数理解析研究所講究録
1134 巻 2000 年 123-135

Defining reflection positive random fields with interactions by polynomials of generalized Euclidean free fields

電気通信大学 吉田 稔 (Minoru W. Yoshida)*

September 19, 1999

0 Introduction

In section 1 the generalized Euclidean free fields are expressed as $\mathcal{S}'$-valued random variables by making use of multiple stochastic integrals.

Using this expression, for space time dimension $d \leq 3$ it is shown that Euclidean random fields defined by Wick powers of the generalized Euclidean free fields satisfy reflection positivity. This main result is stated in Theorem 7 of section 2. In Proposition 9 it is shown (unfortunately) that the reflection positive Euclidean random field defined by Wick power of generalized Euclidean free field has no analytic continuation to any Wightman distribution when $d = 3$.

Section 3 is an appendix.

1 Fundamental lemmas

Let Δ be the d-dimensional Laplacian, and denote $J^{\alpha} = (-\Delta + m^2)^{-\frac{\alpha}{2}}$ for some fixed $m > 0$. Then, for the pseudo-differential operator $((|\xi|^2 + m^2)^{-\frac{\alpha}{2}}$ the Green kernels $J^{\alpha}(x)$ can be given explicitly by modified Bessel functions. Precisly, J^{α} has the following integral representation (cf. [Rö]):

$$J^{\alpha}(x) = \frac{1}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{\alpha}{2})}\int_0^{\infty} \exp\{-\frac{|x|^2}{4s} - m^2 s\} s^{\frac{-d-2+\alpha}{2}} ds, \qquad x \in \boldsymbol{R}^d. \tag{1.1}$$

In the sequel for $\alpha = 1$ we denote

$$J^1 = J.$$

Let $\mathcal{S}(\boldsymbol{R}^d)$ be the Schwartz space of rapidly decreasing test functions equipped with usual topology, as a consequence, it is a Fréchet space. Let $\mathcal{S}'(\boldsymbol{R}^d)$ be the topological dual space of $\mathcal{S}(\boldsymbol{R}^d)$. For each a, b, $d > 0$, we define a linear subspace $B_d^{a,b}$ of $\mathcal{S}'(\boldsymbol{R}^d)$ as follows:

$$B_d^{a,b} = \{(|x|^2 + 1)^{\frac{b}{4}} J^{-a} f : \ f \in L^2(\boldsymbol{R}^d; \lambda^d)\}. \tag{1.2}$$

Then $B_d^{a,b}$ becomes a separable Hilbert space with the scalar product

$$< u|v >= \int_{\boldsymbol{R}^d} J^a((1+|x|^2)^{-\frac{b}{4}} u(x))\, J^a((1+|x|^2)^{-\frac{b}{4}} v(x)) dx, \qquad u,\, v \in B_d^{a,b}. \tag{1.3}$$

*Dept. Systems Engineering The Univ. ELECTRO-COMMUNICATIONS 1-5-1, Chofugaoka, Chofu, Tokyo, 182-8585, JAPAN. e-mail yoshida@cocktail.cas.uec.ac.jp fax +81 424 98 0541. Supported in part by Grant-in-Aid Science Research (No. 10640154), Ministry of Education, Japan.

Let $\mathcal{B}_K$ be the Kolmogorov σ-field of $C(\boldsymbol{R}^d \to \boldsymbol{R})$:

$$\mathcal{B}_K = \text{the smallest } \sigma\text{-field of } C(\boldsymbol{R}^d \to \boldsymbol{R}) \text{ by which } \pi_x,\ x \in \boldsymbol{R}^d \text{ are measurable,}$$

where

$$\pi_x : C(\boldsymbol{R}^d \to \boldsymbol{R}) \ni f \longmapsto f(x) \in \boldsymbol{R}.$$

We obviously have the following (Proposition 1 of [Y]):

Proposition 1 *Let $C(\boldsymbol{R}^d \to \boldsymbol{R})$ be the space of real valued continuous functions defined on $\boldsymbol{R}^d$ equipped with the uniform convergence topology, $C_0(\boldsymbol{R}^d \to \boldsymbol{R})$ be the $\mathcal{LF}$-space of real valued continuous functions defined on $\boldsymbol{R}^d$ with compact supports equipped with the canonical $\mathcal{LF}$-topology (cf. for eg. [Tr]), and $\mathcal{B}(C(\boldsymbol{R}^d \to \boldsymbol{R}))$, $\mathcal{B}(C_0(\boldsymbol{R}^d \to \boldsymbol{R}))$, $\mathcal{B}_K$ and $\mathcal{B}(B_d^{a,b})$ be the Borel σ-fields of $C(\boldsymbol{R}^d \to \boldsymbol{R})$, $C_0(\boldsymbol{R}^d \to \boldsymbol{R})$, the Kolmogorov σ-field of $C(\boldsymbol{R}^d \to \boldsymbol{R})$ and the Borel σ-field of $B_d^{a,b}$ respectively. Then, for any a, $b > 0$ the following identity holds:*

$$\begin{aligned}\mathcal{B}(C_0(\boldsymbol{R}^d \to \boldsymbol{R})) &= \Big\{A \cap C_0(\boldsymbol{R}^d \to \boldsymbol{R}) \ :\ A \in \mathcal{B}(C(\boldsymbol{R}^d \to \boldsymbol{R}))\Big\} \\ &= \Big\{A \cap C_0(\boldsymbol{R}^d \to \boldsymbol{R}) \ :\ A \in \mathcal{B}_K\Big\} = \Big\{A \cap C_0(\boldsymbol{R}^d \to \boldsymbol{R}) \ :\ A \in \mathcal{B}(B_d^{a,b})\Big\}. \end{aligned} \tag{1. 4}$$

By this, the next Proposition 2 follows:

Proposition 2 *Any $C_0(\boldsymbol{R}^d \to \boldsymbol{R})$-valued measurable function defined on a measurable space can be regarded as a $B_d^{a,b}$-valued measurable function for any a, $b > 0$.*

We denote the Fourier and Fourier inverse transform of a function φ respectively by $\mathcal{F}[\varphi]$ and $\mathcal{F}^{-1}[\varphi]$, which are defined by

$$\mathcal{F}[\varphi](\xi) = \int_{\boldsymbol{R}^d} e^{-\sqrt{-1}x\cdot\xi}\varphi(x)dx,$$

$$\mathcal{F}^{-1}[\varphi](\xi) = (2\pi)^{-d}\int_{\boldsymbol{R}^d} e^{\sqrt{-1}x\cdot\xi}\varphi(x)dx \qquad \text{for} \quad \varphi \in \mathcal{S}(\boldsymbol{R}^d).$$

We sometimes denote $\mathcal{F}[\varphi] = \hat{\varphi}$. Let $\eta_1 \in C_0^\infty(\boldsymbol{R}^d)$ be such that

$$0 \le \eta_1(x) \le 1 \qquad \text{and} \qquad \eta_1(x) = \begin{cases} 1 & |x| \le 1 \\ 0 & |x| \ge 2. \end{cases} \tag{1. 5}$$

and let $\eta_k(x) = \eta_1(\frac{x}{k}) \in C_0^\infty(\boldsymbol{R}^d)$, $k = 1, 2, \ldots$. Also define $\rho \in C_0^\infty(\boldsymbol{R}^d)$ as follows:

$$\rho(x) = \begin{cases} C\exp(-\dfrac{1}{1-|x|^2}) & |x| < 1 \\ 0 & |x| \ge 1 \end{cases},$$

where the constant C is taken to satisfy

$$\int_{\boldsymbol{R}^d} \rho(x)dx = 1. \tag{1. 6}$$

Define

$$\rho_k(x) = k^d\rho(kx). \tag{1. 7}$$

For $\alpha > 0$ we define $J_k^\alpha \in \mathcal{S}(\boldsymbol{R}^d)$, $\quad k = 1, 2, \ldots$ by

$$J_k^\alpha(x) = \int_{\boldsymbol{R}^d} J^\alpha(y)\rho_k(x-y)dy$$

and

$$F_k^\alpha(x; y_1, \ldots, y_p) = (\eta_k(x))^p J_k^\alpha(x-y_1)\cdots J_k^\alpha(x-y_p), \tag{1. 8}$$

also let

$$F^\alpha(x; y_1, \ldots, y_p) = J^\alpha(x-y_1)\cdots J^\alpha(x-y_p), \qquad p = 1, 2, \ldots. \tag{1. 9}$$

