Renormalization and Asymptotics

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1   Introduction

My lectures consist of the following two parts:

(1) Introduction to renormalization group (RG) (esp., the Stückelberg-Petermann style RG),

(2) Applications of the RG idea to the asymptotic analysis of differential equations (esp., the new proto-RG approach)

Except for the introduction that prepares the audience to our mode of thinking, the main purpose of the lectures is to report presumably interesting mathematical phenomena encountered by a field worker in the land of nonlinearity. It is up to you to find mathematically meaningful topics buried in the field notebook.

Section 2 corresponds to (1), and Section 3 corresponds to (2). Section 2 is similar to my other introductory articles [1]. The main part of Section 3 is to explain our recent approach to streamline reductive and singular perturbations. Section 4 is devoted to end remarks.

2   Introduction to Renormalization Group Approach

2.A   Nonlinearity and dimensional analysis

Dimensional analysis is based on the principle that any objectively meaningful relation among observables can be written as a relation among dimensionless quantities (= scaling invariants), because the units we use (say, m or inch) have no special meaning (their choice is not imposed by Nature). Therefore, the general form of a relation among observables is:

$$\Pi = f(\Pi_0, \Pi_1, \cdots, \Pi_n), \qquad (2.1)$$

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where Π and Π_i $(i = 0, 1, \cdots, n)$ are dimensionless quantities. According to the standard wisdom of dimensional analysis, we may ignore from this relation the dimensionless quantities much larger or much smaller than unity.

Assume that Π_0 is very large. The standard instruction (wisdom) of dimensional analysis may be expressed as follows, "Since the limit

$$\Pi = \lim_{\Pi_0 \to \infty} f(\Pi_0, \Pi_1, \cdots, \Pi_n) \tag{2.2}$$

'exists,' we may asymptotically ignore the Π_0 effect on Π."

Although the instruction sounds very natural, it forces us to ignore many interesting natural phenomena. We must note that the most typical nonlinear phenomena such as, chaos, turbulence, critical phenomena, biological phenomena, etc., are solely due to the interference between disparate scales (e.g., length scales). In other words, the limit (2.2) may not exist, and when there is no limit, we observe typically nonlinear phenomena due to scale interference.

2.B   Asymptotics and phenomenology

When we wish to study a nonlinear phenomenon, often we wish to describe its aspects relevant to us. Consequently, we wish to describe the phenomenon at our (time and space) scale. This scale is much larger than the so-called microscopic scales of atoms and elementary particles. Let us write the ratio of our scale L_0 and the microscopic scale ℓ as $\zeta = \Pi_0 = L_0/\ell$. We are interested in the $\zeta \to \infty$ limit. Suppose an observable f we are interested in depends on the scale of observation as $f = f(\zeta)$. If the limit converges, $\lim_{\zeta \to \infty} f(\zeta) = c$, then f has a definite value very insensitive to the microscopic details at our observation scale. As mentioned above in many interesting cases this limit does not exist. This implies that at however large a scale we may observe f, the result depends on the microscopic details. That is, f depends on microscopic details sensitively (depends on the details of individual systems for which we observe f) even observed at our scale.

If we could isolate divergent quantities from the observable f, then the remaining part would be insensitive to the microscopic details (= universal to a class of systems for which we observe f) by construction.

The isolated divergent quantities can be understood as phenomenological parameters sensitive to the microscopic details. We should recall that a typical phenomenological law such as the Navier-Stokes equation has the structure consisting of the universal form of the equation + phenomenological parameters (density and shear viscosity).

We call the procedure to absorb divergences in the limit of $\zeta \to \infty$ into adjustable parameters a *renormalization* procedure. If we can remove divergences by this procedure, we say that the system (problem or phenomenon) is *renormalizable*. Although the usage of this word is much looser than that in high energy physics (the reader may well say it is an abuse), the lecturer believes that this is the most practical definition of renormalizability.

An important point worth noticing is that the effects of microscopic details are very large (even divergent), but they are confined to well-defined places (quantities) in many phenomena in Nature. This is a reason why we can understand (can feel that we understant) Nature without paying much attention to Her details. If a phenomenon is not renormalizable, then we cannot expect to understand it in general terms (that is, we cannot have any general theory).

The above consideration tells us how to extract a phenomenological description (if any) of a given phenomenon. We look for structurally unstable aspects of the phenomenon and try to isolate them. If we succeed in this program and if the remaining structure is non-trivial (such as the structure of the Navier-Stokes equation), then we have an interesting phenomenological framework to understand the given phenomenon.

