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Ramified Cauchy problem for a class of Fuchsian operators with tangent characteristics

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In an open neighborhood of $0\in \mathbf{C}_{x}^{n+1}, x=(x_{0}, x')=(x_{0}, x_{1}, \ldots, x_{n})$, we consider the following second order partial differential operator with holomorphic coefficients:

$$a(x, D)=x_{0}(D_{0}+qx_{0}^{q-1}D_{1})D_{0}+\sum_{j=1}^{n}a_{j}(x)D_{j}+b(x).$$

Here q is an integer ≥ 2 and D_{j} denotes the differentiation with respect to $x_{j}\,(0\leq j\leq n)$. It induces the following isomorphism:

$$a(x, D):x_{0}^{2}\mathbf{C}\{x\}\tilde{\rightarrow}x_{0}\mathbf{C}\{x\},$$

where $\mathbf{C}\{x\}$ denotes the stalk at the origin of the sheaf of holomorphic functions.

Following [7], we put

$$T:x_{0}=x_{1}=0,\ K_{0}:x_{1}=0,\ K_{1}:x_{1}-x_{0}^{q}=0.$$

It is easy to see that K_{0} and K_{1} are characteristic hypersurfaces of $a(x, D)$ and that $K_{0}\cap K_{1}=T$.

Define a function $h(x)$ by $h(x)=-x_{0}^{q}/x_{1}$ for $x\not\in T$. If $x_{1}=0$, we set $h(x)=\infty$ by convention. Then it is easy to see that

$$\begin{aligned} S &= \{x; h(x)=0\}\cup T,\\ K_{0} &= \{x; h(x)=\infty\}\cup T,\\ K_{1} &= \{x; h(x)=-1\}\cup T. \end{aligned}$$

We introduce two closed subsets A_{0} and A_{1} of $\mathbf{C}^{n+1}$ by

$$A_{0}=\{x;-1\leq h(x)\leq 0\ \text{or}\ h(x)=\infty\}\cup T\supset S\cup K_{0}\cup K_{1},$$
$$A_{1}=\{x; h(x)\geq 0\ \text{or}\ h(x)=\infty\ \text{or}\ h(x)=-1\}\cup T\supset S\cup K_{0}\cup K_{1}.$$

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We consider the following Cauchy problem:

(1) $$a(x,D)u(x) = x_0 v(x), \quad D_0^j u|_{x_0=0} \equiv 0 \, (j=0,1).$$

Here we assume that there exists an open connected neighborhood Ω of the origin such that $v(x)$ is holomorphic in the universal covering space of $\Omega \setminus (K_0 \cup K_1)$. In a neighborhood of $y \in \Omega \cap (S \setminus T)$, the Cauchy problem (1) admits a unique holomorphic solution.

Our main result is

Theorem 1 *There exists an open connected neighborhood $\mathcal{O}$ of $0 \in \mathbf{C}_x^{n+1}$ such that for $j = 0, 1$ the solution $u(x)$ to (1) extends holomorphically to the universal covering space of $\mathcal{O} \setminus A_j$.*

Notice that the point y can be assumed to be arbitrarily close to the origin by [1].

Remark The above theorem means that $u(x)$ extends holomorphically along any path $\gamma : I \to \mathcal{O}$ with $\gamma(0) = y$ and $\gamma(t) \notin A_j$ for $t > 0$. Here I is the closed interval $[0,1]$.

Consider, for example, a path γ_0 with $\gamma_0(0) = y$ such that $h \circ \gamma_0(t) = 4t \, (0 \le t \le 1/2)$ and that $h \circ \gamma_0(t) \in \mathbf{C}$ rotates many times along the circle $|z| = 2$ as t increases from $1/2$ to 1. This is a situation where $\gamma_0(t)$ moves around K_0. So the theorem ($j = 0$) implies *the ramification of the solution $u(x)$ around K_0.*

Next, let $\varepsilon > 0$ be sufficiently small and consider a path $\gamma_1 : I \to \mathcal{O}$ with $\gamma_1(0) = y$ such that $h \circ \gamma_1(t) = -2(1-\varepsilon)t \, (0 \le t \le 1/2)$ and that $h \circ \gamma_1(t)$ rotates many times along $|z+1| = \varepsilon$ for $t \ge 1/2$. This is a situation where $\gamma_1(t)$ moves around K_1. So the theorem ($j = 1$) implies *the ramification of $u(x)$ around K_1.*