Then we see that the function F_k^α and F^α are symmetric in the last p variables $(y_1, \ldots, y_p)$ and

$$F_k^\alpha \in \mathcal{S}((\boldsymbol{R}^d)^{p+1}), \qquad F_k^\alpha(x; y_1, \ldots, y_p) = 0 \quad \text{for} \quad |x| \geq 2k. \tag{1. 10}$$

The convolution $\rho_k *$ defines a mollifier. Let us recall the following important properties:

$$\rho \in C_0^\infty(\boldsymbol{R}^d), \qquad \hat{\rho}_k(\xi) = \hat{\rho}(\frac{\xi}{k}), \qquad |\hat{\rho}(\xi)| \leq 1, \qquad \hat{\rho}(0) = 1 \quad \text{(by (1.6))}. \tag{1. 11}$$

Hence, $\hat{\rho}_k(\xi)$ converges to 1 uniformly on compact sets: For any $M < \infty$ and any $\epsilon > 0$ there exists an $N < \infty$ and

$$0 \leq 1 - \hat{\rho}_n(\xi) < \epsilon \qquad \forall \xi \quad \text{such that} \quad |\xi| \leq M \quad \text{and} \quad \forall n \geq N. \tag{1. 12}$$

Now, suppose that on a complete probability space $(\Omega, \mathcal{F}, P)$ we are given an isonormal Gaussian process $W = \{W(h), h \in L^2(\boldsymbol{R}^d; \lambda^d)\}$, where λ^d denotes the Lebesgue measure on $\boldsymbol{R}^d$:
W is a centered Gaussian family of random variables such that

$$E[W(h)W(g)] = \int_{\boldsymbol{R}^d} h(x)\, g(x)\lambda^d(dx), \qquad h,\ g \in L^2(\boldsymbol{R}^d; \lambda^d).$$

To be precise, Ω would be the complete separable metric space $\boldsymbol{R}^\infty$ equipped with the metric

$$d(\boldsymbol{x}, \boldsymbol{y}) = \sum_{n=1}^{\infty} 2^{-n} \min\{|x_n - y_n|, 1\}, \qquad \boldsymbol{x} = (x_1, x_2, \cdots), \quad \boldsymbol{y} = (y_1, y_2, \cdots),$$

$$P = N_{0,1}^\infty, \qquad \mathcal{F} = \text{ the completion of the Borel } \sigma\text{-field of } \Omega \text{ with respect to } P\ .$$

Then for every $\mathcal{F}/\mathcal{B}(T)$-measurable mapping $f : \Omega \longrightarrow T$ the measure $\nu = \mu \circ f^{-1}$ becomes a regular probability measure on T, where T is a topological space having a countable open base and $\mathcal{B}(T)$ is the Borel σ-field of T.

In order to give the expressions of multiple stochastic integrals for random variables on $L^2(\Omega, P)$, we regard the Gaussian process W as $L^2(\Omega, P)$-valued Gaussian measure on the parameter space $(\boldsymbol{R}^d, \mathcal{B}(\boldsymbol{R}^d))$ (cf. section 1.1.2 of [Nu]): For $A \in \mathcal{B}(\boldsymbol{R}^d)$ such that $\lambda^d(A) < \infty$ we denote $W(A) = W(\chi_A)$, where χ_A is the indicator function. Now, for $h \in L^2(\boldsymbol{R}^d; \lambda^d)$ the random variable $W(h)$ can be regarded as a stochastic integral, and is denoted by $W(h) = \int_{\boldsymbol{R}^d} h\, dW$.

For expectations of multiple stochastic integrals the following holds:

$$E[\{\int f(y_1, \ldots, y_p)W(dy_1)\cdots W(y_p)\}^2] = p!\|f\|_{L^2}^2 \qquad \text{for} \quad f \in L^2((\boldsymbol{R}^d)^p; (\lambda^d)^p). \tag{1. 13}$$

For each $\alpha > 0$, $p \geq 1$ and $k \geq 1$ we define a random variable $:_k \phi_{\alpha,\omega}^p :$ as follows:

$$:_k \phi_{\alpha,\omega}^p : (x) = \int_{(\boldsymbol{R}^d)^p} F_k^\alpha(x; y_1, \ldots, y_p) W_\omega(dy_1)\cdots W_\omega(dy_p). \tag{1. 14}$$

We can take $:_k \phi^p_{\alpha,\omega}:$ as a $C_0(\boldsymbol{R}^d \to \boldsymbol{R})$-valued random variable, indeed since there exists a bounded open set $D_k = \{x \mid |x| < 2k\} \subset \boldsymbol{R}^d$ and

$$:_k \phi^p_{\alpha,\omega} : (x) = 0 \qquad \text{for} \quad x \in \boldsymbol{R}^d \setminus D_k \quad \forall \omega \in \Omega.$$

Also by the Kolmogorov's continuity criterion the stochastic process $\{:_k \phi^p_{\alpha,\omega} : (x)\}_{x \in \boldsymbol{R}^d}$ admits a continuous modification, we also denote it by $:_k \phi^p_{\alpha,\omega} : (x)$. Hence,

$$:_k \phi^p_{\alpha,\omega} : (\cdot) \in C_0(\boldsymbol{R}^d) \qquad \forall \omega \in \Omega.$$

The following Proposition 3 is the restatement of Proposition 3 of [Y]:

Proposition 3 *Let $g \in L^2(\boldsymbol{R}^d)$ and $K \in L^2((\boldsymbol{R}^d)^{p+1})$. Suppose that K satisfies the following:*

$$K(x; y_1, \cdots, y_p) \qquad \text{is symmetric in the last } p \text{ variables} \quad (y_1, \cdots, y_p);$$

$$\text{there exists a compact set } D \subset \boldsymbol{R}^d \text{ and} \qquad K(x; y_1, \cdots, y_p) = 0 \qquad \text{for} \quad (x, y_1, \cdots, y_p) \in D^c \times (\boldsymbol{R}^d)^p;$$

$$\text{the map} \quad \boldsymbol{R}^d \ni x \longmapsto K(x; \cdot) \in L^2((\boldsymbol{R}^d)^p) \qquad \text{is continuous.}$$

Then, $\int_{(\boldsymbol{R}^d)^p} K(x; y_1, \cdots, y_p)\, W_\omega(dy_1) \cdots W_\omega(dy_p)$ has a measurable modification $I_p(K_x)(\omega)$ which is measurable with respect to two variables (ω, x) such that for all $x \in \boldsymbol{R}^d$

$$\int_{(\boldsymbol{R}^d)^p} K(x; y_1, \cdots, y_p)\, W_\omega(dy_1) \cdots W_\omega(dy_p) = I_p(K_x)(\omega) \quad P-a.s.\, \omega \in \Omega.$$

And the following Fubini type formula holds:

$$\int_{\boldsymbol{R}^d} g(x) I_p(K_x)(\omega)\, dx = \int_{(\boldsymbol{R}^d)^p} \Big(\int_{\boldsymbol{R}^d} g(x)\, K(x; y_1, \cdots, y_p)\, dx \Big)\, W_\omega(dy_1) \cdots W_\omega(dy_p) \qquad P-a.s.\, \omega \in \Omega.$$