2.C ABC of renormalization

Let us illustrate the above considerations in terms of presumably the simplest example, the von Koch curve (please refer to the figure in [1]).

Let ℓ be the 'microscopic unit' of the von Koch curve. Let L be its total length along the curve, and L_0 be its overall size. These lengths make two dimensionless ratios L/L_0 and L_0/ℓ. Therefore, the principle

of dimensional analysis implies

$$\frac{L}{L_0} = f\left(\frac{L_0}{\ell}\right).$$ (2.3)

Everyone knows that f diverges in the $\zeta \to \infty$ limit. Therefore, we cannot follow the standard wisdom of dimensional analysis; we cannot simply ignore ℓ. Indeed,

$$L = L_0^{\ln 4/\ln 3} \ell^{1-\ln 4/\ln 3}.$$ (2.4)

That is, $f(x) = x^{\ln 4/\ln 3-1}$. If we collect various von Koch curves with different ℓ and L_0, we see that their 'true' lengths L are always proportional to $L_0^{\ln 4/\ln 3}$. This is the universal structure of the von Koch curves. The proportionality constant of $L_0^{\ln 4/\ln 3}$ to L is very sensitive to ℓ, and must be treated as an adjustable parameter, if we do not know ℓ. Note that all the features of phenomenology show up here. $L/\ell^{1-\ln 4/\ln 3}$ is the structurally stable quantity that is invariant under the perturbation of the microscopic details of the curve. If we could identify such a quantity, we can isolate the universal aspects (structurally stable aspects) of the phenomenology.

The example is very simple, but this is almost an ideal example to illustrate all the important aspects of the RG approach. An observer knows only the overall size L_0, the scale of observation (resolving power) λ, and the actually measured length $\tilde{L}$ measured with the given resolution. The true length L and $\tilde{L}$ must be proportional (when λ is fixed):

$$\tilde{L} = ZL.$$ (2.5)

Z must be dimensionless and must depend on ℓ/λ. The divergence of the true length in the $\ell \to 0$ limit cannot be observed as long as the curve is observed at the scale λ (i.e., $\tilde{L}$ is finite). Therefore, Z must be chosen so that the divergence of L in this limit is absorbed in Z. Such a coefficient that absorbs divergences is called a *renormalization constant*. In our example, if $\ell \to \ell/3$, then $L \to (4/3)L$, so that in the $\ell \to 0$ limit, the divergence of L should behave as $(4/3)^{-\log_3 \ell} = \ell^{1-\ln 4/\ln 3}$. The renormalization group constant Z is so chosen to remove the divergence $\ell^{1-\ln 4/\ln 3}$ (i.e., to remove this divergence from ZL) as $\propto (\lambda/\ell)^{1-\ln 4/\ln 3}$.

λ is a quantity introduced by the observer, unrelated to the system (the von Koch curve) itself. Therefore, the 'true' length L should not depend on λ (a belief in the reality of the world). In other words, if ℓ

and L_0 are fixed, L does not change, even if λ is altered.[2]

$$\lambda \frac{\partial L}{\partial \lambda} = 0. \tag{2.6}$$

On the other hand, the quantity that the macroscopic observer knows are L_0, $\tilde{L}$, and λ, so that she would conclude dimensional analytically as

$$\frac{\tilde{L}}{\lambda} = f\left(\frac{L_0}{\lambda}\right). \tag{2.7}$$

This and (2.5) imply that

$$L = Z^{-1}\lambda f\left(\frac{L_0}{\lambda}\right). \tag{2.8}$$

Introducing this into (2.6), we obtain

$$f(x) - \alpha f(x) - x f'(x) = 0, \tag{2.9}$$

where

$$\alpha \equiv \partial \ln Z / \partial \ln \lambda. \tag{2.10}$$

The equation (2.6) or its consequence (2.9) is called a renormalization group (RG) equation. If α converges in the $\ell \to 0$ limit, then this equation becomes an equation governing the universal aspect of the problem. In the present example, the limit exists:

$$\alpha = 1 - \frac{\ln 4}{\ln 3}. \tag{2.11}$$

Solving (2.9), we get

$$f(x) \propto x^{1-\alpha}, \tag{2.12}$$

i.e.,

$$\tilde{L} \propto L_0^{1-\alpha} \lambda^\alpha \propto L_0^{\ln 4/\ln 3}. \tag{2.13}$$

Thus, we have recovered the phenomenological relation mentioned above.

