{x : f(x) > M/2\}.$$

Then we shall have

$$\int_{S\cap D} f(x)\ dx > (M/2)\, m(S \cap D) > 0$$

where m denotes the Lebesgue measure of the open set $S \cap D$. Therefore, the expected payoff to the HIDER is zero no matter what mixed strategy it uses.

0.2 The Interval Model

For the interval model we take as the continuous region the interval $I = [0, n]$ with length n and assume the SEEKER can examine an open subinterval of length s. Without loss of generality we can replace replace n with 1 and s with s/n, and we shall usually assume that $n = 1$. The HIDER can distribute h units of material in any way such that its upper boundary takes the shape of a continuous function f on I with $\int_0^1 f(t)\ dt = h$. For simplicity we shall say that the HIDER *chooses the function* f. If at the beginning of a turn the HIDER chooses the function f and the SEEKER chooses the subinterval $A = [t, t+s]$ then at the end of the turn the SEEKER will be left with $\int_{I\setminus A} f(t)\ dt$ units. The HIDER wins, i. e. receives payoff 1, if at the end of any turn it has N units of material remaining. The HIDER loses (payoff 0) if after T turns it has failed to accumulate N units. We can, without loss of generality, assume that $N = 1$ by scaling the amount of material that the HIDER can conceal, and we shall usually do this.

0.2.1 Single Stage Search, Strategies

Before developing the solution of the CAGCR in which the there is only a single turn describe some strategies that the SEEKER and HIDER can use.

The Parameter p and the Covering Strategy for the SEEKER

If the SEEKER can search an interval of length s let $p(s) = p$ denote the smallest number of closed subintervals of length s that are needed to cover I. Then $p = 1/s$ if $1/s$ is an integer and $p = [1/s] + 1$ otherwise, where $[\,]$ denotes the greatest integer function. If $s < 1$ then $p \geq 2$. As we shall illustrate the parameter p is important to the analysis of the CAGCR on the interval.

Definition 3 *The* covering strategy *of the SEEKER is to choose with probability* $1/p$ *an interval covering one of the* p *intervals* $[k/p, (k+1)/p]$, $k = 0, 1, 2 \ldots, p-1$.

The SEEKER is capable of undertaking the covering strategy since the length of $[k/p, (k+1)/p]$ is $1/p \leq s$.

Point Strategy for the HIDER

Since the boundary of the HIDER's material must be a continuous function it is not possible for the HIDER to concentrate M units of its material at a single point t_0. However, the HIDER can approximate such a distribution by using a function $f(t)$ that is 0 when t is not in the interval $(t_0 - \varepsilon, t_0 + \varepsilon)$ and whose graph is an isosceles triangle over the interval with altitude M/ε. If t is one of the endpoints of I we can adjust the function so that its graph is a right triangle rather than an isosceles.

Definition 4 *The point strategy* $P(t_0, M)$ *is the strategy of concentrating* M *units of material at the point* t_0.

If we say that the HIDER uses the point strategy $P(t_0, M)$ we actually mean that the HIDER approximates such a strategy using, for example, one of the triangular functions described above. The HIDER cannot receive the full payoff v obtained by a point strategy, but only payoffs that converge to v. By the strategy

$$P(t_0, M_1) + P(t_1, M_2) + \ldots + P(t_k, M_k)$$

we mean the HIDER concentrates M_1 units at the point t_0 and M_1 units at the point $t_1, \ldots,$ and M_k units at t_k.

Solution of the One Stage Game on the Interval

By $I_s(h)$ we denote the value of the single stage CAGCR on the unit interval I in which the HIDER conceals h units by choosing a function f with $\int_I f(t)\,dt = h$ and the SEEKER chooses a subinterval A with length s. The HIDER wins if $\int_{I\setminus A} f(t)\,dt \geq 1$ and loses otherwise. We assume that $s < 1$ because otherwise the SEEKER must find everything so $I_s(h)$ has to be 0. The following proposition shows that the situation is deterministic except for a narrow range of values of h.

Proposition 5 *If* $h \geq \frac{p}{p-1}$ *then the HIDER can win with certainty by using the strategy* $\sum_{k=0}^{p-1} P\left(\frac{k}{p-1}, \frac{h}{p}\right)$ *so that* $I_s(h) = 1$.