Proposition 4 *For each $k \in \boldsymbol{N}$ and $r \geq 1$ there exists $M_{k,r}$ and*

$$\int_\Omega \int_{\boldsymbol{R}^d} \left| :_k \phi^p_{\alpha,\omega} : (x) \right|^r dx P(d\omega) < M_{k,r}. \tag{1. 15}$$

Also for each k and l let

$$U^{k,l}(x_1, \dots, x_l) \equiv E\big[(:_k \phi^p_{\alpha,\cdot} : (x_1)) \cdots (:_k \phi^p_{\alpha,\cdot} : (x_l))\big],$$

then

$$U^{k,l} \in C_0((\boldsymbol{R}^d)^l \to \boldsymbol{R}). \tag{1. 16}$$

Proof. (1.15) follows from Lemma 10 in Appendix.
(1.16) can be shown as follows: For symmetric functions $f(y_1, \cdots, y_p) \in L^2((\boldsymbol{R}^d)^p; \lambda^{dp})$ and $g(y_1, \cdots, y_q) \in L^2((\boldsymbol{R}^d)^q; \lambda^{dq})$ the multiple stochastic integrals $I_p(f) = \int_{(\boldsymbol{R}^d)^p} f(y_1, \cdots, y_p) W(dy_1) \cdots W(dy_p)$ and $I_q(g) = \int_{(\boldsymbol{R}^d)^q} g(y_1, \cdots, y_q) W(dy_1) \cdots W(dy_q)$ satisfy

$$I_p(f) I_q(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \otimes_r g), \tag{1. 17}$$

$$E[I_p(f)I_q(g)] = \begin{cases} 0 & p \neq q \\ p! < f, g >_{L^2((\boldsymbol{R}^d)^p)} & p = q \end{cases}, \tag{1. 18}$$

where

$$(f \otimes_r g)(y_1, \ldots, y_{p+q-2r}) = \int_{(\boldsymbol{R}^d)^r} f(y_1, \ldots, y_{p-r}, \boldsymbol{y}) g(y_{p+1}, \ldots, y_{p+q-r}, \boldsymbol{y}) d\boldsymbol{y}$$

(cf. section1.1 of [Nu]). By (1.14) for each x since $:_k \phi^p_\alpha : (x) = I_p(F^\alpha_k(x; \cdot))$, using (1.17) and (1.18) over again, then we see that $U^{k,l}(x_1, \ldots, x_l)$ is a linear combination of

$$\Big(((F^\alpha_k(x_1; \cdot) \otimes_{r_1} F^\alpha_k(x_2; \cdot)) \otimes_{r_2} F^\alpha_k(x_3; \cdot)) \otimes_{r_3} F^\alpha_k(x_4; \cdot) \cdots \Big) \otimes_{r_{l-1}} F^\alpha_k(x_l; \cdot) \in C_0((\boldsymbol{R}^d)^l \to \boldsymbol{R}),$$

where integers $r_1, \ldots, r_{l-1}$ satisfy

$$0 \leq r_1 \leq p, \qquad 0 \leq r_k \leq p \wedge (kp - 2r_1 - \cdots - 2r_{k-1}), \quad k = 2, \ldots, l-1.$$

Hence

$$U^{k,l} \in C_0((\boldsymbol{R}^d)^l \to \boldsymbol{R}).$$

■

Statements i), ii) and iii) of the following Proposition 5 are the results of Theorem 1 in [Y], of which proof is given in Appendix.

Proposition 5 *Suppose that a positive integer p and positive real numbers a, b and α satisfy*

$$\min\left(1, \frac{2a}{d}\right) + p \times \min\left(1, \frac{2\alpha}{d}\right) > p, \qquad b > d, \tag{1. 19}$$

and let $\{:_k \phi^p_{\alpha,\omega} :\}$ be the sequence of $C_0(\boldsymbol{R}^d \to \boldsymbol{R})$-valued random variables defined by (1.14). Then the following hold:
i)

$$\lim_{k,m\to\infty} \int_\Omega \left\| :_k \phi^p_{\alpha,\omega} : - :_m \phi^p_{\alpha,\omega} : \right\|^2_{B^{a,b}_d} P(d\omega) = 0. \tag{1. 20}$$

ii) There exists a P-null set $\mathcal{N}$, a subsequence $\{:_{k_j} \phi^p_{\alpha,\omega} :\}$ of $\{:_k \phi^p_{\alpha,\omega} :\}$ and a $B^{a,b}_d$-valued random variable $: \phi^p_{\alpha,\omega} :$ such that

$$\lim_{k_j\to\infty} \| :_{k_j} \phi^p_{\alpha,\omega} : - : \phi^p_{\alpha,\omega} : \|_{B^{a,b}_d} = 0, \qquad \forall \omega \in \Omega \setminus \mathcal{N}. \tag{1. 21}$$

iii) For $\varphi \in \mathcal{S}(\boldsymbol{R}^d)$ there exists a P-null set $\mathcal{N}_\varphi$ which may depend on φ and

$$<: \phi^p_{\alpha,\omega} :, \varphi>_{\mathcal{S}',\mathcal{S}} = l_{p,\omega}(\varphi) \qquad \forall \omega \in \Omega \setminus \mathcal{N}_\varphi \tag{1. 22}$$

and

$$\lim_{k\to\infty} \| <:_k \phi^p_{\alpha,\cdot} :, \varphi>_{\mathcal{S}',\mathcal{S}} - <: \phi^p_{\alpha,\cdot} :, \varphi>_{\mathcal{S}',\mathcal{S}} \|_{L^2(\Omega;P)} = 0, \tag{1. 23}$$

where

$$l_{p,\omega}(\varphi) \quad = \quad \int_{(\boldsymbol{R}^d)^p} \Big(\int_{\boldsymbol{R}^d} \varphi(x)\, J^\alpha(x - y_1) \cdots J^\alpha(x - y_p)\, dx \Big) W_\omega(dy_1) \cdots W_\omega(dy_p).$$

iv) There exists a constant C and

$$\Big\{ E\big[|<: \phi^p_\alpha : \varphi >_{\mathcal{S}',\mathcal{S}}|^r \big] \Big\}^{\frac{1}{r}} \leq C(r-1)^{\frac{p}{2}} \|\varphi\|_{L^2(\boldsymbol{R}^d;\lambda^d)} \qquad \forall \varphi \in \mathcal{S}(\boldsymbol{R}^d) \quad \forall r \geq 2. \tag{1. 24}$$

Proof. Since i)-iii) are the results in Theorem 1 of [Y], we only show iv) roughly. By (1.22) $<: \phi^p_{\alpha,\omega} :, \varphi>_{S',S}$ has an expression of multiple stochastic integral $l_{p,\omega}(\varphi)$, for this applying (1.18) and a standard multiple convolution argument (cf. Theorem V.2 of [Si] and (1.22) of [Y]) we have

$$\begin{aligned} E\big[|<: \phi^p_\alpha : \varphi >_{S',S}|^2\big] &= \int_{(\boldsymbol{R}^d)^p} \Big(\int_{\boldsymbol{R}^d} \varphi(x) F^\alpha(x; y_1, \ldots, y_p) dx\Big)^2 dy_1 \cdots dy_p \\ &\leq C' \|\varphi\|^2_{L^2(\boldsymbol{R}^d; \lambda^d)} \qquad \text{for all} \quad \varphi \in \mathcal{S}(\boldsymbol{R}^d). \end{aligned}$$