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[2] As we will know later, it is often advantageous to use a more structured differential operator instead of the simple derivative $\partial/\partial\lambda$.

2.D ABC of Perturbative RG

The von Koch curve does not need any approximate means, but again this is a very good example to illustrate a perturbative RG approach.

In the above when ℓ is shrunk to $\ell/3$, the total length L increases to $4L/3$. Although 4/3 is fairly different from unity, to use a perturbative approach let us write this 4/3 as e^ϵ and pretend that $\epsilon > 0$ is sufficiently small, so that $e^\epsilon \simeq (1+\epsilon)$. If we complete n construction steps of the von Koch curve, to order ϵ,

$$L_0 \to L = (1 + n\epsilon)L_0. \tag{2.14}$$

The equation is reliable only when $\epsilon n \ll 1$. That is, this equation cannot uniformly be used with respect to ϵ.[3] If we write n in terms of ℓ,

$$L = \left\{ 1 + \epsilon \log_3\left(\frac{L_0}{\ell}\right) \right\} L_0 \tag{2.15}$$

to order ϵ. Expanding the renormalization constant as $Z = 1 + A\epsilon + \cdots$, we determine A so that the divergence in the $\ell \to 0$ is removed order by order in ϵ. To prepare for this, we introduce a length scale λ and rewrite (2.15) as

$$L = \left[1 + \epsilon \left\{ \log_3\left(\frac{L_0}{\lambda}\right) + \log_3\left(\frac{\lambda}{\ell}\right) \right\} \right] L_0. \tag{2.16}$$

Consequently, (2.5) may be expanded as

$$\tilde{L} = ZL = \left\{ 1 + \epsilon \left[A + \log_3\left(\frac{\lambda}{\ell}\right) \right] + \epsilon \log_3\left(\frac{L_0}{\lambda}\right) \right\} L_0. \tag{2.17}$$

Therefore, if we choose $A = -\log_3(\lambda/\ell)$, the divergence to order ϵ may be absorbed into Z. The resultant equation

$$\tilde{L} = \left\{ 1 + \epsilon \ln_3\left(\frac{L_0}{\lambda}\right) \right\} L_0 \tag{2.18}$$

is called the renormalized perturbation result (to order ϵ). If we introduce

$$Z = 1 - \epsilon \frac{1}{\ln 3} \ln \frac{\lambda}{\ell} \tag{2.19}$$

into the definition (2.10) of α, we obtain $\alpha = -\epsilon/\ln 3$ (the order ϵ result), so that (2.13) implies $\tilde{L} \propto L_0^{1+\epsilon/\ln 3}$. If we set $\epsilon = \ln 4 - \ln 3$, then the result happens to be exact.

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[3]In this sense, the term proportional to n corresponds to the secular term in differential equations.

3 Renormalization Group Theoretical Reduction

As we have seen from the simple von Koch curve, RG can be used as a tool of asymptotic analysis. Needless to say, RG is a well-known tool for extraction of phenomenology, and the latter is essentially a sort of asymptotic description. Therefore, the observation just mentioned is not surprising, but are not all the asymptotic analyses in the world just applications of RG? To begin with, let us check the idea with the study of large space-time scale of differential equations.

3.A Simple Example

Let us review the simplest example:

$$\epsilon \frac{d^2 y}{dt^2} + \frac{dy}{dt} + y = 0, \tag{3.1}$$

where $\epsilon > 0$ is assumed to be small. Expanding the solution formally as

$$y = y_0 + \epsilon y_1 + \cdots, \tag{3.2}$$

we obtain to order ϵ

$$y = A_0 e^{-t} - \epsilon A_0 t e^{-t} + O(\epsilon^2). \tag{3.3}$$

The first order term in ϵ is the secular term. Note the perfect parallelism between this example and the von Koch perturbation result. Splitting the secular divergence as $(t - \tau) + \tau$, we absorb τ into A_0, which is modified to $A(\tau)$.[4] This new coefficient is determined to agree with the observation at present, i.e., at t. Thus, (3.3) turns into the renormalized perturbation result

$$y = A(\tau)e^{-t} - \epsilon(t - \tau)A(\tau)e^{-t} + O(\epsilon^2). \tag{3.4}$$

In this equation t need not be small because we may choose τ sufficiently close to t. τ is the parameter that does not exist in the original problem, so that $\partial y/\partial \tau = 0$. This is the RG equation:

$$\frac{dA}{d\tau} = -\epsilon A. \tag{3.5}$$

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[4] We do not introduce the renormalization constant for simplicity, but to go beyond the lowest nontrivial order, it is advisable to use one.