Sketch of proof

Following [7], we will give an integral representation of the solution $u(x)$.
Let $\Delta_m \, (m \ge 1)$ be the standard m-dimensional simplex $\subset \mathbf{R}^m$:

$$\Delta_m = \{t \in \mathbf{R}^m; \, 0 \le t_1 \le \cdots \le t_m \le 1\}, \; t = (t_1, \ldots, t_m).$$

The system of coordinates of $\mathbf{C}^m$ is $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m)$ and the singular m-simplex $S_m = S_m(x_0)$, depending on the parameter $x_0 \in \mathbf{C}$, is defined by

$$S_m(x_0) : \, t \in \Delta_m \mapsto x_0 t \in \mathbf{C}_\sigma^m.$$

We set $d\sigma_{(m)} = d\sigma_1 \wedge \cdots \wedge d\sigma_m$.

We introduce multiphase functions following [7]. Put $k_0(x) = \varphi_0(x) = x_1$, $k_1(x) = x_1 - x_0^q$ and for $m \geq 1$,

$$\varphi_m(\sigma, x) = k_m(x) + \sum_{j=1}^{m} (-1)^{j+1} \sigma_j^q,$$

where $k_m = k_0$ if m is even and $k_m = k_1$ if m is odd. They satisfy the eikonal equation for $a(x, D)$ and we have

$$\varphi_{m+2}|_{\sigma_{m+2}=\sigma_{m+1}=x_0} = \varphi_{m+1}|_{\sigma_{m+1}=x_0} = \varphi_m \quad (m \geq 0).$$

We give the solution $u(x)$ to (1) near the point y in the form of a series of the type

$$u(x) = \sum_{m=2}^{\infty} I_m(x) \tag{2}$$

with

$$\begin{aligned} I_m(x) &= \int_{S_m} u_m(\sigma_1, \varphi_m(\sigma, x), \sigma', x'') d\sigma_{(m)} \\ &= \int_{\Delta_m} u_m(x_0 t_1, \varphi_m(x_0 t, x), x_0 t_2, \ldots, x_0 t_{m-1}, x'') x_0^m dt_1 \wedge \cdots \wedge dt_m, \end{aligned}$$

where $u_m = u_m(\zeta, \sigma', x'')$ is a holomorphic function. As a matter of fact, if m is odd, then $u_m \equiv 0$ and $I_m(x) \equiv 0$.

We can prove that u_m $(m = 4, 6, 8, \ldots)$ is given by

$$u_2(\zeta, x'') = v(\zeta, x''), \tag{3}$$

$$\sigma_{m-1} u_m(\zeta, \sigma_2, \ldots, \sigma_{m-1}, x'') = R(\alpha, \beta, \zeta, x'', \partial_1, D'') u_{m-2} \quad (m \geq 4) \tag{4}$$

for $\alpha = \sigma_{m-1}, \beta = \psi_{2,m-2}(\sigma_2, \ldots, \sigma_{m-2})$, where $R(\alpha, \beta, \zeta, x'', \partial_1, D'')$ is a first order operator with holomorphic coefficients in a neighborhood of the origin of $\mathbf{C}^2_{\alpha,\beta} \times \mathbf{C}^2_\zeta \times \mathbf{C}^{n-1}_{x''}$, $x'' = (x_2, \ldots, x_n)$.

Therefore we have

$$\begin{aligned} &|u_m(x_0 t_1, \varphi_m(x_0 t, x), x_0 t_2, \ldots x_0 t_{m-1}, x'') x_0^m| \\ \leq\ & \frac{|x_0|^{1+\frac{m}{2}}}{t_3 t_5 t_7 \cdots t_{m-3} t_{m-1}} c^{m+1} \cdot \frac{m}{2} \cdot \left(\frac{m}{2}\right)! \end{aligned}$$

Then the convergence of the series (2) is a consequence of the following lemma.

Lemma 1 *For* $m = 4, 6, 8, \ldots,$ *we have*

$$j_m = \int_{\Delta_m} \frac{dt_1 dt_2 \cdots dt_{m-1} dt_m}{t_3 t_5 t_7 \cdots t_{m-3} t_{m-1}} = \left\{ \left(\frac{m}{2} \right)! \right\}^{-2} \left(\frac{m}{2} + 1 \right)^{-1}. \tag{5}$$

We extend the solution u analytically by deforming the singular simplex $S_m(x_0)$.

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