By this and Theorem I.22 of [Si] (cf. also (1.17) and (1.18)) (1.24) follows. ■

In the seque we shall denote $:_k \phi^1_{\alpha,\omega}:$ and $: \phi^1_{\alpha,\omega} :$ by ${}_k\phi_{\alpha,\omega}$ and $\phi_{\alpha,\omega}$ respectively. Recall that for each $x \in \boldsymbol{R}^d$ $\eta_k(x) J^\alpha_k(x - \cdot) \in \cap_{p \geq 1} L^p(\boldsymbol{R}^d; \lambda^p)$, and

$${}_k\phi_{\alpha,\omega}(x) = \int_{\boldsymbol{R}^d} \eta_k(x) J^\alpha_k(x - y) W_\omega(dy) \in L^2(\Omega; P).$$

Also we have to recall that by Theorem 1.1.2 of [Nu] for each $x \in \boldsymbol{R}^d$ the real valued random variable $:_k \phi^p_{\alpha,\omega} : (x)$ defined by (1.14), which satisfies (1.20) and (1.21), is the p-th Wick power of the $L^2(\Omega; P)$-random variable ${}_k\phi_{\alpha,\omega}(x)$:

$$\begin{aligned} :_k \phi^p_{\alpha,\omega} : (x) &= \int_{(\boldsymbol{R}^d)^p} \big(\eta_k(x) J^\alpha_k(x - y_1)\big) \cdots \big(\eta_k(x) J^\alpha_k(x - y_p)\big) W_\omega(dy_1) \cdots W_\omega(dy_p) \\ &= p! \sum_{m=0}^{[\frac{p}{2}]} \frac{1}{m!(p-2m)!} \big({}_k\phi_{\alpha,\omega}(x)\big)^{p-2m} \Big(-\frac{1}{2} b_{\alpha,k}(x)\Big)^m, \end{aligned} \tag{1. 25}$$

where

$$b_{\alpha,k}(x) = \big(\eta_k(x)\big)^2 \int_{\boldsymbol{R}^d} \big(J^\alpha_k(x - z)\big)^2 dz.$$

2 Main results

In this section we shall show that the Euclidean random field $: \phi^p_{\alpha,\omega} :$ defined by (1.21) has the property of reflection positivity by making use of the propositions given in the preceding section. We adopt the definition of reflection positivity for Euclidean random fields given in section 5 of [AGW], and use same terminologies: $\boldsymbol{R}^d_+ \equiv \{x \in \boldsymbol{R}^d \,|\, x = (x^0, \vec{x}) \in \boldsymbol{R} \times \boldsymbol{R}^{d-1}, x^0 > 0\}$, $\mathcal{S}((\boldsymbol{R}^d_+)^n)$ be the real Schwartz-functions on $\boldsymbol{R}^{dn}$ with supports in $(\boldsymbol{R}^d_+)^n$ and a time reflection operator θ is defined by $\theta(x^0, \vec{x}) = (-x^0, \vec{x})$.

Proposition 6 *For $\alpha \in (0, 1]$ and $d \in \boldsymbol{N}$ let $\phi_{\alpha,\omega}$ be $\mathcal{S}'(\boldsymbol{R}^d \to \boldsymbol{R})$-valued random variable defined by (1.21) for $p = 1$. Then for $a^n_r \in \boldsymbol{R}$, $\varphi^{n,r}_l \in \mathcal{S}(\boldsymbol{R}^d_+ \to \boldsymbol{R})$, $r = 1, \ldots, N_n$, $l = 1, \ldots, n$, $n = 1, \ldots, N$ $(N_n, N \in \boldsymbol{N})$ and $a \in \boldsymbol{R}$, the following holds:*

$$\begin{aligned} E\Big[&\Big\{\sum_{n=1}^N \big(\sum_{r=1}^{N_n} a^n_r < \theta\varphi^{n,r}_1, \phi_\alpha > \cdots < \theta\varphi^{n,r}_n, \phi_\alpha >\big) + a\Big\} \\ &\times \Big\{\sum_{n=1}^N \big(\sum_{r=1}^{N_n} a^n_r < \varphi^{n,r}_1, \phi_\alpha > \cdots < \varphi^{n,r}_n, \phi_\alpha >\big) + a\Big\}\Big] \geq 0. \end{aligned} \tag{2. 1}$$

Proof. The Euclidean random field defined by $\phi_{\alpha,\omega}$ is a generalized Euclidean free field, and the corresponding sequence of Schwinger functions S^α_n satisfy the following:

$$S^\alpha_n(\varphi_1 \otimes \cdots \otimes \varphi_n) \equiv E[< \varphi_1, \phi_\alpha > \cdots < \varphi_n, \phi_\alpha >], \tag{2. 2}$$

$$S_0^\alpha \equiv 1, \qquad \varphi_r \in \mathcal{S}(\boldsymbol{R}^d \to \boldsymbol{R}), \quad r = 1, \ldots, n.$$

$$\sum_{n=0}^{N}\sum_{m=0}^{N} S_{n+m}^\alpha(\theta f_n \otimes f_m) \geq 0 \qquad \text{for all} \quad f_0 \in \boldsymbol{R},\ f_n \in \mathcal{S}((\boldsymbol{R}_+^d)^n),\ n = 1, \ldots, N. \tag{2. 3}$$

If we take $f_n = \sum_{r=1}^{N_n} a_r^n \varphi_1^{n,r} \otimes \cdots \otimes \varphi_n^{n,r}$ and $f_0 = a$ in (2.3), then by (2.2) the desired result follows. ■

Theorem 7 *Let $\alpha \in (0,1]$ and $d \in \boldsymbol{N}$. Also let p be a positive integer satisfying $\min(1, \frac{2\alpha}{d}) > \frac{p-1}{p}$. Then the $\mathcal{S}'(\boldsymbol{R}^d \to \boldsymbol{R})$-valued random variable $: \phi_{\alpha,\omega}^p :$ defined by (1.21) satisfies the property of reflection positivity: for $\varphi_l^{n,r} \in \mathcal{S}(\boldsymbol{R}_+^d \to \boldsymbol{R})$, $r = 1, \ldots, N_n$, $l = 1, \ldots, n$, $n = 1, \ldots, N$ $(N_n, N \in \boldsymbol{N})$ and $a \in \boldsymbol{R}$,*

$$E\Big[\Big\{\sum_{n=1}^{N}(\sum_{r=1}^{N_n} < \theta\varphi_1^{n,r}, : \phi_\alpha^p :> \cdots < \theta\varphi_n^{n,r}, : \phi_\alpha^p :>) + a\Big\}$$
$$\times \Big\{\sum_{n=1}^{N}(\sum_{r=1}^{N_n} < \varphi_1^{n,r}, : \phi_\alpha^p :> \cdots < \varphi_n^{n,r}, : \phi_\alpha^p :>) + a\Big\}\Big] \geq 0. \tag{2. 4}$$

Proof. Since $C_0^\infty(\boldsymbol{R}^d \to \boldsymbol{R})$ is dense in $\mathcal{S}(\boldsymbol{R}^d \to \boldsymbol{R})$, by (1.24) and Hölder's inequality it suffices to prove (2.4) for $\mathcal{D}_+ \equiv C_0^\infty(\boldsymbol{R}^d \to \boldsymbol{R}) \cap \mathcal{S}(\boldsymbol{R}_+^d \to \boldsymbol{R})$.