The renormalized perturbation (3.4) simplifies, if we set $\tau = t$:

$$y = A(t)e^{-t}. \tag{3.6}$$

From (3.5), we see that $A(t)$ obeys the following 'amplitude equation'

$$\frac{dA(t)}{dt} = -\epsilon A(t). \tag{3.7}$$

Solving this for A and using it in (3.6), we get the result that agrees with the one obtained by the conventional singular perturbation method.[5]

From this simple example, we may have two claims:

(1) The secular term is a divergence that should be renormalized, and the renormalized perturbation result is the conventional singular perturbation result.

(2) The RG equation is an equation governing the global behavior of the solution. The equation obtained by the reductive perturbation is the RG equation.

The correctness of these claims has been demonstrated with various examples by 1994 [4]. There are, however, two unsatisfactory features in our results.

First of all, our 'demonstration' is only through numerous examples: What is the general theorem that guarantees these claims in a much more abstract and clean fashion? I have no idea.[6]

The other unsatisfactory feature is practical. Looking at the simple example, we must conclude that the core of the singular perturbation theory is the reductive perturbation theory: if we know the reductive perturbation result, solving the resultant equation, we can obtain the singular perturbation result. Therefore, a procedure that requires an explicit perturbation result to obtain the RG equation is theoretically inelegant and practically inconvenient.

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[5] $\tau = t$ simplifies the computation drastically, but some people questions the legitimacy of the procedure. Generally, the result of the renormalized perturbation may be written as

$$y(t) = f(t; \epsilon\tau) + \epsilon(t - \tau)g(t) + O(\epsilon^2), \tag{3.8}$$

if we introduce the RG equation result. Since f is differentiable with respect to the second variable, with the aid of Taylor's formula

$$y(t) = f(t; \epsilon t) + \epsilon(\tau - t)\partial_2 f(t, \epsilon t) + \epsilon(t - \tau)g(t) + O(\epsilon^2). \tag{3.9}$$

Here, ∂_2 denotes the partial differentiation with respect to the second variable. The second and the third terms of this formula must cancel each other as seen from the construction of the RG equation. That is, to remove the secular term by setting $\tau = t$ is always correct.

[6] It is not hard to estimate the errors of the resultant formulas. It can be done, for example, by following a standard method used in the justification of amlitude equations.

We will see that this problem is largely overcome by the proto RG approach [5].

3.B Resonance and Proto RG Equation

To explain our new approach, let us use the Rayleigh equation

$$\frac{d^2y}{dt^2} + y = \epsilon \frac{dy}{dt}\left(1 - \frac{1}{3}\left(\frac{dy}{dt}\right)^2\right). \tag{3.10}$$

We solve this perturbatively as

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots. \tag{3.11}$$

Its zeroth order reads

$$y_0 = Ae^{it} + A^*e^{-it}, \tag{3.12}$$

where A is a complex constant. The equation governing y_1 is

$$\left(\frac{d^2}{dt^2} + 1\right)y_1 = iA(1 - |A|^2)e^{it} + \frac{i}{3}A^3e^{3it} + cc, \tag{3.13}$$

where cc denotes the complex conjugate. From the structure of (3.13), we see that the solution has the following structure:

$$y_1 = P_1 e^{it} + Q_1 e^{3it} + cc. \tag{3.14}$$

By inspection we know $P_1(t, A)$ is first order in t, and $Q_1(t, A)$ is a constant. From (3.13) we see

$$L_t P_1 = iA(1 - |A|^2), \tag{3.15}$$

$$R_t Q_1 = \frac{1}{3}iA^3, \tag{3.16}$$

where

$$L_t \equiv \frac{d^2}{dt^2} + 2i\frac{d}{dt}, \tag{3.17}$$

$$R_t \equiv \frac{d^2}{dt^2} + 6i\frac{d}{dt} - 8. \tag{3.18}$$

The renormalized perturbation result reads

$$y(t) = A_R(\tau)e^{it} + \epsilon[P_1(t, A_R(\tau)) - \hat{P}_1(\tau, A_R(\tau))]e^{it} + \cdots. \tag{3.19}$$