Proof. The SEEKER's search interval can cover at most one of the points $\frac{k}{p-1}$: $k = 0, 1, \ldots, p-1$ and the amount of material deposited at the remaining points is

$$(p-1)\frac{h}{p} \geq (p-1)\frac{p}{(p-1)p} = 1$$

so that the HIDER will certainly win. □

Theorem 6 *If* $1 \leq h < \frac{p}{p-1}$ *then* $I_s(h) = \frac{p-1}{p}$*.An optimal strategy for the SEEKER is the covering strategy, and an optimal strategy for the HIDER is to choose one of the point strategies* $P\left(\frac{j}{p-1}, \frac{h}{p}\right) : j = 1, 2, \ldots, p-1$ *with probability* $\frac{1}{p}$.

Proof. Suppose the SEEKER adopts the covering strategy and the HIDER uses the pure strategy f. Let A_k denote the interval $\left[\frac{k}{p}, \frac{k+1}{p}\right]$ for $k = 0, 1, \ldots, p-1$. The sets $I \backslash A_k : k = 0, 1, \ldots, p-1$ cover I a total of $p-1$ times so we have

$$\sum_{k=0}^{p-1} \int_{I \backslash A_k} f(t)\ dt = (p-1)h < \frac{(p-1)p}{p-1} = p.$$

This means that $\int_{I \backslash A_k} f(t)\ dt < 1$ for at least one k. Thus the HIDER can win with probability no greater than $\frac{p-1}{p}$because the SEEKER's interval will cover one of the intervals A_k.

The HIDER's search interval can cover at most one of the points $\frac{j}{p-1} : j = 0, 1, \ldots, p-1$ so that if the HIDER chooses one of these points with probability $\frac{1}{p}$ and concentrates its material there, the HIDER will miss it with probability $\frac{p-1}{p}$. Thus the HIDER can obtain at least this amount. □

0.2.2 Multistage CAGCR on an Interval with Rearrangement

When we study games with more than one turn the following question arises: Can the HIDER rearrange its material at the beginning of every turn, or is it prohibited from doing so. If the HIDER can rearrange its material this results in an essentially new game for the SEEKER on each turn. We shall analyze such games in this section.

Distributions of Material

If the HIDER conceals h units of material on some given turn, we are concerned, not only with the average amount of material that will remain after the search, but with the probability distribution of various amounts of that material. We shall say that the remaining material has a *p-fold* distribution if there are w_i units of material remaining with probability $\frac{1}{p}$ for $i = 1, 2, \ldots, p$; and $\sum_i w_i \leq (p-1)h$. We do not require all of the amounts w_i to be distinct. We shall now show that no matter what the HIDER does the SEEKER can force it to end the turn with a p-fold distribution and that no matter what the SEEKER does the HIDER can obtain any p-fold distribution that it requires.

Theorem 7 *If the HIDER conceals h units of material and the SEEKER uses the covering strategy then at the end of the turn the HIDER will be left with a distribution* $(v_1, v_2, \ldots v_p)$ *where HIDER retains the amount* v_i *with probability* $\frac{1}{p}$ *and* $\sum_i v_i \leq (p-1)\,h$. *If the HIDER uses with probability* $\frac{1}{p!}$ *one of the strategies*

$$\sum_{j=0}^{p-1} P\left(\frac{j}{p-1}, u_{\pi(j)}\right) \text{ where } \sum_{j=0}^{p-1} u_j = h$$

as π *ranges over the permutations of* $(0, 1, \ldots, p-1)$ *then for* $j = 0, 1, \ldots p-1$ *the HIDER can obtain with probability* $\frac{1}{p}$ *the amount*

$$w_j = (h - u_j)\,,$$

and the distribution (w_j) *is a p-fold distribution.*

Proof. Suppose the HIDER distributes its material with the function f. If the SEEKER uses the covering strategy it will cover the interval $\left[\frac{j}{p}, \frac{j+1}{p}\right]$, $j = 0, \ldots, p-1$ with its interval I of length $s > \frac{1}{p}$ with probability $\frac{1}{p}$. Thus the HIDER will be left with the amount

$$v_j = h - \int_I f(x)\ dx \leq h - \int_{\frac{j}{p}}^{\frac{j+1}{p}} f(x)\ dx$$

with probability $\frac{1}{p}$. Since

$$\sum_{j=0}^{p-1} v_j \leq ph - \sum_{j=0}^{p-1} \int_{\frac{j}{p}}^{\frac{j+1}{p}} f(x)\ dx = ph - \int_0^1 f(x)\ dx = (p-1)\,h$$

our first assertion is valid.