By Proposition 5 (from (1.23) similar to the proof of (1.24) we also have L^r convergence of $< \varphi, :_k \phi_\alpha^p :>$ to $< \varphi, : \phi_\alpha^p :>$ for all $r \geq 2$) we see that

$$\lim_{k\to\infty}\Big\{E\Big[\Big\{\sum_{n=1}^{N}(\sum_{r=1}^{N_n} < \theta\varphi_1^{n,r}, :_k \phi_\alpha^p :> \cdots < \theta\varphi_n^{n,r}, :_k \phi_\alpha^p :>) + a\Big\}$$
$$\times \Big\{\sum_{n=1}^{N}(\sum_{r=1}^{N_n} < \varphi_1^{n,r}, :_k \phi_\alpha^p :> \cdots < \varphi_n^{n,r}, :_k \phi_\alpha^p :>) + a\Big\}\Big]\Big\}$$
$$= E\Big[\Big\{\sum_{n=1}^{N}(\sum_{r=1}^{N_n} < \theta\varphi_1^{n,r}, : \phi_\alpha^p :> \cdots < \theta\varphi_n^{n,r}, : \phi_\alpha^p :>) + a\Big\}$$
$$\times \Big\{\sum_{n=1}^{N}(\sum_{r=1}^{N_n} < \varphi_1^{n,r}, : \phi_\alpha^p :> \cdots < \varphi_n^{n,r}, : \phi_\alpha^p :>) + a\Big\}\Big]. \tag{2. 5}$$

Also by (1.15) and Hölder's inequality we have

$$E\Big[\Big\{\sum_{n=1}^{N}\Big(\sum_{r=1}^{N_n}\Big(\int_{\boldsymbol{R}^d} |\theta\varphi_1^{n,r}(x) :_k \phi_\alpha^p : (x)|dx\Big) \cdots \Big(\int_{\boldsymbol{R}^d} |\theta\varphi_n^{n,r}(x) :_k \phi_\alpha^p : (x)|dx\Big)\Big) + |a|\Big\}$$
$$\times\Big\{\sum_{n=1}^{N}\Big(\sum_{r=1}^{N_n}\Big(\int_{\boldsymbol{R}^d} |\varphi_1^{n,r}(x) :_k \phi_\alpha^p : (x)|dx\Big) \cdots \Big(\int_{\boldsymbol{R}^d} |\varphi_n^{n,r}(x) :_k \phi_\alpha^p : (x)|dx\Big)\Big) + |a|\Big\}\Big] < \infty. \tag{2. 6}$$

Since for other p, N and N_n the proofs can be carried out similarly, in order to clarify the discussion and notations we only prove (2.4) for $N = 1$, $N_1 = 1$ and $p = 2$. By (1.14) and (1.18) since $E[:_k \phi_\alpha^p : (x)] = 0$, by (2.6) and Fubini's lemma we have

$$E[\{< \theta\varphi, :_k \phi_\alpha^p :> +a\}\{< \varphi, :_k \phi_\alpha^p :> +a\}]$$
$$= E\Big[\Big\{\int_{\boldsymbol{R}^d} \varphi(\theta x) :_k \phi_\alpha^p : (x)dx + a\Big\}\Big\{\int_{\boldsymbol{R}^d} \varphi(x) :_k \phi_\alpha^p : (x)dx + a\Big\}$$
$$= \int_{\boldsymbol{R}^{2d}} \varphi(x)\varphi(x')E\Big[\big(:_k \phi_\alpha^p : (x)\big)\big(:_k \phi_\alpha^p : (\theta x')\big)\Big]dxdx' + a^2. \tag{2. 7}$$

By (2.5) it suffices to prove that the right hand side of (2.7) is not less than 0 for $\varphi \in \mathcal{D}_+$ when k is large enough. For $\varphi \in \mathcal{D}_+$ let δ be the distance between $supp[\varphi]$ and the boundary of $\boldsymbol{R}^d_+$. For each $M \in \boldsymbol{N}$ we take $\{D^M_j\}_{j=1,\ldots,M}$, a covering of $supp[\varphi]$, such that distance between $\overline{D^M_j}$ and the boundary of $\boldsymbol{R}^d_+$ is not less than $\delta - \frac{1}{M}$, $\overline{\bigcup_{j=1}^M D^M_j}$ is compact, $\bigcup_{j=1}^M D^M_j \supset supp[\varphi]$, $D^M_j \cap D^M_{j'} = \emptyset \; (j \neq j')$, $(|supp[\varphi]| - 1)/M \leq |D^M_j| \leq (|supp[\varphi]| + 1)/M$; $\bigcap_{M=1}^\infty \bigcup_{j=1}^M D^M_j = supp[\varphi]$.

For each M and j we denote x^M_j as a point in the partition D^M_j. By (1.16) we know that $U^{k,2} \in C_0((\boldsymbol{R}^d)^2 \to \boldsymbol{R})$, and the RHS of (2.7) is the limit of a Riemann sum:

$$\int_{\boldsymbol{R}^{2d}} \varphi(x)\varphi(x')E\Big[\big(:_k \phi^p_\alpha : (x)\big)\big(:_k \phi^p_\alpha : (\theta x')\big)\Big]dxdx' + a^2 = \lim_{M\to\infty} \mathcal{E}_k(M) + a^2,$$

where

$$\mathcal{E}_k(M) = \sum_{j=1}^M \sum_{i=1}^M \varphi(x^M_j)\varphi(x^M_i)U^{k,2}(\theta x^M_j, x^M_i)|D^M_j||D^M_i|.$$

It can be seen that if $\frac{1}{k} + \frac{1}{M} < \delta$, then

$$\mathcal{E}_k(M) + a^2 \geq 0. \tag{2. 8}$$

Indeed since $E[:_k \phi^p_\alpha : (x^M_j)] = 0$ (cf. (1.14) and (1.18)), by Proposition 4 we see that

$$\mathcal{E}_k(M) + a^2 = E\Big[\{\sum_{j=1}^M \varphi(x^M_j)(:_k \phi^p_\alpha : (\theta x_j))|D^M_j| + a\}\{\sum_{i=1}^M \varphi(x^M_i)(:_k \phi^p_\alpha : (x_i))|D^M_i| + a\}\Big]. \tag{2. 9}$$

And for $p = 2$ by (1.25) we have

$$\begin{aligned}
\sum_{i=1}^M \varphi(x^M_i)(:_k \phi^2_\alpha : (x^M_i))|D^M_i| + a &= \sum_{i=1}^M \varphi(x^M_i)\Big\{\big({}_k\phi_\alpha(x^M_i)\big)^2 - b_{\alpha,k}(x^M_i)\Big\}|D^M_i| + a \\
&= \sum_{i=1}^M \varphi(x^M_i)\Big\{\big(< \eta_k(x^M_i)\rho_k(x^M_i - \cdot), \phi_\alpha >\big)^2 - b_{\alpha,k}(x^M_i)\Big\}|D^M_i| + a \\
&= \sum_{i=1}^M a_i < \varphi^i_1, \phi_\alpha >< \varphi^i_2, \phi_\alpha > + a',
\end{aligned} \tag{2. 10}$$

where

$$a_i = \varphi(x^M_i)(\eta_k(x^M_i))^2|D^M_i|, \qquad \varphi^i_1(\cdot) = \varphi^i_2(\cdot) = \rho_k(x^M_i - \cdot), \qquad a' = a - \sum_{i=1}^M \varphi(x^M_i)b_{\alpha,k}(x^M_i)|D^M_i|.$$

Also since $b_{\alpha,k}(\theta x) = b_{\alpha,k}(x)$ and $\rho(\theta x - y) = \rho(x - \theta y)$ we have

$$\sum_{i=1}^M \varphi(x^M_i)(:_k \phi^2_\alpha : (\theta x^M_i))|D^M_i| + a = \sum_{i=1}^M a_i < \theta\varphi^i_1, \phi_\alpha >< \theta\varphi^i_2, \phi_\alpha > + a'. \tag{2. 11}$$

For $\varphi \in \mathcal{D}_+$ since $\rho_k(x^M_i - \cdot) \in \mathcal{D}_+$ for all $i = 1, \ldots, M$ when $\frac{1}{k} + \frac{1}{M} < \delta$ (cf. (1.7)), by (2.10) and (2.11) from Proposition 6 the RHS of (2.9) is non-negative for such k and M, and (2.4) has been proved for $N = 1$, $N_1 = 1$ and $p = 2$. By the above proof, it is obvious that the other cases can be proved by a similar way.