Here, $\hat{P}_1$ is the singular part of P_1 (note that if P_1 does not have an additive constant term, then $P_1 = \hat{P}_1$). Let L_τ be L_t with its t being replaced by τ. Then,

$$0 = L_\tau y = [L_\tau A_R - \epsilon L_\tau \hat{P}_1(\tau, A_R)]e^{it}. \tag{3.20}$$

That is,

$$\left(\frac{d^2}{dt^2} + 2i\frac{d}{dt}\right) A_R(\tau) = \epsilon i A_R(1 - |A_R|^2). \tag{3.21}$$

From this we see that differentiation with respect to τ raises the order by ϵ. Therefore, to order ϵ we may ignore the second derivative. Replacing τ with t, we obtain to order ϵ

$$\frac{dA_R}{dt} = \frac{1}{2}\epsilon A_R(1 - |A_R|^2). \tag{3.22}$$

This is the RG equation (the amplitude equation) to the same order. Thus, we call (3.21) the *proto RG equation*. If we obtain the proto RG equation, then the RG equation can be obtain by an algebraic procedure. Note that to obtain the proto RG equation to order ϵ we do not need any explicit perturbative result. This feature becomes important when the problems become complicated (e.g., partial differential equations).

Instead of $\partial y/\partial \tau = 0$, to use $L_\tau y = 0$ is the proto RG approach. Is this approach effective for higher order results? For nonlinear problems we need slightly more information than required by the first order result. Still, the approach is much simpler than the conventional perturbation calculation.

3.C Amplitude Equation – RG Theoretical Reduction

As we have seen above the essence of singular perturbation theory is the reductive perturbation. The proto RG equation approach makes the reduction process transparent. Let us apply this to the 2D Swift-Hohenberg equation:

$$\frac{\partial u}{\partial t} = \epsilon(u - u^3) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)^2 u. \tag{3.23}$$

The unperturbed solution is $Ae^{ikx} + cc$, where A is a complex constant. We assume the perturbative solution as

$$u = Ae^{ikx} + A^*e^{-ikx} + \epsilon u_1 + \epsilon^2 u_2 + \cdots. \tag{3.24}$$

The first order term obeys

$$\frac{\partial u_1}{\partial t} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)^2 u_1 = (1 - 3|A|^2)Ae^{ikx} - A^3 e^{3ikx} + cc. \tag{3.25}$$

Its solution has the following form:[7]

$$u_1 = P_1(t, r)e^{ikx} + Q_1(t, r)e^{3ikx},$$

(3.26)

where P_1 is singular (unbounded and/or non-integrable), because $r = (x, y)$. e^{ikx} is the zero solution of (3.25). Since

$$\left[\frac{\partial}{\partial t} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2 \right)^2 \right] P_1 e^{ikx} = (1 - 3|A|)^2) A e^{ikx},$$

(3.27)

we have

$$\left[\frac{d}{dt} + \frac{\partial^4}{\partial x^4} + 4ik\frac{\partial^3}{\partial x^3} + 2\frac{\partial^2}{\partial y^2}\frac{\partial^2}{\partial x^2} + \left(-4k^2\frac{\partial^2}{\partial x^2} + 4ik\frac{\partial^2}{\partial y^2}\frac{\partial}{\partial x} + \frac{\partial^4}{\partial y^4} \right) \right] P_1$$

$$\equiv LP_1 = (1 - 3|A|)^2) A$$

(3.28)

Similarly, we obtain

$$\left[\frac{d}{dt} + \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} + 6ik\frac{\partial}{\partial x} - 8k^2 \right)^2 \right] Q_1$$

$$\equiv RQ_1 = -A^3.$$

(3.29)

P_1 cannot be a constant, but Q_1 can.

The renormalized perturbation result has the following form:

$$u = A_R(\tau, \rho)e^{ikx} + \epsilon[P_1(t, r) - P_1(\tau, \rho)]e^{ikx} + Q_1 e^{3ikx} \cdots.$$

(3.30)

Consequently, the proto RG equation to order ϵ is

$$\left(\frac{\partial}{\partial \tau} + L_{\tau,\rho} A_R(\tau, \rho) \right) = \epsilon(1 - 3|A_R|^2) A_R.$$

(3.31)