The second assertion follows since the SEEKER can cover at most one of the p points at which the HIDER has placed its material and since the HIDER is choosing the order of the amounts at random, no one u_j is more likely to be covered than another. □

Iteration Relations

Let us denote by $I(\mathbf{h}, \mathbf{s})$ the value of the T stage CAGCR with $h = (h_1, h_2, \ldots, h_T)$ and $s = (s_1, s_2, \ldots, s_T)$ in which the HIDER conceals h_i units on turn i and the SEEKER searches an interval of length s_i. Denote by $f(u)$ the function

$$f(u) = I\left((u + h_2, h_3, \ldots h_T), (s_2, s_3, \ldots, s_T)\right).$$

Theorem 8 *The value of* $I(\mathbf{h}, \mathbf{s})$ *is the solution to the following optimization problem where* $p = p(s_1)$.

$$\begin{array}{c} \textit{Maximize } M = \frac{1}{p}\sum_{j=0}^{p-1} f(u_j) \\ \textit{subject to } u_j \geq 0,\ u_j \leq h,\ \sum_{j=0}^{p-1} u_j \leq (p-1)\,h_1. \end{array} \tag{0.1}$$

An optimal strategy for the SEEKER is the covering strategy. An optimal strategy for the HIDER is to choose with probability $\frac{1}{p!}$ one of the permutation (v_j) of the numbers $(h-u_j)$ and play the strategy $\sum_{j=0}^{p-1} P\left(\frac{j}{p-1}, v_j\right)$.

Proof. If the SEEKER uses the covering strategy then the HIDER will end the turn with a p-fold distribution of values (w_j). The expected payoff will then be $\frac{1}{p}\sum_{j=0}^{p-1} f(w_j) \leq M$. If the HIDER chooses the suggested strategy then it will end the turn with the p-fold distribution of values (u_j) so it will obtain $\frac{1}{p}\sum_{j=0}^{p-1} f(u_j) = M$. □

Corollary 9 *For every integer T the function $I(h,s)$ where $\mathbf{h} = (h_1, h_2, \ldots, h_T)$ and $\mathbf{s} = (s_1, s_2, \ldots, s_T)$ assumes only finitely many values.*

Proof. We proove the result by induction on T. For $T = 1$ the result is true by Theorem 6. If it is true for $T-1$ then it is true for T since if $I(\mathbf{h}, \mathbf{s})$ takes on p values when $\mathbf{h}$ and $\mathbf{s}$ have length the number of possible subsums of $\sum_{j=0}^{p-1} f(u_j)$ is at most 2^p so the number of solutions to the optimization problem is at most 2^p. □

Suppose we let $\{0 = M_0 < M_1 < \ldots < M_k = 1\}$ denote the finite collection of values that $I(\mathbf{h}, \mathbf{s})$ assumes when $\mathbf{h}$ and $\mathbf{s}$ have length $T-1$, and let t_j denote the smallest value such that $f(t_j) = I((t_j + h_2, h_3, \ldots h_T), (s_2, s_3, \ldots, s_T)) \geq M_j$. It may happen that t_j will be 0 if h_2 is large enough. Then we transform the nonlinear optimization problem 0.1 into the following integer programming problem.

Theorem 10 *The value of $I(\mathbf{h}, \mathbf{s})$ is the solution to the following optimization problem where $p = p(s_1)$.*

$$\begin{array}{c} \textit{Maximize } M = \frac{1}{p}\sum_{j=0}^{k} \varepsilon_j M_j \\ \textit{subject to } \varepsilon_j \textit{ is a positive integer; } \sum_{j=0}^{k} \varepsilon_j \leq p;\ \sum_{j=0}^{k} \varepsilon_j t_j \leq (p-1)\,h. \end{array} \tag{0.2}$$

Proof. *Suppose the solution for 0.1 is M attained for the values $(u_0, u_1, \ldots u_{p-1})$. Let ε_j equal the number of values of i for which $f(u_i) = M_j$. Then since there are p values of u_i it follows that $\sum_{j=0}^{k} \varepsilon_j \leq p$. Since $t_j \leq u_i$ whenever $f(u_i) = M_j$ it follows that*

$$\sum_{j=0}^{k} \varepsilon_j t_j \leq \sum_{j=0}^{p-1} u_j \leq (p-1)\,h.$$

Finally we have

$$M = \frac{1}{p}\sum_{j=0}^{p-1} f(u_j) = \frac{1}{p}\sum_{j=0}^{k} \varepsilon_j M_j. \quad \square$$