■

The following Corollary 8 is a direct consequence of Theorem 7:

Corollary 8 *Let $\alpha \in (0,1]$ and $d \in \boldsymbol{N}$. Also let p be a positive integer satisfying* $\min(1, \frac{2\alpha}{d}) > \frac{p-1}{p}$. *Then, the sequence of Schwinger functions $S_n^{\alpha,p}$ defined by*

$$S_n^{\alpha,p}[\varphi_1 \otimes \cdots \otimes \varphi_n] \equiv E[< \varphi_1, : \phi_\alpha^p > \cdots < \varphi_n, : \phi_\alpha^p >],$$
$$S_0^{\alpha,p} \equiv 1, \qquad \varphi_r \in \mathcal{S}(\boldsymbol{R}^d \to \boldsymbol{R}), \quad r = 1, \ldots, n$$

satisfy the property of reflection positivity (2.3).

It is interesting whether$S_n^{\alpha,p}$ can be analytically continued to some Wightman distribution or not. In order to define Wightman distribution corresponding to the Schwinger function $S_n^{\alpha,p}$ (if it is possible), roughly speaking we have to consider Laplace Fourier inverse transform of $S_n^{\alpha,p}$ (cf [AGW]). In the cases when $p \geq 2$, the inverse image of $S_n^{\alpha,p}$ involves convolutions of inverse image of $S_2^{\alpha,1}$. By this consideration we have the following Proposition 9, which is a result in [Y2].

Proposition 9 *By Corollary 8, for $d = 3$, $\alpha = 1$ and $p = 2$ the Schwinger functions $\{S_n^{1,2}\}$ satisy reflection positivity. But they can not be analytically continued to some Wightman distributions.*

Proof.(cf. [Y2]) The inverse image of Laplace Fourier transform T of $S_n^{1,1}$ for $d = 3$ does not admit convolution $T * T$ even in any sense of distributions. By the above mentioned considerations, the result follows. ■

Remarks and Notes 1 *i) In the early 70th multiple stochastic integrals were applied for the consideration of the Free Markov field by [Ne].*
ii) Considerations about Banach spaces in which $\phi_{1,\omega}$ takes values were made in [Re].
iii) In [Y] continuous maps $F : S' \to S'$ such that $F(\phi_{\alpha,\omega}) = J^\gamma(\eta_M : \phi_{\alpha,\omega}^p :)$ are considered, and for $p = 2$ it is shown that the map is $H - C^1$ (cf. [ÜZ] and [Ku]).
iv) For $d = 3$, $p = 2$ and $\alpha = 1$ the Euclidean random field $: \phi^2 :$ is reflection positive by Theorem 7. It may be interesting to consider the so called stochastic quantization (cf. [AR]) for this random field.
v) The result derived here and the results in a expository paper [Y] can be applied to considerations of various types of Schwinger functions, for eg. convoluted generalized white noise discussed in [AGW]. These applications will be made in future work.

3 Appendix

Lemma 10 *Suppose that a positive integer p and positive real numbers a, b and α satisfy*

$$\min(1, \frac{2a}{d}) + p\min(1, \frac{2\alpha}{d}) > p \qquad b > d. \tag{3. 1}$$

Let

$$G(y_1, \ldots, y_p; y) = \int_{\boldsymbol{R}^d} (1 + |x|^2)^{-\frac{b}{4}} J^a(x-y) F^\alpha(x; y_1, \ldots, y_p) dx,$$
$$G_k(y_1, \ldots, y_p; y) = \int_{\boldsymbol{R}^d} (1 + |x|^2)^{-\frac{b}{4}} J^a(x-y) F_k^\alpha(x; y_1, \ldots, y_p) dx,$$
$$G_{k,n}(y_1, \ldots, y_p; y) = \int_{\boldsymbol{R}^d} (1 + |x|^2)^{-\frac{b}{4}} J_n^a(x-y) F_k^\alpha(x; y_1, \ldots, y_p) dx.$$

Then

$$\lim_{k\to\infty} \|G - G_k\|_{L^2((\boldsymbol{R}^d)^{p+1};(\lambda^d)^{p+1})} = 0, \tag{3. 2}$$
$$\lim_{n\to\infty} \|G_k - G_{k,n}\|_{L^2((\boldsymbol{R}^d)^{p+1};(\lambda^d)^{p+1})} = 0. \tag{3. 3}$$

Proof. Since

$$\int_{(\boldsymbol{R}^d)^{p+1}}\{\int_{\boldsymbol{R}^d} g(x)J^a(x-y)F^\alpha(x;y_1,\dots,y_p)dx\}^2 d\boldsymbol{y} = \int_{\boldsymbol{R}^d}\int_{\boldsymbol{R}^d} J^{2a}(z-x)(J^{2\alpha}(z-x))^p g(z)g(x)dxdz,$$

by Theorem V.2 (cf. also Theorem V.3) in [Si], we see that under the assumption (3.1) for p, a and α there exists a constant C and

$$\int_{(\boldsymbol{R}^d)^{p+1}}\left\{\int_{\boldsymbol{R}^d} g(x)J^a(x-y)F^\alpha(x;y_1,\dots,y_p)dx\right\}^2 d\boldsymbol{y} \le C\|g\|^2_{L^2} \qquad \text{for all} \quad g\in L^2(\boldsymbol{R}^d;\lambda^d). \quad (3.\ 4)$$

In (3.4) if we set $g(x)=(1+|x|^2)^{-\frac{b}{4}}(1-(\eta_k(x))^p)$ and denote

$$G^0_k(y_1,\dots,y_p;y) = \int_{\boldsymbol{R}^d}(1+|x|^2)^{-\frac{b}{4}}J^a(x-y)(\eta_k(x))^p J^\alpha(x-y_1)\cdots J^\alpha(x-y_p)dx,$$

then

$$\int_{(\boldsymbol{R}^d)^{p+1}}|G(y_1,\dots,y_p;y)-G^0_k(y_1,\dots,y_p;y)|^2 d\boldsymbol{y} \le C\int_{|x|>k}(1+|x|^2)^{-\frac{b}{2}}dx. \qquad (3.\ 5)$$

Also for q such that

$$0<\frac{1}{q}<\frac{2\alpha}{d} \qquad (3.\ 6)$$

there exists a constant C and the following holds:

$$\begin{aligned}
&\int_{(\boldsymbol{R}^d)^{p+1}}|G^0_k(y_1,\dots,y_p;y)-G_k(y_1,\dots,y_p;y)|^2 d\boldsymbol{y}\\
&= (2\pi)^{-d}\int_{(\boldsymbol{R}^d)^{p+1}}|(\hat{\eta}_k)^p * \hat{f}(\sum_{i=1}^p\xi_i+\xi)|^2\prod_{i=1}^p(m^2+|\xi_i|^2)^{-\alpha}|1-\hat{\rho}_k(\xi_i)|^2(m^2+|\xi|^2)^{-a}\\
&\times d\xi_1\cdots d\xi_p d\xi\\
&\le C\|(\eta_k)^p f\|^2_{L^2}\left(\int_{\boldsymbol{R}^d}(|x|^2+m^2)^{-\alpha q}|1-\hat{\rho}_k(x)|^{2q}dx\right)^{\frac{1}{q}}, \text{ where } f(x)=(1+|x|^2)^{-\frac{b}{4}}. \qquad (3.\ 7)
\end{aligned}$$

Since $b>d$, by (3.5), (3.6), (3.7), (1.11) and (1.12) we see that (3.2) holds.