Here, $L_{\tau,\rho}$ is L with the replacement $t \to \tau$, $r \to \rho$. L contains superfluous terms. To remove such terms, generally speaking, how to observe (at what space-time scale to observe) the system must be specified. In the present example, if we choose $t \sim x^2 \sim y^4 \sim 1/\epsilon$, we obtain

$$\left(\frac{\partial}{\partial t} - 4k^2\frac{\partial^2}{\partial x^2} + 4ik\frac{\partial^2}{\partial y^2}\frac{\partial}{\partial x} + \frac{\partial^4}{\partial y^4} \right) A_R(t, r) = \epsilon(1 - 3|A_R|^2) A_R.$$

(3.32)

That is, the usual Newell-Whitehead equation results. The choice of the orders above may look arbitrary, but, actually, in this case there is no other choice. For example, if we assume $y^4 \sim y^2x^2 \sim t \sim 1/\epsilon$,

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[7]in the formal algebraic sense; to characterize this form mathematically is a problem.

then $\partial^2/\partial x^2$ and $\partial^3/\partial x \partial y^2$ dominate the left-hand side and cannot balance with the order ϵ terms on the right-hand side. In this way we see that (3.32) is the unique order ϵ result.

The second order proto RG equation reads

$$LA_R = \epsilon(1 - 3|A_R|^2)A_R + \epsilon^2 \frac{3}{64k^4}|A_R|^4 A_R. \tag{3.33}$$

(3.31) is the equation obtained by Graham [6], but as seen clearly in (3.33) the equation is not consistent to order ϵ (as first realized by [7]). If we wish to retain all the differential operators in (3.31), as seen in (3.33), we need a higher order correction to the nonlinear term.

The reader might have asked what happens if the RG equation is reduced further. For example, the Boltzmann equation may be obtained as an RG equation [8], but the equation can further be reduced to the Navier-Stokes equation [9], if observed at further larger space-time scale. However, if we look at the system at the same scale, no further reduction should be possible. For example, in the case of the Newell-Whitehead equation, we can easily show that its RG reduction gives the same equation. In other words, it is the fixed point of the system reduction.

We can derive phase equations, equations of motion for singularities of a field (such as the interface equation, equation of motion for kinks and defects), etc., as RG equations. So far the assertion seems to hold that all the named phenomenological equations are RG equations.

3.D   All orders

Let us study the 'simplest' example of the singular perturbation problem again:

$$\left(\epsilon \frac{d^2}{dt^2} + \frac{d}{dt} + 1\right)y = 0. \tag{3.34}$$

Expanding as $y = y_0 + \epsilon y_1 + \cdots + \epsilon^n y_n + \cdots$, we have

$$\frac{dy_n}{dt} + y_n = -\frac{d^2 y_{n-1}}{dt^2}. \tag{3.35}$$

Writing the lowest order result as $y_0 = Ae^{-t}$, the solution of this equation can be written in the following form $y_n = AP_n e^{-t}$, where P_n is governed by:

$$\frac{dP_n}{dt} = -\frac{d^2 P_{n-1}}{dt^2} + 2\frac{dP_{n-1}}{dt} - P_{n-1} \tag{3.36}$$

Its initial condition is $P_n(0) = 0$ that allows us to identify P_n and its singular part $\hat{P}_n$.

Using these results, the perturbation result reads

$$y(t) = A[1 + \epsilon P_1(t) + \cdots + \epsilon^n P_n(t) + \cdots]e^{-t}. \tag{3.37}$$

Since our problem is linear, P_n does not depend on A. If we renormalize A as usual as $A = Z A_R(\tau)$:

$$y(t) = A_R(\tau)Z[1 + \epsilon P_1(t) + \cdots + \epsilon^n P_n(t) + \cdots]e^{-t}, \tag{3.38}$$

and if we assume (without any loss of generality) that $t - \tau$ is higher order infinitesimal than any power of ϵ to simplify the calculation, we obtain

$$Z^{-1} = 1 + \epsilon P_1(\tau) + \cdots + \epsilon^n P_n(\tau) + \cdots. \tag{3.39}$$

That is, for linear problems, renormalization is the same as the naive renormalization we are familiar with in, e.g., solid state physics:

$$A_R = A[1 + \epsilon P_1(\tau) + \cdots + \epsilon^n P_n(\tau) + \cdots]. \tag{3.40}$$

The renormalized coefficient obeys the following equation according to (3.36)

$$\frac{dA_R}{d\tau} = \epsilon\left(-\frac{d^2 A_R}{d\tau^2} + 2\frac{dA_R}{d\tau} - A_R\right). \tag{3.41}$$

Solving this order by order in ϵ, P_n is determined. Note, however, (3.41) is obtained by introducing $y = A_R(t)e^{-t}$ into the original problem (3.34). That is, (3.41) is the proto RG equation (to all orders). From this the RG equation can be obtained by solving it for $dA_R/d\tau$ order by order. To the lowest order

$$\frac{dA_R}{d\tau} = -\epsilon A_R. \tag{3.42}$$

Using this to the right-hand side of (3.41), we obtain to order ϵ^2

$$\frac{dA_R}{d\tau} = -(\epsilon + 2\epsilon^2)A_R. \tag{3.43}$$

The observation is basic that differentiation raises the power of ϵ by one:

$$\frac{d^2 A_R}{d\tau^2} = -\epsilon\frac{dA_R}{d\tau} = \epsilon^2 A_R. \tag{3.44}$$

In this way, for example, to order ϵ^3 we have

$$\frac{dA}{d\tau} = -(\epsilon + 2\epsilon^2 + 5\epsilon^3)A. \tag{3.45}$$

3.E   Merit of Proto RG Approach in Linear Cases

The reader may say that linear problems are so simple that such a calculation has no merit. However, there are many linear ordinary differential equation problems that cannot directly be solved by the RG approach proposed in [4].[8] For example,[9] if we solve

$$\frac{d^2y}{dt^2} + (2 + \epsilon)\frac{dy}{dt} + y = 0 \tag{3.46}$$

perturbatively, the zeroth order solution has the form $(A + Bt)e^{-t}$. There is already a secular term that complicates the identification of the divergence. However, there is no difficulty at all in the proto RG approach. Let $y = A(t)e^{-t}$. Then, the proto RG equation (although we need not name such a trivial equation) reads

$$\frac{d^2A}{dt^2} = \epsilon\left(A - \frac{dA}{dt}\right). \tag{3.47}$$

From this, the lowest order RG equation is

$$\frac{d^2A}{dt^2} = \epsilon A. \tag{3.48}$$

Notice that the procedure is quite mechanical.

Nishiura [10] mentions other 'difficult' examples such as

$$\frac{dy}{dt} = -\epsilon^2 y + \epsilon y^2. \tag{3.49}$$

This example helps us to make an implicit assumption in our RG method explicit. Its proto RG equation is

$$\frac{dA}{dt} = \epsilon A^2 - \epsilon^2 A. \tag{3.50}$$

The examples we have discussed so far allow us to assume that A is of order unity. However, in this example, the solution we are interested in is of order ϵ. That is, although we claim that the RG approach does not require any a priori knowledge, we need at least such an estimate. Therefore, both terms on the right hand side of (3.50) are comparable, so no further reduction is possible. That is, we must interpret that the proto RG equation is the RG equation itself for this example.

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[8]In this paper, problems were avoided with the aid of the approach via the canonical form of the equation. With the canonical form, our simple RG always works.

[9]This example was stressed by F. Furtado.

As a not-so-trivial example of reducing the proto RG to the RG equation, let us consider the bifurcation problem of the Mathieu equation: the problem is to find the range of ω such that

$$\frac{d^2y}{dt^2} + y = -\epsilon[\omega + 2\cos(2t)]y \tag{3.51}$$

does not have a bounded solution. Although this is not an autonomous equation, for linear problems, it is easy to see that the proto RG method works to all orders. The unperturbed solution reads

$$y_0 = Ae^{it} + A^*e^{-it}. \tag{3.52}$$

The easiest method that still allows us to avoid explicit calculation of perturbative results from this equation is to expand as

$$y = A(t)e^{it} + \epsilon B(t)e^{3it} + \epsilon^2 C(t)e^{5it} + cc. \tag{3.53}$$

This form is easily guessed from the fact that e^{2it} appears with ϵ in (3.51). The procedure is to get the equations (they may also be called proto RG equations) for the coefficients, and then to reduce them to the equation of A alone. For example, to order ϵ^2 we have

$$\frac{d^2A}{dt^2} + 2i\frac{dA}{dt} = -\epsilon(\omega A + A^*) - \epsilon^2 B \tag{3.54}$$

The equation for B is

$$\epsilon\left(\frac{d^2B}{dt^2} + 6i\frac{dB}{dt} - 8B\right) = -\epsilon A - \epsilon^2 B + \epsilon^3 C \tag{3.55}$$

Since derivatives give higher order powers of ϵ, we see from this $B = A/8$ to order ϵ. Hence, to order ϵ^2 the proto RG equation is reduced to

$$\frac{d^2A}{dt^2} + 2i\frac{dA}{dt} = -\epsilon(\omega A + A^*) - \epsilon^2\frac{A}{8}. \tag{3.56}$$

It is easy to reduce this further to a first order differential equation, from which the bifurcation condition can be read off.