Similar to (3.7), since

$$\|G_k-G_{k,n}\|^2_{L^2((\boldsymbol{R}^d)^{p+1};(\lambda^d)^{p+1})} \le C\|(\eta_k)^p f\|^2_{L^2}\left(\int_{\boldsymbol{R}^d}(|x|^2+m^2)^{-as}|1-\hat{\rho}_n(x)|^{2s}dx\right)^{\frac{1}{s}}$$

holds for s such that $0<\frac{1}{s}<\frac{2a}{d}$, (3.3) can be proved. ■

Lemma 11 *Suppose that p, a, b and α satisfy (3.1) and let $\{:_k \phi^p_{\alpha,\omega}:\}$ be the continuous modification of (1.14) and $I_k(\omega,z)$ be a measurable modification of*

$$\int_{(\boldsymbol{R}^d)^p}\left(\int_{\boldsymbol{R}^d} J^a(x-z)(1+|x|^2)^{-\frac{b}{4}}F^\alpha_k(x;y_1,\dots,y_p)dx\right)W_\omega(dy_1)\cdots W_\omega(dy_p),$$

that is measurable with respect to two variables (ω,z), then for each k there exists a measurable set $\mathcal{O}_k\in\mathcal{F}\otimes\mathcal{B}(\boldsymbol{R}^d)$ such that $P\otimes\lambda^d(\mathcal{O}_k)=0$ and

$$\int_{\boldsymbol{R}^d} J^a(x-z)(1+|x|^2)^{-\frac{b}{4}} :_k \phi^p_{\alpha,\omega}:(x)\,dx = I_k(\omega,z) \qquad \forall(\omega,z)\in(\Omega\times\boldsymbol{R}^d)\setminus\mathcal{O}_k. \qquad (3.\ 8)$$

Proof. Let $J_n^a(x) = (\rho_n * J^a)(x)$, then for each $z \in \boldsymbol{R}^d$ $J_n^a(z-\cdot)(1+|\cdot|^2)^{-\frac{b}{4}} \in L^2$. Also, note that the function F_k^α satisfies the condition for K in Proposition 3. Hence, if we let $I_p(K_x)(\omega)$ be the measurable version of $\int_{(\boldsymbol{R}^d)^p} F_k^\alpha(x; y_1, \ldots, y_p) W_\omega(dy_1) \cdots W_\omega(dy_p)$, which is an element of $L^2(\boldsymbol{R}^d \times \Omega; \lambda^d \otimes P)$, and for each fixed z let $g(x) = J_n^a(z-x)(1+|x|^2)^{-\frac{b}{4}}$, then we can apply Proposition 3. On the other hand, the continuous modification $:_k \phi_{\alpha,\omega}^p :$ is an equivalent process of $\int_{(\boldsymbol{R}^d)^p} F_k^\alpha(x; y_1, \ldots, y_p) W_\omega(dy_1) \cdots W_\omega(dy_p)$ that satisfies $:_k \phi_{\alpha,\omega}^p :\in C_0(\boldsymbol{R}^d \to \boldsymbol{R})$ $\forall \omega \in \Omega$, and it is also a $\mathcal{B}(\boldsymbol{R}^d) \otimes \mathcal{F}$-measurable function. Thus, $:_k \phi_{\alpha,\omega}^p : (x)$ can play the roll of $I_p(K_x)(\omega)$ in Proposition 3 and we see the following: for each $z \in \boldsymbol{R}^d$

$$\int_{\boldsymbol{R}^d} J_n^a(x-z)(1+|x|^2)^{-\frac{b}{4}} :_k \phi_{\alpha,\omega}^p : (x)\, dx = I_{k,n}(\omega, z) \qquad P-a.s.\, \omega \in \Omega \tag{3. 9}$$

where

$$I_{k,n}(\omega, z) = \int_{(\boldsymbol{R}^d)^p} \left(\int_{\boldsymbol{R}^d} J_n^a(x-z)(1+|x|^2)^{-\frac{b}{4}} F_k^\alpha(x; y_1, \ldots, y_p) dx \right) W_\omega(dy_1) \cdots W_\omega(dy_p).$$

By Kolmogorov's continuity criterion we can assume $I_{k,n}(\omega, z)$ to be a continuous process:

$$I_{n,k}(\omega, \cdot) \in C_0(\boldsymbol{R}^d \to \boldsymbol{R}).$$

Then, since $:_k \phi_{\alpha,\omega}^p :\in C_0(\boldsymbol{R}^d \to \boldsymbol{R})$ the equality (3.9) holds storonger sense: for each n there exists a P-null set $\mathcal{N}_n$ and

$$\int_{\boldsymbol{R}^d} J_n^a(x-z)(1+|x|^2)^{-\frac{b}{4}} :_k \phi_{\alpha,\omega}^p : (x)\, dx = I_{k,n}(\omega, z) \qquad \forall z \in \boldsymbol{R}^d,\ \forall \omega \in \Omega \setminus \mathcal{N}_n. \tag{3. 10}$$

Next, by (1.1), since $J^a \in L^1$, applying Lebesgue's convergence theorem we see that for each $z \in \boldsymbol{R}^d$

$$\lim_{n\to\infty} \int_{\boldsymbol{R}^d} J_n^a(x-z)(1+|x|^2)^{-\frac{b}{4}} :_k \phi_{\alpha,\omega}^p : (x)\, dx = \int_{\boldsymbol{R}^d} J^a(x-z)(1+|x|^2)^{-\frac{b}{4}} :_k \phi_{\alpha,\omega}^p : (x) dx \ \ \forall \omega \in \Omega.$$

Then by (3.10)

$$\lim_{n\to\infty} I_{k,n}(\omega, z) = \int_{\boldsymbol{R}^d} J^a(x-z)(1+|x|^2)^{-\frac{b}{4}} :_k \phi_{\alpha,\omega}^p : (x) dx \quad \forall z \in \boldsymbol{R}^d,\ \forall \omega \in \Omega \setminus (\cup_n \mathcal{N}_n). \tag{3. 11}$$

On the other hand, by (1.13), (1.1) and Lemma 10 from Bochner Von Neumann measurability theorem, we can take a measurable modification $I_k(\omega, z)$ of

$$\int_{(\boldsymbol{R}^d)^p} \left(\int_{\boldsymbol{R}^d} J^a(x-z)(1+|x|^2)^{\frac{b}{4}} F_k^\alpha(x; y_1, \ldots, y_p) dx \right) W_\omega(dy_1) \cdots W_\omega(dy_p),$$

then by Fubini's lemma, and again by (1.13) and Lemma 1 we have

$$\begin{aligned} &\lim_{n\to\infty} \int_\Omega \int_{\boldsymbol{R}^d} \{I_{k,n}(\omega, z) - I_k(\omega, z)\}^2\, dz\, P(d\omega) \\ &= \lim_{n\to\infty} p! \int_{\boldsymbol{R}^d} \int_{(\boldsymbol{R}^d)^p} \left\{ \int_{\boldsymbol{R}^d} (J_n^a(x-z) - J^a(x-z))(1+|x|^2)^{-\frac{b}{4}} F_k^\alpha(x; y_1, \ldots, y_p) dx \right\}^2 dy dz = 0. \end{aligned}$$