3.F   Beyond All Orders

As we have seen in the preceding subsection, the (proto) RG method works to all orders for linear problems. It is not hard to see that even for nonlinear resonant problems, the procedure given here can be consistently

performed order by order to all orders. However, it is clear that the method explained cannot give the other solution of (3.34) whose leading order behavior is $e^{-t/\epsilon}$.

One (and the conventional) way to retain such a solution is to scale the variable as $t = \epsilon s$. Then, the perturbation term becomes non-singular. However, we wish to reduce the amount of insight needed to solve problems as much as possible, so that we avoid rescaling of the variables.

Although there might be other reasons, one chief reason why we cannot obtain the fundamental set of the singularly perturbed ordinary differential equation is that the unperturbed equation has a lower order, so that the dimension of the solution space is reduced. In other words, we cannot impose the auxiliary conditions that the original perturbed equation can accomodate. For example, (3.34) with $\epsilon = 0$ is a first order equation, so that there is no way to impose two independent auxiliary conditions.

From this point of view, (3.34) is not the simplest example. The simplest example seems to be

$$\epsilon \frac{dy}{dt} + y = 0. \tag{3.57}$$

Its general solution is $y = Ae^{-t/\epsilon}$. If we perform the expansion $y = y_0 + \epsilon y_1 + \cdots$, then we obtain $y = 0$, which is consistent with the asymptotic expansion of the exact solution. The problem of the simplest example is that the zeroth order equation is not even an ODE, so that not a single auxiliary condition can be imposed.

This observation suggests that, if we could impose the same number of auxiliary conditions to the perturbed and unperturbed equations, we might be able to overcome the difficulty. The most natural approach seems to be as follows. An initial condition may be imposed with the aid of the delta function as

$$\epsilon \frac{dy}{dt} + y = \alpha\delta(t) \tag{3.58}$$

with a homogeneous initial condition $y(0) = 0$. The zeroth order equation reads

$$y_0 = \alpha\delta(t). \tag{3.59}$$

The perturbation equations read

$$y_n = -\frac{dy_{n-1}}{dt}, \tag{3.60}$$

so that

$$y = \sum_{n=0}^{\infty} \alpha \left(-\epsilon\frac{d}{dt}\right)^n \delta(t). \tag{3.61}$$

To sum this highly singular series, we use the Borel summation method. Let

$$B(s) \equiv \sum_{n=0}^{\infty} \alpha \frac{1}{n!} \left(-s \frac{d}{dt} \right)^n \delta(t) = \alpha \delta(t - s). \tag{3.62}$$

Then, the Borel summation result reads

$$y = \frac{1}{\epsilon} \int_0^{\infty} B(s) e^{-s/\epsilon} ds = \frac{\alpha}{\epsilon} e^{-t/\epsilon}. \tag{3.63}$$

Thus, we have obtained the result beyond all orders from a perturbative calculation.

From the above calculation, it is tempting to conjecture that perturbative calculations, appropriately organized, can give us all the information about the original equation. Consequently, the results beyond all orders can also be obtained perturbatively. A crucial ingredient seems to be to retain the degrees of freedom (flexibility of introducing sufficiently many auxiliary conditions) in the original problem in the perturbative processes.

4 End Remarks

The outstanding problems in the related fields of these lectures seems (other than mathematically unsatsisfactory aspects already mentioned above):

(1) Clarify the relation between the Stückelberg-Petermann RG and the Wilson-Kadanoff RG. As the reader knows, the latter has been rigorized for several systems, but the so-called field theoretical schemes have not been. The ODE example that can be solved in both ways should be an ideal laboratory for this problem.

(2) The relation between the original equation and the reduced equation has been studied, but it is desirable that there is a method closely related to the idea of RG. Similar things may be said for all the problems in this field of asymptotic analysis; is there any RG-related unified logic for rigorous results?

(3) Practically, we are interested in much more complicated systems like proteins: describe the long-term (1-1000 sec) dynamics of a protein molecule consisting of 200 amino acid residues (with surrounding water molecules). Philosophically, RG-like means should work, but in practice, we have no idea to implement it. A patient step-by step trial and error approach seems mandatory [13].

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