Hence, for each k there exists a subsequence $\{I_{k,n_j}(\omega, z)\}_{j=1,2,\ldots}$ of $\{I_{k,n}(\omega, z)\}_{n=1,2,\ldots}$ and a measurable set $\mathcal{O}_k'$ satisfying $P \otimes \lambda^d(\mathcal{O}_k') = 0$ and the following holds:

$$\lim_{j\to\infty} I_{k,n_j}(\omega, z) = I_k(\omega, z) \qquad \forall (\omega, z) \in (\Omega \times \boldsymbol{R}^d) \setminus \mathcal{O}_k'.$$

Now, by this and (3.11) we obtain (3.8). ∎

Proof of Proposition 5-i), ii), iii). By making use of the expression (3.8) given by Lemma 11, noting (1.1), by (1.13) and Fubini's lemma we then have the following:

$$\begin{aligned}&\int_\Omega \int_{\boldsymbol{R}^d} \Big| \int_{\boldsymbol{R}^d} J^a(x-z)(1+|x|^2)^{-\frac{b}{4}}(:_k \phi^p_{\alpha,\omega} : (x) - :_m \phi^p_{\alpha,\omega} : (x))dx\Big|^2 dz P(d\omega) \\ &= p! \int_{\boldsymbol{R}^d} \int_{(\boldsymbol{R}^d)^p} \left(\int_{\boldsymbol{R}^2} J^a(x-z)(1+|x|^2)^{-\frac{b}{4}}(F^\alpha_k(x,y_1,\dots,y_p) - F^\alpha_m(x,y_1,\dots,y_p))dx \right)^2 \\ &\times dy_1 \cdots dy_p dz. \end{aligned} \tag{3. 12}$$

By Lemma 10 the right hand side of (3.12) vanishes as $k,\ m \to \infty$:

$$\lim_{k,m\to\infty} \int_\Omega \int_{\boldsymbol{R}^d} \Big| \int_{\boldsymbol{R}^d} J^a(x-z)(1+|x|^2)^{-\frac{b}{4}}(:_k \phi^p_{\alpha,\omega} : (x) - :_m \phi^p_{\alpha,\omega} : (x))dx\Big|^2 dz P(d\omega) = 0. \tag{3. 13}$$

This proves (1.20). Hence, by Proposition 2 the sequence $\{:_k \phi^p_{\alpha,\omega} :\}_{k=1,2,\dots}$ forms a Cauchy sequence in the Banach space

$$L^2(\Omega \to B^{a,b}_d; P) = \left\{ f \mid f : \Omega \ni \longmapsto f(\omega) \in B^{a,b}_d,\ \int_\Omega \| f(\omega)\|^2_{B^{a,b}_d} P(d\omega) < \infty \right\},$$

and there exists a $: \phi^p_{\alpha,\cdot} : \in L^2(\Omega \to B^{a,b}_d; P)$ such that

$$\lim_{k\to\infty} \int_\Omega \big\| :_k \phi^p_{\alpha,\omega} : - : \phi^p_{\alpha,\omega} : \big\|^2_{B^{a,b}_d} P(d\omega) = 0. \tag{3. 14}$$

By this, (1.21) holds for some subsequence $\{:_{k_j} \phi^p_{\alpha,\omega} :\}$.

For $\varphi \in \mathcal{S}(\boldsymbol{R}^d)$, using the similar expression as (3.8) for $<:_k \phi^p_{\alpha,\omega} :, \varphi>_{\mathcal{S}',\mathcal{S}}$, passing the similar discussion for (3.12) we have

$$\lim_{k\to\infty} \int_\Omega | <:_k \phi^p_{\alpha,\omega} :, \varphi >_{\mathcal{S}',\mathcal{S}} - l_{p,\omega}(\varphi)|^2 P(d\omega) = 0. \tag{3. 15}$$

Thus, there exists a P-null set $\mathcal{N}_\varphi$ that may depend on φ and for some subsequence $\{:_{\tilde{k}_j} \phi^p_{\alpha,\omega} :\}$ of $\{:_{k_j} \phi^p_{\alpha,\omega} :\}$ the following holds:

$$\lim_{\tilde{k}_j\to\infty} <:_{\tilde{k}_j} \phi^p_{\alpha,\omega} :, \varphi >_{\mathcal{S}',\mathcal{S}} = l_\omega(\varphi) \qquad \forall \omega \in \Omega \setminus \mathcal{N}_\varphi. \tag{3. 16}$$

On the other hand, obviously the convergence of $:_{k_j} \phi^p_{\alpha,\omega}$: to $: \phi^p_{\alpha,\omega}$: with respect to $B^{a,b}_d$-norm implies the weak convergence:

$$<: \phi^p_{\alpha,\omega} :, \varphi >_{\mathcal{S}',\mathcal{S}} = \lim_{k_j\to\infty} <:_{k_j} \phi^p_{\alpha,\omega} :, \varphi >_{\mathcal{S}',\mathcal{S}} \qquad P-a.e.\ \omega \in \Omega. \tag{3. 17}$$

Hence, by (3.15), (3.16) and (3.17) we see that

$$<: \phi^p_{\alpha,\omega} :, \varphi>_{\mathcal{S}',\mathcal{S}} = l_{p,\omega}(\varphi) \qquad \forall \omega \in \Omega \setminus \mathcal{N}_\varphi$$

and

$$\lim_{k\to\infty} \int_\Omega | <:_k \phi^p_{\alpha,\omega} :, \varphi>_{\mathcal{S}',\mathcal{S}} - <: \phi^p_{\alpha,\omega} :, \varphi>_{\mathcal{S}',\mathcal{S}}|^2 P(d\omega) = 0.$$

∎

References

[AGW] Albeverio, S., Gottschalk, H., Wu, J.-L.: *Convoluted generalized white noise, Schwinger functions and their analytic continuation to Wightman functions.* Rev. Math. Phys. 8 (1996), 763-817.

[AH-K] Albeverio, S., Høegh-Krohn, R.: *Euclidean Markov fields and relativistic quantum fields from stochastic partial differential equations in four dimensions.* Physics Letters B 177, 175-179, (1986).

[AR] Albeverio, S., Röckner, M.: *Classical Dirichlet forms on topological vector spaces - construction of an associated diffusion process.* Probab. Th. Rel. Fields 83, 405-434 (1989).

[Ku] Kusoka, S.: *The non-linear transformation of Gaussian measure on Banach space and its absolute continuity (I).* J. Fac. Sci. Univ. Tokyo IA 29, 567-597 (1982).

[Ne] Nelson, E.: *The free Markov field.* J. Funct. Anal. 12, 221-227 (1973).

[Nu] Nualart, D.: *The Malliavin calculus and related topics.* Springer-Verlag, New York/Heidelberg/Berlin, 1995.

[Re] Reed, M.: *Functional analysis and probability theory.* In: Lecture Notes in Physics Vol. 25, 2-43, Springer-Verlag, New York/Heidelberg/Berlin, 1973.

[Rö] Röckner, M.: *Traces of harmonic functions and a new path space for the free quantum field.* J. Funct. Anal. 79, 211-249 (1988).

[Si] Simon, B.: *The $P(\Phi)_2$ Euclidean (Quantum) Field Theory.* Princeton Univ. Press, Princeton, NJ., 1974.

[Tr] Treves, F.: *Topological vector spaces, distributions and kernals.* Academic Press, New York, 1967.

[ÜZ] Üstünel, A.S., Zakai, M.: *The change of variable formula on Wiener space.* Lect. Notes Math, Vol. 1655, pp. 24-39, Springer-Verlag, Berlin/Heidelberg/New York, 1997.

[Y1] Yoshida, M.W.: *Abstract Wiener spaces on spaces of measures defined by multiple stochastic integrals and non-linear continuous maps on them.* To appear from Bulletin of The Univ. of Electro- Commun. Vol. 12 (1999)

[Y2] Yoshida, M.W.: *Consideration of scalar quantum field theory with definite and indefinite metric through new Examples for all dimensions* In Preparation